Mathematical Methods for Modern Physics

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Outline I

Lecture 1 : Simplicial Homology I

Simplices

- $\Delta^n \subset \mathbb{R}^{n+1}$ given by $(t_0, \cdots, t_n)$ with $t_i \geq 0$ and $\sum t_i = 1$
- $\breve{\Delta}^n$ same with $t_i > 0$
- Standard linear maps from the faces

$$\begin{aligned} m_i : \Delta^{n-1} &\longrightarrow \Delta^n \qquad\qquad (i = 0, \cdots, n) \\ (t_0, \cdots, t_{n-1}) &\mapsto (t_0, \cdots, 0, \cdots, t_{n-1}) \end{aligned}$$

  with 0 in i–th position
- For $0 \leq j \leq n-1$ and $0 \leq i \leq n$ we have

$$\begin{aligned} m_i \circ m_j &= m_j \circ m_{i-1} \qquad (i > j) \\ &= m_{j+1} \circ m_i \qquad (i \leq j) \end{aligned}$$

Definition of Finite Δ–Complex

- X topological space
- A finite list of maps $\sigma_\alpha : \Delta^{n_\alpha} \longrightarrow X$ such that

- σ_α one to one from $\breve{\Delta}^{n_\alpha}$ to $e_\alpha \equiv \sigma_\alpha\left(\breve{\Delta}^{n_\alpha}\right)$
- The sets e_α have vanishing overlap and cover X
- If σ_α is in the list, then so is $\sigma_\alpha \circ m_i$ for $i = 0, \cdots, n_\alpha$
- $A \subset X$ is open (closed) in $X \Leftrightarrow \sigma_\alpha^{-1}(A)$ is open (closed) in Δ^{n_α}

Homology

- X is a Δ–complex
- $\Delta_n(X)$ formal linear combinations with integer coefficients of maps σ_α with $n_\alpha = n$

$$\sum_{\alpha \text{ with } n_\alpha = n} k_\alpha \ \sigma_\alpha \qquad\qquad (k_\alpha \in \mathbb{Z})$$

Free abelian group with basis given by maps σ_α with $n_\alpha = n$. Elements are called n–chains

Lecture 1 : Simplicial Homology III

- Boundary maps

$$\partial : \Delta_n(X) \to \Delta_{n-1}(X)$$
$$\partial\sigma_\alpha = \sum_{i=0}^{n_\alpha} (-)^i \; \sigma_\alpha \circ m_i$$

- Basic fact

$$\partial^2 = 0$$

Proof (with $n = n_\alpha$)

$$\partial^2\sigma_\alpha = \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-)^{i+j} \sigma_\alpha \circ m_i \circ m_j$$
$$= \sum_{n \geq i > j \geq 0} (-)^{i+j} \sigma_\alpha \circ m_i \circ m_j + \sum_{n-1 \geq j \geq i \geq 0} (-)^{i+j} \sigma_\alpha \circ m_{j+1} \circ m_i$$

In the second term, $j+1 \to i$ and $i \to j$. We obtain the first term, up to an overall $(-)$ sign

Lecture 1 : Simplicial Homology IV

- Chain complex

$$\cdots \longrightarrow C_{n+1} \stackrel{\partial_{n+1}}{\longrightarrow} C_n \stackrel{\partial_n}{\longrightarrow} C_{n-1} \longrightarrow \cdots$$

with

$$\partial_n \circ \partial_{n+1} = 0$$

We have

C_n	Chains
$\ker \partial_n$	Cycles
$\operatorname{Im} \partial_{n+1}$	Boundaries

and

$$\operatorname{Im} \partial_{n+1} \subset \ker \partial_n \subset C_n$$

Lecture 1 : Simplicial Homology V

- Homology groups of chain complex

$$H_n(C) = \ker \partial_n \ / \ \operatorname{Im} \partial_{n+1}$$

Two cycles are homologous if they differ by a boundary

$$\partial\alpha = 0 \qquad\qquad [\alpha] = [\beta] \quad \text{if} \quad \alpha = \beta + \partial\gamma$$

- Simplicial homology of the complex $\Delta_n(X)$ denoted by

$$H_n^{\Delta}(X)$$

Two Dimensional Examples

- Point and S_1

$$0 \rightarrow \mathbb{Z} \rightarrow 0 \qquad\qquad H_0(\text{point}) = \mathbb{Z}$$

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \qquad\qquad H_0(S_1) = H_1(S_1) = \mathbb{Z}$$

Lecture 1 : Simplicial Homology VI

- Torus

$$\partial U = \partial D = a + b - c$$
$$\partial a = \partial b = \partial c = 0$$

with chain complex

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$$

and

$$H_0 = \mathbb{Z}$$
$$H_1 = \mathbb{Z}^2 \qquad \text{(using the base } a, b, a + b - c \text{ is obvious)}$$
$$H_2 = \mathbb{Z} \qquad \text{(generated by } U - D\text{)}$$

Lecture 1 : Simplicial Homology VII

- Real projective plane $\mathbb{R}P^2 = S_2 / (x \sim -x)$

$$
\begin{aligned}
\partial U &= a - b + c \\
\partial D &= -a + b + c \\
\partial a = \partial b &= w - v \qquad\qquad \partial c = 0
\end{aligned}
$$

with chain complex

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0$$

and

$$
\begin{aligned}
H_0 &= \mathbb{Z} \\
H_1 &= \mathbb{Z}_2 \qquad (\ker \partial_1 \text{ generated by } \tilde{c} = a - b \text{ and } c \\
& \qquad\qquad \operatorname{Im} \partial_2 \text{ by } c + \tilde{c} \text{ and } c - \tilde{c}) \\
H_2 &= 0
\end{aligned}
$$

Lecture 1 : Simplicial Homology VIII

- Surface of genus g with κ crosscaps. Chain complex

$$0 \rightarrow \mathbb{Z}^{4g-2+4\kappa} \xrightarrow{\partial_2} \mathbb{Z}^{6g+6\kappa-3} \xrightarrow{\partial_1} \mathbb{Z}^{1+\kappa} \rightarrow 0$$

If $\kappa = 0$ then

$\partial\left(U_1 - U_2 + \cdots\right) = 0$ unique generator of H_2

The c_i are homologous to a_i, b_i

$\partial_1 = 0$

and we get homology

$$H_2 = \mathbb{Z} \qquad H_1 = \mathbb{Z}^{2g} \qquad H_0 = \mathbb{Z}$$

If $\kappa > 0$ choose $g = 0$ since

$$(g, \kappa) \sim (g-1, \kappa+2) \qquad (\kappa > 0)$$

Then

$$\ker \partial_2 = 0$$

$$\ker \partial_1 \quad \text{generated by} \begin{cases} c_i & 2\kappa - 1 \\ a_i - d_i,\ b_i - d_i & 2\kappa \\ d_{i+1} - d_i & \kappa - 2 \end{cases}$$

$\operatorname{Im} \partial_2$ generated by $4\kappa - 2$ terms of the form $c + a - d\,,\ c + b - d$

and (reinserting g)

$$H_2 = 0 \qquad H_1 = \mathbb{Z}^{2g+\kappa-1} \oplus \mathbb{Z}_2 \qquad H_0 = \mathbb{Z}$$

- Exercise : If one starts with chains $\sum k_\alpha\ \sigma_\alpha$ with coefficients in an arbitrary abelian group G one obtains the homology groups with coefficients in G

$$H_n^\Delta(X, G)$$

Compute the homology groups for the above spaces for $G = \mathbb{Z}_2$, $\mathbb{R}$.

Lecture 2 : Cohomology and Homological Algebra I

Simplicial Cohomology

- Give chain complex $\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} \cdots$ and abelian group G define cochains

$$
\begin{aligned}
C^n(G) &= \mathrm{Hom}(C_n, G) = C_n^\star \\
&= \text{group maps from } C_n \text{ to } G
\end{aligned}
$$

- Coboundary map δ with $\delta^2 = 0$

$$
\begin{aligned}
&\delta : C^n(G) \to C^{n+1}(G) \\
&C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\varphi} G \\
&\delta\varphi = \varphi\partial
\end{aligned}
$$

- Cocycle and coboundary

$$
\begin{aligned}
\delta\varphi &= 0 && (\varphi \text{ vanishes on boundaries}) \\
\varphi &= \delta\psi && (\varphi \text{ vanishes on cycles})
\end{aligned}
$$

Lecture 2 : Cohomology and Homological Algebra II

- Cohomology of the cochain complex

$$\cdots \xrightarrow{\delta} C^n(G) \xrightarrow{\delta} C^{n+1}(G) \xrightarrow{\delta} \cdots$$

  defined by

$$H^n(C, G) = \ker \delta / \operatorname{Im} \delta$$

- Simplicial cohomology

$$H^n_\Delta(X, G)$$

  when $C_n = \Delta_n(X)$. A cochain $\varphi \in C^n$ is like giving an element $\varphi(\sigma_\alpha) \in G$ for each σ_α with $n_\alpha = n$, since those form a basis for $\Delta_n(X)$

- In general

$$H^n \neq H_n^\star$$

  The above is true if $G = \mathbb{R}, \mathbb{C}$.

Lecture 2 : Cohomology and Homological Algebra III

Basic Homological Algebra

- Chain map $f : A_n \to B_n$

$$
\begin{array}{ccccccc}
\cdots \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} \cdots \\
 & \downarrow f & & \downarrow f & \\
\cdots \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} \cdots
\end{array}
$$

Squares commute so that

$$f \circ \partial = \partial \circ f$$

- Maps

$$
\begin{array}{rcl}
\text{cycles} & \longrightarrow & \text{cycles} \\
\text{boundaries} & \longrightarrow & \text{boundaries}
\end{array}
$$

and therefore

$$H_n(A) \xrightarrow{f_\star} H_n(B)$$

- $f, \ g : A_n \longrightarrow B_n$ chain maps. A chain homotopy between f and g is a map

$$P : A_n \longrightarrow B_{n+1}$$
$$P\partial + \partial P = g - f$$

  In homology

$$g_\star = f_\star$$

  since, on cycles, g and f differ by a boundary

- A chain

$$\cdots \longrightarrow \quad A_{n+1} \quad \stackrel{\partial_{n+1}}{\longrightarrow} \quad A_n \quad \stackrel{\partial_n}{\longrightarrow} A_{n-1} \longrightarrow \cdots$$

  is called exact sequence if it has vanishing homology

$$\ker \partial_n = \operatorname{Im} \partial_{n+1}$$

- Short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

means

$$\alpha \text{ injective}$$
$$\beta \text{ surjective}$$
$$\operatorname{Im}\alpha = \ker\beta$$

Lecture 2 : Cohomology and Homological Algebra VI

- Let

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$$

a short exact sequence of chains. There exists a map

$$\partial : H_n(C) \to H_{n-1}(A)$$

such that the long sequence below is exact

$$\cdots \to H_n(A) \xrightarrow{i_\star} H_n(B) \xrightarrow{j_\star} H_n(C) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

In the proof we shall refer to the following two diagrams

$$\begin{array}{ccccccccc} 0 & \to & A_n & \xrightarrow{i} & B_n & \xrightarrow{j} & C_n & \to & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \to & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{j} & C_{n-1} & \to & 0 \end{array}$$

and

$$
\begin{array}{ccccc}
 & & \tilde{b} & \rightarrow & \tilde{c} \\
 & & & & \downarrow \\
\tilde{a} & & b & \rightarrow & c \\
 & & \downarrow & & \downarrow \\
a & \rightarrow & \partial b & & 0 \\
\downarrow & & \downarrow & & \\
\partial a & \rightarrow & 0 & &
\end{array}
$$

To define ∂ let $c \in C_n$ with $\partial c = 0$. Using the surjectivity of j we have $c = jb$ with ∂b such that $j\partial b = 0$. Since $\ker j = \operatorname{Im} i$ we have $\partial b = ia$ with $\partial a = 0$ since $i\partial a = \partial^2 b = 0$ and i is injective. Then

$$\partial [c] = [a]$$

To show that the above is well defined, assume $c = \partial \tilde{c}$. Then $\tilde{c} = j\tilde{b}$ and $b = \partial \tilde{b} + i\tilde{a}$ for some $\tilde{a}$. But then $ia = \partial b = i\partial \tilde{a}$ and $a = \partial \tilde{a}$.

Exactness of the long homology sequence is shown by proving $\ker \partial \subset \operatorname{Im} j_\star$, $\ker j_\star \subset \operatorname{Im} i_\star$, $\ker i_\star \subset \operatorname{Im} \partial$ and the opposite inclusions. As an example, let us show the first inclusion. With reference to the above constrution, assume

$$a = \partial \tilde{a}$$

Then

$$\partial \left(b - i\tilde{a} \right) = 0 \qquad\qquad j \left(b - i\tilde{a} \right) = jb = c$$

- Five Lemma. In the commutative diagram below, if the rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is also an isomorphism

$$\begin{array}{ccccccccc} A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' & \rightarrow & E' \end{array}$$

Examples

Homology of the simplex Δ^N

- We will prove that (clear for $N = 0$)

$$H_0^\Delta(\Delta^N) = \mathbb{Z}$$
$$H_n^\Delta(\Delta^N) = 0 \qquad (1 \leq n \leq N)$$

- Let $A_n = \Delta_n(\Delta^N)$ and $B_n = \Delta_n(\Delta^{N+1})$
- Define two maps

$$i : A_n \to B_n$$
$$P : A_n \to B_{n+1}$$

where i is the inclusion and P is defined by

$$[v_0, \cdots, v_n] \mapsto [w, v_0, \cdots, v_n, w]$$

We have

$$i\partial = \partial i \qquad \text{(map of chains)}$$
$$\partial P = -P\partial + i \qquad \text{(chain homotopy between } i \text{ and } 0\text{)}$$

and

$$B_0 = iA_0 \oplus \mathbb{Z} \qquad (\mathbb{Z} \text{ generated by } [w])$$
$$B_n = iA_n \oplus PA_{n-1} \qquad (n \geq 1)$$

- Let $b \in B_n$ with $\partial b = 0$. If $n \geq 1$ then

$$b = ia + Pa' = \partial Pa + P\left(\partial a + a'\right)$$

  Also

$$\partial b = i\left(\partial a + a'\right) - P\partial a' = 0$$

  implies $\partial a + a' = 0$ and $\partial a' = 0$.
- If $n = 0$ then

$$b = ia + k\,[w] = \partial Pa + k\,[w]$$

Homology of the sphere $S_N \simeq \partial\Delta^{N+1}$

- Chain complex of Δ^{N+1}

$$0 \to \Delta_{N+1} = \mathbb{Z} \stackrel{\partial_{N+1}}{\to} \Delta_N = \mathbb{Z}^{N+2} \stackrel{\partial_N}{\to} \Delta_{N-1} \to \cdots$$

with

$$\ker \partial_N = \operatorname{Im} \partial_{N+1} = \mathbb{Z}$$

But $\ker \partial_N$ computes the N homology of $\partial\Delta^{N+1}$ which equals Δ^{N+1} aside from a single simplex of dimension $N+1$. Therefore the non–vanishing homology groups of the sphere are

$$H_N(S_N) = \mathbb{Z}$$
$$H_0(S_N) = \mathbb{Z}$$

Lecture 3 : de Rham Cohomology I

Forms

- M a manifold. A k–form is written localy as

$$\omega = \frac{1}{k!}\,\omega_{i_1\cdots i_k}(x)\;dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \omega_I(x)\;dx^I$$

  $\Omega^k(M)$ space of smooth k–forms on M (with $0 \le k \le \dim_{\mathbb{R}} M$)

- Assiciative wedge product defined by

$$\left(dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\wedge\left(dx^{j_1}\wedge\cdots\wedge dx^{j_q}\right) = dx^{i_1}\wedge\cdots\wedge dx^{j_q}$$

- Exterior derivative

$$d:\Omega^k(M)\to\Omega^{k+1}(M)$$

  defined by

$$d\left(\omega_I(x)\;dx^I\right) = \partial_i\omega_I(x)\;dx^i\wedge dx^I$$

Lecture 3 : de Rham Cohomology II

- Basic properties

$$d\left(\alpha \wedge \beta\right) = d\alpha \wedge \beta + \left(-\right)^{\alpha} \alpha \wedge d\beta$$

$$d^2 = 0 \qquad \left(\partial_i \partial_j f\left(x\right) \; dx^i \wedge dx^j = 0 \right)$$

- Pullback

$$f : N \to M$$
$$f^\star : \Omega^k\left(M\right) \to \Omega^k\left(N\right)$$

locally defined by

$$\left(f^\star \omega\right)_{j_1 \cdots j_k}\left(y\right) = \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} \; \omega_{i_1 \cdots i_k}\left(x\left(y\right)\right)$$

and satisfying

$$f^\star\left(d\alpha\right) = df^\star\left(\alpha\right)$$
$$\left(f \circ g\right)^\star = g^\star \circ f^\star$$
$$f^\star\left(\alpha \wedge \beta\right) = f^\star\left(\alpha\right) \wedge f^\star\left(\beta\right)$$

Lecture 3 : de Rham Cohomology III

de Rham Cohomology

- Complex

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim_{\mathbb{R}} M}(M) \rightarrow 0$$

- Cohomology

$$H^n(M) = \frac{\ker d}{\operatorname{Im} d} \qquad \text{(closed form / exact forms)}$$

- Given $f : N \rightarrow M$ the map $f^\star$ descends in cohomology (chain map)

$$f^\star : H^n(M) \rightarrow H^n(N)$$

Lecture 3 : de Rham Cohomology IV

- Cohomology ring. The wedge produce on forms descends in cohomology

$$H^{\star}(M) = \oplus_k H^k(M)$$

$$H^k \times H^q \xrightarrow{\wedge} H^{k+q} \qquad [\alpha] \wedge [\beta] \mapsto [\alpha \wedge \beta]$$

Compatible with pullback

$$f^{\star}([\alpha] \wedge [\beta]) = f^{\star}[\alpha] \wedge f^{\star}[\beta]$$

- Cohomology ring with compact support

$$H_c^{\star}(M) = \oplus_k H_c^k(M)$$

using forms with compact support $\Omega_c^k(M)$ with
$d : \Omega_c^k(M) \to \Omega_c^{k+1}(M)$
Note : Pullbacks do not send forms with compact support in forms with compact support

Mayer–Vietoris

- If $A \subset M$ is open with $i : A \to M$ inclusion, we have the chain maps

$$i^\star : \Omega^\star(M) \to \Omega^\star(A) \qquad \text{restriction map}$$
$$i_\star : \Omega_c^\star(A) \to \Omega_c^\star(M) \qquad \text{extension map}$$

- Assume

$$M = A \cup B \qquad (A, B \text{ open})$$

- Chain maps

$$0 \to \Omega^\star(M) \to \Omega^\star(A) \oplus \Omega^\star(B) \overset{i_A^\star - i_B^\star}{\to} \Omega^\star(A \cap B) \to 0$$
$$0 \to \Omega_c^\star(A \cap B) \to \Omega_c^\star(A) \oplus \Omega_c^\star(B) \overset{j_{A\star} - j_{B\star}}{\to} \Omega_c^\star(M) \to 0$$

with i_A, i_B and j_A, j_B inclusions

$$A \cap B \overset{i_A}{\to} A \overset{j_A}{\to} M \qquad\qquad A \cap B \overset{i_B}{\to} B \overset{j_B}{\to} M$$

Lecture 3 : de Rham Cohomology VI

- Short exact sequences. To show surjectivity of $i_A^\star - i_B^\star$ choose a partition of unity ρ_A, ρ_B. Given a form ω on $A \cap B$ it comes from

$$\rho_B \omega \oplus -\rho_A \omega$$

  Surjectivity of $j_{A\star} - j_{B\star}$. A form ω on M comes from

$$\rho_A \omega \oplus -\rho_B \omega$$

- Long exact sequences

$$\cdots \to H^k(M) \to H^k(A) \oplus H^k(B) \to H^k(A \cap B) \to H^{k+1}(M) \to \cdots$$

$$\cdots \to H_c^k(A \cap B) \to H_c^k(A) \oplus H_c^k(B) \to H_c^k(M) \to H_c^{k+1}(A \cap B) \to \cdots$$

Poincaré Lemmas

- Basic statement

$$H^k(M \times \mathbb{R}^n) = H^k(M)$$

$$H_c^k(M \times \mathbb{R}^n) = H_c^{k-n}(M)$$

Lecture 3 : de Rham Cohomology VII

- Projection and zero section

$$\mathbb{R}^n \times \mathbb{R} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{R}^n \qquad\qquad \pi \circ s = 1_{\mathbb{R}^n}$$

- Map

$$K : \Omega^k\left(\mathbb{R}^n \times \mathbb{R}\right) \to \Omega^{k-1}\left(\mathbb{R}^n \times \mathbb{R}\right)$$

defined by

$$a_I\left(x, t\right)\ dx^I \mapsto 0$$
$$a_I\left(x, t\right)\ dx^I dt \mapsto \left(\int_0^t a_I\left(x, s\right) ds\right) dx^I$$

- Basic fact

$$s^\star \circ \pi^\star - 1 = 0$$
$$\pi^\star \circ s^\star - 1 = \left(-\right)^k \left(dK - Kd\right) \qquad \text{(chain homotopy)}$$

Therefore in cohomology $s^\star$ and $\pi^\star$ are inverses and the cohomologies conincide

Lecture 3 : de Rham Cohomology VIII

- Sample computation

$$(dK - Kd)\left(a_I dx^I\right) = -K\left(\partial_i a_I dx^i dx^I\right) - K\left(\partial_t a_I \; dt dx^I\right)$$
$$= (-)^{k-1}\left(\int_0^t \partial_t a_I\right) dx^I = (-)^{k-1}\left(a_I(x,t) - a_I(x,0)\right) dx^I$$

- Let

$$e = e(t)\, dt \quad \text{with compact support}$$
$$\int e = 1$$

and

$$E(t) = \int_{-\infty}^{t} e(s)\, ds$$

Lecture 3 : de Rham Cohomology IX

- Chain maps

$$\Omega_c^k(\mathbb{R}^n) \underset{\pi_\star}{\overset{e_\star}{\rightleftarrows}} \Omega_c^{k+1}(\mathbb{R}^n \times \mathbb{R})$$

given by

$$\phi \overset{e_\star}{\mapsto} \phi \wedge e$$

and

$$a_I dx^I \overset{\pi_\star}{\mapsto} 0$$

$$a_I dx^I dt \overset{\pi_\star}{\mapsto} \left(\int_{-\infty}^{\infty} a_I(x,s)\, ds \right) dx^I$$

- Map

$$K : \Omega_c^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega_c^{k-1}(\mathbb{R}^n \times \mathbb{R})$$

defined by

$$a_I(x,t)\;\; dx^I \mapsto 0$$

$$a_I(x,t)\;\; dx^I dt \mapsto \left(\int_{-\infty}^{t} a_I\;\; ds - E(t) \int_{-\infty}^{\infty} a_I\;\; ds \right) dx^I$$

- Again (exercise)

$$\pi_\star \circ e_\star - 1 = 0$$

$$e_\star \circ \pi_\star - 1 = (-)^k (dK - Kd) \qquad \text{(chain homotopy)}$$

Homotopy invariance

- Let

$$M \underset{\pi}{\overset{s_t}{\rightleftarrows}} M \times \mathbb{R} \xrightarrow{F} N$$

- The maps

$$f_t = F \circ s_t : M \to N$$

define a smooth family parameterized by t

- In cohomology the map $s_t^\star = (\pi^\star)^{-1}$ is independent of t and so is

$$f_t^\star = s_t^\star \circ F^\star : H^\star(N) \to H^\star(M)$$

Lecture 3 : de Rham Cohomology XI

- Two spaces M and N are homotopic if we have two maps $f : M \longrightarrow N$ and $g : N \longrightarrow M$ with $g \circ f$ and $f \circ g$ smoothly deformable to the identity on M and N respectively. Homotopic spaces have the same cohomology
- $A \subset M$ is a deformation retract if there is a smooth family of maps $f_t : M \longrightarrow M$ with $f_t|_A = 1_A$ and with $f_0 = 1_M$ and $f_1(M) = A$. Then A and M are homotopic
- Example : Spheres S_N

$$S_N = A \cup B \text{ with } A \cap B \sim S_{N-1}$$

Long exact sequence

$$\cdots \longrightarrow H^{N-1}(A) \oplus H^{N-1}(B) \longrightarrow H^{N-1}(A \cap B) \longrightarrow$$
$$\longrightarrow H^N(S_N) \longrightarrow H^N(A) \oplus H^N(B) \longrightarrow \cdots$$

implies

$$H^{N-1}(S_{N-1}) = H^N(S_N)$$

Lecture 4: Poincaré Duality and Künneth Theorem I

Integration and Stokes Theorem

- N manifold with boundary if you can cover it with coordinate patches (U_α, x_α) with U_α diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$ (given by $(x_1, \cdots, x_n)$ with $x_n \geq 0$)
- ∂N given by points corresponding to $\partial \mathbb{H}^n$ $(x_n = 0)$ with local coordinates $(x_1, \cdots, x_{n-1})$
- N orientable if you can choose coordinates with

$$\det \frac{\partial y}{\partial x} > 0$$

- Let $\omega \in \Omega_c^n(N)$. Given an oriented (U_α, x_α) and a partition of unity ρ_α define

$$\int_N \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega \qquad \int_{U_\alpha} \eta \equiv \int_{\mathbb{R}^n, \mathbb{H}^n} \eta_{1 \cdots n}(x_\alpha)\, dx_\alpha^1 \cdots dx_\alpha^n$$

- If (V_β, y_β) has the same orientation and χ_β is a corresponding partition of unity we have

$$\int_{U_\alpha} \rho_\alpha \chi_\beta \omega = \int_{V_\beta} \rho_\alpha \chi_\beta \omega$$

since

$$dy_\beta^1 \wedge \cdots \wedge dy_\beta^n = \det\left(\frac{\partial y_\beta}{\partial x_\alpha}\right) dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$
$$dy_\beta^1 \cdots dy_\beta^n = \left|\det\left(\frac{\partial y_\beta}{\partial x_\alpha}\right)\right| dx_\alpha^1 \cdots dx_\alpha^n$$

- Summing over α, β we obtain

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega$$

- Stokes Theorem

$$\int_N d\omega = \int_{\partial N} \omega$$

where, given oriented coordinates $x_1, \cdots, x_n$ on N with $x_n \geq 0$, the orientation on ∂N is given by $(-)^n \; x_1, \cdots, x_{n-1}$

Using linearity it suffices to show it for $\mathbb{R}^n, \mathbb{H}^n$. For instance

$$\begin{aligned} \omega &= f \; dx^1 \wedge \cdots \wedge dx^{n-1} \\ d\omega &= (-)^{n-1} \; \partial_n f \; dx^1 \wedge \cdots \wedge dx^n \\ \int_{\mathbb{H}^n} d\omega &= (-)^{n-1} \; \int_{x^n \geq 0} \partial_n f \; dx^1 \cdots dx^n \\ &= (-)^n \; \int_{x^n = 0} f \; dx^1 \cdots dx^{n-1} = \int_{\partial \mathbb{H}^n} \omega \end{aligned}$$

Lecture 4: Poincaré Duality and Künneth Theorem IV

$\dim H^n < 0$

- M with good finite cover $U_1 \cdots U_p$ (of finite type) and

$$A = U_1 \cup \cdots \cup U_{p-1} \qquad \text{(of finite type)}$$
$$B = U_p$$

- $A \cap B$ of finite type (covered by $U_i \cap U_p$ with $i = 1, \cdots, p-1$)
- Long exact sequences

$$H^{k-1}(A \cap B) \to H^k(M) \to H^k(A) \oplus H^k(B)$$
$$H_c^{k+1}(A \cap B) \leftarrow H_c^k(M) \leftarrow H_c^k(A) \oplus H_c^k(B)$$

- Left and right factors above have a finite dimension by induction on p. By exactness

$$\dim H^k(M) < \infty \qquad \dim H_c^k(M) < \infty$$

Poincaré Duality

- M orientable of finite type (with $\dim M = n$)
- $M = A \cup B$ with ρ_A, ρ_B partition of unity
- Integration maps

$$H^k(M) \times H_c^{n-k}(M) \to \mathbb{R}$$
$$[\alpha] \times [\beta] \mapsto \int_M \alpha \wedge \beta \qquad \text{(well defined using Stokes)}$$

or equivalently

$$H^k(M) \to H_c^{n-k\ \star}(M)$$
$$[\alpha] \mapsto \int_M \alpha \wedge$$

The above map is an isomorphism

- Look at the diagram

$$\begin{array}{ccc} H^k(M) & \rightarrow & H_c^{n-k}{}^\star(M) \\ \downarrow & & \downarrow \\ H^k(A) \oplus H^k(B) & \rightarrow & H_c^{n-k}{}^\star(A) \oplus H_c^{n-k}{}^\star(B) \\ \downarrow & & \downarrow \\ H^k(A \cap B) & \rightarrow & H_c^{n-k}{}^\star(A \cap B) \\ \downarrow & & \downarrow \\ H^{k+1}(M) & \rightarrow & H_c^{n-k-1}{}^\star(M) \end{array}$$

We shall show that it is a commutative diagram up to signs. The theorem then follows by the five–lemma and induction on the size of the finite cover

The only subdle point is the last square. Let $[\gamma] \in H^k(A \cap B)$ and $[\omega] \in H_c^{n-k-1}(M)$

Lecture 4: Poincaré Duality and Künneth Theorem VII

- The class $d^\star[\gamma]$ is defined by

$$\begin{aligned} &d(\rho_B\gamma) && \text{on } A \\ &d(-\rho_A\gamma) && \text{on } B \end{aligned}$$

  which coincide and have support on $A \cap B$ and define an element of $H^{k+1}(M)$
- The class $d_\star[\omega]$ is defined by

$$\begin{aligned} d(\rho_A\omega) &\in H_c^{n-k}(A) \\ d(-\rho_B\omega) &\in H_c^{n-k}(B) \end{aligned}$$

  which coincide and have support on $A \cap B$ and define an element of $H_c^{n-k}(A \cap B)$

Lecture 4: Poincaré Duality and Künneth Theorem VIII

- We must show that

$$\int_M d^\star [\gamma] \wedge [\omega] = \pm \int_{A\cap B} [\gamma] \wedge d_\star [\omega]$$

This follows from

$$\int_A \rho_A d(\rho_B \gamma) \wedge \omega + \int_B \rho_B d(-\rho_A \gamma) \wedge \omega$$
$$= \pm \int_A \rho_B \gamma \wedge d(\rho_A \omega) \pm \int_B \rho_A \gamma \wedge d(-\rho_B \omega)$$
$$= \pm \int_{A\cap B} (\rho_A + \rho_B) \gamma \wedge d(\rho_A \omega) = \pm \int_{A\cap B} \gamma \wedge d(\rho_A \omega)$$

Künneth Theorem

- Consider the space $M \times N$ with M of finite type
- Look at projections

$$\begin{array}{ccc} M \times N & \xrightarrow{\eta} & N \\ \downarrow \pi & & \\ M & & \end{array}$$

Lecture 4: Poincaré Duality and Künneth Theorem IX

- The map

$$\begin{aligned} H^{\star}(M) \times H^{\star}(N) &\to H^{\star}(M \times N) \\ [\alpha] \times [\beta] &\mapsto [\pi^{\star}\alpha \wedge \eta^{\star}\beta] \end{aligned} \qquad \text{(well defined ! Check)}$$

  is an isomorphism

  Proof similar in spirit to that used to show Poincaré duality, relying on the Meyer–Vietoris sequence and induction on size of the finite cover of M

Lecture 5: Čech Cohomology I

Sheafs

- Sheaf $\mathcal{F}$ on X
  - U open $\mapsto$ $\mathcal{F}(U)$ abelian group
  - $V \subset U$ $\mapsto$ restriction maps $\mathcal{F}_U^V : \mathcal{F}(U) \to \mathcal{F}(V)$

  such that

$$\mathcal{F}_V^W \circ \mathcal{F}_U^V = \mathcal{F}_U^W \qquad \text{(for } W \subset V \subset U\text{)}$$
$$\mathcal{F}_U^U = 1$$

  and such that, if $U = \bigcup_i U_i$, then
  - given $f \in \mathcal{F}(U)$ such that $f|_{U_i} = 0$ then $f = 0$
  - given $f_i \in \mathcal{F}(U_i)$ such that $f_i = f_j$ on $U_i \cap U_j$, then there is an $f \in \mathcal{F}(U)$ with $f_i = f|_{U_i}$
- Examples of interest to us
  - Constant sheafs with $\mathcal{F}(U) = G$ fixed abelian group $(\mathbb{Z}, \mathbb{R}, \mathbb{C}, \cdots)$ and $\mathcal{F}_U^V = 1_G$
  - Smooth and holomorphic sections of vector bundles

Lecture 5: Čech Cohomology II

- Map of sheafs $f : \mathcal{F} \to \mathcal{G}$ are maps

$$f_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

compatible with restrictions

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \downarrow \mathcal{F}_U^V & & \downarrow \mathcal{G}_U^V \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

Čech Cohomology

- U_α open cover of X with $\alpha \in I$ ordered countable set

Lecture 5: Čech Cohomology III

- Čech cochains

$$C^p(U,\mathcal{F}) = \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}\left(U_{\alpha_0\cdots\alpha_p}\right)$$

with

$$U_{\alpha_0\cdots\alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$$

A cochain is the following data

$$\omega_{\alpha_0\cdots\alpha_p} \in \mathcal{F}\left(U_{\alpha_0\cdots\alpha_p}\right)$$

Convention: extend $\omega_{\alpha_0\cdots\alpha_p}$ to all indices by requiring antisymmetry

- Coboundary map

$$\delta : C^p \to C^{p+1}$$

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-)^i \; \omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}} \qquad \text{(restriction maps suppressed)}$$

$$\delta^2 = 0$$

Lecture 5: Čech Cohomology IV

- Cohomology

$$H^{\star}(U, \mathcal{F})$$

Relation to Simplicial Cohomology

- X finite simplicial complex (double baricentric subdivision of a Δ–complex)
- U_α with $\alpha = 1, \cdots, N$ one of the ordered vertices of X is the open–star of α (union of the interiors $\breve{\Delta}$ of all simplices which contain α)
- U_α is a good finite cover and

$$\begin{array}{ccl} U_\alpha & \leftrightarrow & \text{Vertices} \\ U_{\alpha\beta} & \leftrightarrow & \text{1–simplices} \ \ (U_{\alpha\beta} \neq \emptyset \ \text{ iff the 1–simplex } \alpha\text{–}\beta \text{ is} \\ & & \qquad\qquad\qquad \text{part of the simplicial complex } X) \\ & \cdots & \end{array}$$

Lecture 5: Čech Cohomology V

- Cochains coincide

$$C^p(U, G) = \mathrm{Hom}(\Delta_n(X), G)$$

where G is the contant sheaf. Also coboundaries coincide and therefore

$$H^p_{\text{Čech}}(U, G) = H^p_{\Delta}(X, G)$$

Čech–deRham Complex

- Good cover U_α of X with partition of unity ρ_α
- Double complex

$$K^{p,q} = C^p(U, \Omega^q)$$

$$\delta : K^{p,q} \to K^{p+1,q}$$

$$d : K^{p,q} \to K^{p,q+1}$$

Lecture 5: Čech Cohomology VI

- Čech–deRham complex

$$K^n = \bigoplus_{p+q=n} K^{p,q}$$
$$D = \delta + (-)^p \, d$$

with

$$D^2 = \delta^2 + d^2 + (-)^p \, \delta d + (-)^{p+1} \, d\delta = (-)^p \, [\delta, d] = 0$$

- Čech–deRham cohomology

$$H^\star_{CD} = \frac{\ker D}{\operatorname{Im} D}$$

Lecture 5: Čech Cohomology VII

- Double inclusion

$$
\begin{array}{ccccc}
0 \to \Omega^2 \xrightarrow{r} & K^{0,2} & K^{1,2} & K^{2,2} \\
0 \to \Omega^1 \xrightarrow{r} & K^{0,1} & K^{1,1} & K^{2,1} \\
0 \to \Omega^0 \xrightarrow{r} & K^{0,0} & K^{1,0} & K^{2,0} \\
 & \uparrow i & \uparrow i & \uparrow i \\
 & C^0(U,\mathbb{R}) & C^1(U,\mathbb{R}) & C^2(U,\mathbb{R}) \\
 & \uparrow & \uparrow & \uparrow \\
 & 0 & 0 & 0
\end{array}
$$

induce maps in cohomology

$$
\begin{aligned}
r^\star &: H^\star(X) \to H^\star_{CD} \\
i^\star &: H^\star(U,\mathbb{R}) \to H^\star_{CD}
\end{aligned}
$$

- Colums are exact since U is good and on the intersections we use Poincaré's Lemma (it is exact in dimension zero at $K^{k,0}$ since we are quotenting by constant functions $C^k(U,\mathbb{R})$)

Lecture 5: Čech Cohomology VIII

- The rows are exact. Define the map

$$P : K^{p,q} \to K^{p-1,q}$$
$$(P\omega)_{\alpha_0 \cdots \alpha_{p-1}} = (-)^p \sum_{\alpha_p} \omega_{\alpha_0 \cdots \alpha_p} \rho_{\alpha_p}$$

We have that

$$P\delta + \delta P = 1$$

and each cocycle is a coboundary

- Proof

$$(P\delta\omega)_{\alpha_0 \cdots \alpha_p} = (-)^{p+1} \sum_{\alpha_{p+1}} \sum_{i=0}^{p+1} (-)^i \, \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}} \, \rho_{\alpha_{p+1}}$$

$$(\delta P\omega)_{\alpha_0 \cdots \alpha_p} = (-)^p \sum_{i=0}^{p} \sum_{\alpha_p} (-)^i \, \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}} \, \rho_{\alpha_{p+1}}$$

All terms cancel aside from the term with $i = p + 1$ in the first sum which equals ω since $\sum_{\alpha_{p+1}} \rho_{\alpha_{p+1}} = 1$

Lecture 5: Čech Cohomology IX

- The maps $r^\star$ and $i^\star$ are isomorphisms. We have therefore

$$H^\star_{\text{deRham}} \overset{r^\star}{\underset{\sim}{\to}} H^\star_{CD} \overset{i^\star}{\underset{\sim}{\leftarrow}} H^\star_{\check{\text{C}}\text{ech}} \simeq H^\star_\Delta$$

- $r^\star$ surjective : Let $\omega \in K^2$ with $D\omega = 0$ (the general case is analogous)

$$\begin{array}{llll} \omega_1 & & & \\ \alpha_1 & \omega_2 & & \\ & \alpha_2 & \omega_3 \overset{\delta}{\to} 0 & \end{array} \qquad\qquad \begin{array}{lll} \eta \to \tilde{\omega}_1 & & \\ & 0 & \\ & & 0 \end{array}$$

  Since $\delta\omega_3 = 0$ choose α_2 so that $\delta\alpha_2 = -\omega_3$. Then $\omega + D\alpha_2$ has no elements in $K^{2,0}$. Analogously I can choose α_1 so that $\omega + D(\alpha_1 + \alpha_2)$ has only a non–vanishing element $\tilde{\omega}_1 \in K^{0,2}$. Since $\delta\tilde{\omega}_1 = 0$ it must be the image of a globally defined closed 2–form η

- $r^\star$, $i^\star$ injective and $i^\star$ surjective are proved in similar ways

Basic Construction

- Manifold M and open cover U_α
- Smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{R})$$

such that

$$g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} \qquad (\text{on } U_\alpha \cap U_\beta \cap U_\gamma)$$

$$(\text{this implies } g_{\alpha\alpha} = 1 \text{ and } g_{\alpha\beta} = g_{\beta\alpha}^{-1})$$

- Building blocks

$$E_\alpha = U_\alpha \times \mathbb{R}^n$$

with equivalence relation

$$(x, v) \in E_\alpha \sim (y, w) \in E_\beta \quad \text{if} \quad x = y \text{ and } v = g_{\alpha\beta} w$$

- Total space

$$\pi : E \longrightarrow M$$

- A section s_α is given by

$$s_\alpha : U_\alpha \to \mathbb{R}^n$$
$$s_\alpha = g_{\alpha\beta}\, s_\beta$$

- Given a map $f : N \to M$ the open cover $V_\alpha = f^{-1}(U_\alpha)$ and maps $g_{\alpha\beta} \circ f$ define the pullback vector bundle on N

$$\begin{array}{ccc} f^{-1}E & \to & E \\ \downarrow \pi & & \downarrow \pi \\ N & \stackrel{f}{\to} & M \end{array}$$

- Complex bundles : replace $\mathbb{R}$ with $\mathbb{C}$
  Holomorphic bundles : M complex manifold, replace $\mathbb{R}$ with $\mathbb{C}$ and smooth with holomorphic

Lecture 6: Vector Bundles III

- Two vector bundles $\left(U_\alpha, g_{\alpha\beta}\right)$ and $\left(U_\alpha, h_{\alpha\beta}\right)$ on M are equivalent if there are smooth maps

$$\lambda_\alpha : U_\alpha \longrightarrow GL\left(n, \mathbb{R}\right)$$

such that

$$g_{\alpha\beta} = \lambda_\alpha \; h_{\alpha\beta} \; \lambda_\beta^{-1}$$

If the open covers are different, pass to a common refinement first. Various equivalent representations $\left(U_\alpha, g_{\alpha\beta}\right)$ are called trivializations

Basic Examples

- Trivial Bundle

$$E = M \times \mathbb{R}^n$$

Lecture 6: Vector Bundles IV

- Tangent bundle TN with transition functions

$$\left(g_{\alpha\beta}\right)_{ij} = \frac{\partial x_\alpha^i}{\partial x_\beta^j}$$

where x_α^i are coordinates on U_α. Sections $V_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ are vector fields

$$\sum_i V_\alpha^i \frac{\partial}{\partial x_\alpha^i}$$

- Holomorphic tangent bundle T_N with N a complex manifold and with transition functions

$$\left(g_{\alpha\beta}\right)_{ij} = \frac{\partial z_\alpha^i}{\partial z_\beta^j}$$

where z_α^i are holomorphic coordinates on U_α

Orientable Bundles

- A real vector bundle is orientable if it has a trivialization with transition functions $g_{\alpha\beta}$ such that

$$\det g_{\alpha\beta} > 0$$

- Two simple facts
  - A manifold M is orientable if $TM \longrightarrow M$ is an orientable vector bundle
  - If M is an orientable manifold and $E \longrightarrow M$ and orientable vector bundle, then E is an orientable manifold
- Basic facts
  - A real vector bundle always admits an $O(n)$ trivialization
  - A complex vector bundle always admits a $U(n)$ trivialization
  - A real orientable vector bundle always admits an $SO(n)$ trivialization

Lecture 6: Vector Bundles VI

Operations on Vector Bundles

- Basic operations on vector spaces

$$V \oplus W$$

$$V \otimes W \qquad \left(\text{also } \operatorname{Sym}^k V \text{ and } \textstyle\bigwedge^k V\right)$$

$$V^\star$$

extend to operations on vector bundles $V, W \to M$, with transition functions given by

$$g_{\alpha\beta} \oplus h_{\alpha\beta}$$

$$g_{\alpha\beta} \otimes h_{\alpha\beta}$$

$${}^t(g_{\alpha\beta}^{-1})$$

Important is the line bundle $\bigwedge^{\dim V} V$ with transition functions

$$\det\left(g_{\alpha\beta}\right)$$

Lecture 6: Vector Bundles VII

- Complex conjugation $\bar{E}$ of a complex vector bundle E has transition functions $g^{\star}_{\alpha\beta}$
- The complexification $E_{\mathbb{C}}$ of a real vector bundle E has $\dim_{\mathbb{C}} E_{\mathbb{C}} = \dim_{\mathbb{R}} E$ and the same transition functions using the inclusion

$$\begin{array}{ccc} GL(n,\mathbb{R}) & \rightarrow & GL(n,\mathbb{C}) \\ \uparrow & & \uparrow \\ O(n) & \rightarrow & U(n) \end{array}$$

- The realization $E_{\mathbb{R}}$ of a complex vector bundle E has $\dim_{\mathbb{R}} E_{\mathbb{R}} = 2\dim_{\mathbb{C}} E$ with transition functions

$$\left(g_{\alpha\beta}\right)_{\mathbb{R}} = M^{-1}\begin{pmatrix} \operatorname{Re} g_{\alpha\beta} & -\operatorname{Im} g_{\alpha\beta} \\ \operatorname{Im} g_{\alpha\beta} & \operatorname{Re} g_{\alpha\beta} \end{pmatrix} M \quad \text{with} \quad M = \begin{pmatrix} 1 \\ 001\cdots \\ \cdots \\ 01 \\ 0001 \\ \cdots \end{pmatrix}$$

Lecture 6: Vector Bundles VIII

and $M \in O(2n)$, defining the map (since $\det(g_{\alpha\beta})_{\mathbb{R}} = \left|\det g_{\alpha\beta}\right|^2$)

$$\begin{array}{ccc} GL(n,\mathbb{C}) & \rightarrow & GL(2n,\mathbb{R}) \\ \uparrow & & \uparrow \\ U(n) & \rightarrow & SO(2n) \end{array}$$

Therefore $E_{\mathbb{R}}$ is orientable

- Exercises: show the isomorphisms as complex bundles
  - $(E_{\mathbb{R}})_{\mathbb{C}} \simeq E \oplus \bar{E}$
  - $\bar{E} \simeq E^{\star}$

More Examples

- Cotangent bundle $T^{\star}M$ with sections one–forms
- Bundles $TM \oplus \cdots \oplus TM \oplus T^{\star}M \oplus \cdots \oplus T^{\star}M$ with sections tensors
- $\bigwedge^k T^{\star}M$ with sections k–forms
- Holomorphic cotangent bundle $T^{\star}_M$ and $\bigwedge^k T^{\star}_M$
- Canonical line bundle $K_M = \bigwedge^{\dim_{\mathbb{C}} M} T^{\star}_M$

Lecture 6: Vector Bundles IX

- A basic relations

$$TM_{\mathbb{C}} = T_M \oplus \bar{T}_M$$

Connection and curvature

- $E \xrightarrow{\pi} M$ vector bundle with trivialization U_α and $g_{\alpha\beta}$
- Section

$$s_\alpha : U_\alpha \to K^n \qquad (K = \mathbb{R}, \mathbb{C} \text{ with } n = \dim_K E)$$
$$s_\alpha = g_{\alpha\beta} s_\beta$$

- A connection are one–forms A_α on U_α with values in $\mathfrak{gl}(K, n)$ such that

$$(d + A_\alpha)\, s_\alpha \equiv D s_\alpha$$

is a section of $E \otimes T^\star M$ so that

$$D s_\alpha = g_{\alpha\beta} D s_\beta$$

This implies

$$A_\alpha = g_{\alpha\beta}\, A_\beta\, g_{\beta\alpha} + g_{\alpha\beta}\, d g_{\beta\alpha}$$

Lecture 6: Vector Bundles X

- Curvature

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$$

so that

$$F_\alpha = g_{\alpha\beta}\ F_\beta\, g_{\beta\alpha}$$

$$DF_\alpha = dF_\alpha + A_\alpha \wedge F_\alpha - F_\alpha \wedge A_\alpha = 0 \qquad \text{(Bianchi Identity)}$$

First Chern Class

- M of finite type with U_α a good cover and ρ_α partition of unity
- $L \xrightarrow{\pi} M$ complex line bundle with $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1) \in \mathbb{C}^\star$ (since $U(1) = SO(2)$ it is like considering real orientable vector bundles with $\dim_{\mathbb{R}} = 2$)
- Define

$$\omega_{\alpha\beta} = -\frac{1}{2\pi i} g_{\alpha\beta}\, dg_{\beta\alpha} \;\in K^{1,1}$$

One has

$$\omega_{\alpha\beta} \propto \; d \ln g_{\alpha\beta} \qquad \to \qquad d\omega = 0$$

$$g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} \qquad \to \qquad \delta\omega = 0$$

Lecture 7: Characteristic Classes II

- Define

$$\theta_{\alpha\beta} = \frac{1}{2\pi i} \ln g_{\alpha\beta} \quad \in \quad K^{1,0} \qquad \text{(choice of } \ln \text{ possible since } U_\alpha \text{ is good)}$$

$$A_\alpha = -2\pi i \, \textstyle\sum_\beta \omega_{\alpha\beta} \, \rho_\beta \quad \in \quad K^{0,1} \qquad \text{(it defines a connection)}$$

so that

$$d\theta = \frac{\delta A}{2\pi i} = \omega$$

- ω is cohomologous to

1. $$-\frac{1}{2\pi i} dA = -\frac{1}{2\pi i} F$$
2. $$(\delta\theta)_{\alpha\beta\gamma} = \theta_{\alpha\beta} + \theta_{\beta\gamma} - \theta_{\alpha\gamma} = n_{\alpha\beta\gamma}$$

1. Cohomology class in $H^2(M, \mathbb{C})$. If we change connection to $A + a$, then $a_\alpha = a_\beta$ defines a global one–form and $F \to F + da$ changes by a boundary
2. $n_{\alpha\beta\gamma} \in \mathbb{Z}$ constants on $U_\alpha \cap U_\beta \cap U_\gamma$. Integer class in $H^2(M, \mathbb{Z})$

Denote with

$$c_1(L)$$

Chern Classes

- $E \xrightarrow{\pi} M$ complex vector bundle with $\dim_{\mathbb{C}} \ E = n$ and with connection A and curvature F. We define the total Chern class of E as

$$c(E) = \det\left(1 - \frac{1}{2\pi i}F\right) = c_0(E) + c_1(E) + \cdots + c_n(E) \quad \in \quad H^\star(M)$$

  where

$$c_0(E) = 1 \qquad\qquad c_i(E) \in H^{2i}(M)$$

- Classes independent of connection
  1. For an infinitesimal variation $A_\alpha \to A_\alpha + \epsilon_\alpha$ one has $\epsilon_\alpha = g_{\alpha\beta}\ \epsilon_\beta\, g_{\beta\alpha}$ and $F_\alpha \to F_\alpha + D\epsilon_\alpha$ with $D\epsilon_\alpha = d\epsilon_\alpha + A_\alpha \wedge \epsilon_\alpha + \epsilon_\alpha \wedge A_\alpha$

2. The variation of

$$\mathrm{Tr}\left(F^n\right)$$

is proportional to (using Bianchi identity)

$$\mathrm{Tr}(D\epsilon\ F^{n-1}) = \mathrm{Tr}(D(\epsilon\ F^{n-1})) = d\,\mathrm{Tr}(\epsilon\ F^{n-1})$$

3. Given two connections A and A' so is the convex combination $xA + (1-x)\,A'$

Basic Properties

- Naturality : given $E \longrightarrow M$ complex vector bundle and $f : N \longrightarrow M$ one has

$$c(f^{-1}E) = f^{\star}c\,(E)$$

since $f^{\star}A_{\alpha}$ defines a connection on $f^{-1}E \longrightarrow N$

- Whitney sum rule

$$c\,(E \oplus F) = c\,(E)\,c\,(F)$$

Given connections A_{α} and B_{α} for E and F, choose $A_{\alpha} \oplus B_{\alpha}$ as connection for $E \oplus F$

Lecture 7: Characteristic Classes V

- Splitting principle : Given vector bundles $E_i \longrightarrow M$ there is a $\sigma : N \longrightarrow M$ such that

$$\sigma^{-1}E_i \text{ is a sum of line bundles}$$
$$\sigma^\star : H^\star(M) \longrightarrow H^\star(N) \text{ is injective}$$

Suppose $P(c(E_i))$ is a polynomial on the Chern classes, and suppose that we have shown that $P = 0$ when the E_i's are sums of line bundles. Then in general

$$\begin{aligned} \sigma^\star P(c(E_i)) &= P(c(\sigma^{-1}E_i)) && \text{(naturality)} \\ &= 0 && \text{(the } \sigma^{-1}E_i \text{ are sums of line bundles)} \end{aligned}$$

Since $\sigma^\star$ is injective we conclude

$$P(c(E_i)) = 0$$

Some computations

- Given two line bundles L_1 and L_2 one has (trivial check)

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \qquad c_1(L_1^\star) = -c_1(L_1)$$

- Let $E = L_1 \oplus \cdots \oplus L_n$ with $c(L_i) = 1 + x_i$. Then

$$c(E) = \prod_i (1 + x_i)$$
$$c_i(E) = \frac{1}{k!} \sum_{i_\alpha \neq i_\beta} x_{i_1} \cdots x_{i_k}$$

- Let $F = \tilde{L}_1 \oplus \cdots \oplus \tilde{L}_m$ with $c\left(\tilde{L}_i\right) = 1 + y_i$. Then

$$c(E \otimes F) = \prod_{i,j} \left(1 + x_i + y_j\right) = 1 + \sum_{i,j} \left(x_i + y_j\right) + \cdots$$
$$= 1 + m\ c_1(E) + n\ c_1(F) + \cdots$$

- If $m = 1$ then

$$c(E \otimes F) = \prod_i (1 + x_i + y) = \sum_i c_i(E)\ \ c^{n-i}(F)$$

- Exercise : Show that $c_i(E^\star) = (-)^i c_i(E)$ and compute Chern classes as symmetric polynomials in the x_i and explicitly for low degrees for

$$\bigotimes^k E = \bigoplus \left(L_{i_1} \otimes \cdots \otimes L_{i_k}\right)$$
$$\bigwedge^k E = \bigoplus_{i_1 < \cdots < i_k} \left(L_{i_1} \otimes \cdots \otimes L_{i_k}\right)$$
$$\mathrm{Sym}^k\, E = \bigoplus_{i_1 \leq \cdots \leq i_k} \left(L_{i_1} \otimes \cdots \otimes L_{i_k}\right)$$

Lecture 7: Characteristic Classes VIII

More Complex Classes

- Classes defined using the splitting principle

$$\mathrm{Td}(E) = \prod_i \frac{x_i}{1 - e^{-x_i}} \qquad \mathrm{Td}(E \oplus F) = \mathrm{Td}(E)\,\mathrm{Td}(F)$$

$$\mathrm{L}(E) = \prod_i \frac{x_i}{\tanh x_i} \qquad \mathrm{L}(E \oplus F) = \mathrm{L}(E)\,\mathrm{L}(F)$$

$$\widehat{\mathrm{A}}(E) = \prod_i \frac{x_i/2}{\sinh(x_i/2)} \qquad \widehat{\mathrm{A}}(E \oplus F) = \widehat{\mathrm{A}}(E)\,\widehat{\mathrm{A}}(F)$$

$$\mathrm{ch}(E) = \sum_i e^{x_i} \qquad \mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$$

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$$

Pontrjagin Classes

- Given a real vector bundle $E \to M$ of $\dim_{\mathbb{R}} = n$ we define the Pontrjagin classes as

$$p(E) = c(E_{\mathbb{C}})$$

Lecture 7: Characteristic Classes IX

- Since $E_{\mathbb{C}} = E_{\mathbb{C}}^{\star}$ and since $c_i\left(E_{\mathbb{C}}^{\star}\right) = (-)^i\, c\left(E_{\mathbb{C}}\right)$ we have that

  $$2c_{2i+1}\left(E_{\mathbb{C}}\right) = 0 \qquad \text{(pure torsion of order 2)}$$

  The above classes are usually discarded and one defines

  $$p = p_0 - p_1 + p_2 - \cdots = c_0 + c_2 + c_4 + \cdots$$
  $$p_i\left(E\right) = (-)^i\, c_{2i}\left(E_{\mathbb{C}}\right)$$

- Since $\left(E \oplus F\right)_{\mathbb{C}} = E_{\mathbb{C}} \oplus F_{\mathbb{C}}$ we have

  $$p\left(E \oplus F\right) = p\left(E\right) p\left(F\right)$$

- For a complex manifold M

  $$TM_{\mathbb{C}} = T_M \oplus \bar{T}_M$$
  $$p\left(TM\right) = c\left(T_M\right) c\left(\bar{T}_M\right) = \prod_i (1 - x_i^2)$$

Euler Class

- Real orientable vector bundle E of $\dim_{\mathbb{R}} E = 2n$ with $SO(2n)$ transition functions
- Choose $\mathfrak{so}(2n)$ connection with curvature F_α
- Euler class

$$e(E) = \mathrm{Pf}\left(\frac{F_\alpha}{2\pi}\right)$$

where

$$\mathrm{Pf}(X) = \frac{1}{2^n n!}\sum_\sigma (-)^\sigma X_{\sigma_1\sigma_2}\cdots X_{\sigma_{2n-1}\sigma_{2n}}$$
$$\mathrm{Pf}(X)^2 = \det(X)$$

The class is closed and independent of the connection in cohomology

Lecture 7: Characteristic Classes XI

- For a complex vector bundle F of $\dim_{\mathbb{C}} F = n$

$$c_n(F) = e(F_{\mathbb{R}})$$

Choose $U(N)$ transition functions and $\mathfrak{u}(n)$ connection with curvature f_α. Then $F_{\mathbb{R}}$ was $\mathfrak{so}(2n)$ curvature

$$F_\alpha = (f_\alpha)_{\mathbb{R}}$$

Clearly

$$\det\left(\frac{F_\alpha}{2\pi}\right) = \left|\det\left(\frac{if_\alpha}{2\pi}\right)\right|^2 = \mathrm{Pf}\left(\frac{F_\alpha}{2\pi}\right)^2$$

To check phase consider case $n = 1$ with $f_\alpha = -2\pi i$. Then $\frac{F_\alpha}{2\pi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\det\left(\frac{if_\alpha}{2\pi}\right) = \mathrm{Pf}\left(\frac{F_\alpha}{2\pi}\right) = 1$.

- Exercise : Compute the Euler class of the tangent bundle TM of an orientable manifold M of dimension $2n$ as a function of the Riemann curvature $R_{\mu\nu}{}^{\alpha}{}_{\beta}$ and the volume form $\sqrt{\det g_{\mu\nu}}dx^1 \cdots dx^{2n}$ for $n = 1, 2$.

Dolbeault Cohomology

- Since $T^{\star}M_{\mathbb{C}} = T^{\star}_{M} \oplus \bar{T}^{\star}_{M}$ we have that

$$\Omega^{n}_{\mathbb{C}}(M) = \bigoplus_{p+q=n} \Omega^{p,q}(M)$$

with $\Omega^{p,q}$ forms with p dz's and q $d\bar{z}$'s

- Differentials

$$d = dz^{a}\,\partial_{a} + d\bar{z}^{\bar{a}}\,\bar{\partial}_{\bar{a}} = \partial + \bar{\partial}$$

with

$$\partial : \Omega^{p,q} \to \Omega^{p+1,q}$$
$$\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$$

and

$$\partial^{2} = \bar{\partial}^{2} = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

Lecture 8: Complex Manifolds II

- Dolbeault cohomology

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\ker \bar{\partial} : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}}{\operatorname{im} \bar{\partial} : \Omega^{p,q-1} \longrightarrow \Omega^{p,q}}$$

In particular

$$H^{p,0}_{\bar{\partial}}(M) \ \text{holomorphic } (p,0)\text{–forms}$$

Exact Sequences in Čech cohomology

- A sequence of sheaf maps

$$\longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{G} \longrightarrow$$

is exact with respect to a covering U_i if the induced sequence

$$\longrightarrow \mathcal{F}(U_{i_0 \cdots i_p}) \xrightarrow{\alpha} \mathcal{H}(U_{i_0 \cdots i_p}) \xrightarrow{\beta} \mathcal{G}(U_{i_0 \cdots i_p}) \longrightarrow$$

is exact for each $U_{i_0 \cdots i_p}$

Lecture 8: Complex Manifolds III

- Given a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$$

we have a long exact sequence in cohomology

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{H}) \to H^0(\mathcal{G}) \to$$
$$\to H^1(\mathcal{F}) \to \cdots$$

- Given a sheaf map $\mathcal{F} \xrightarrow{\alpha} \mathcal{H}$ define the kernel sheaf $\ker(\alpha)$ by

$$\ker(\alpha)(U) = \ker \alpha_U : \mathcal{F}(U) \to \mathcal{H}(U)$$

A long sequence

$$\to \mathcal{F}_{n-1} \xrightarrow{\alpha_{n-1}} \mathcal{F}_n \xrightarrow{\alpha_n} \mathcal{F}_{n+1} \to$$

is exact if and only if $\alpha_n \circ \alpha_{n+1} = 0$ and if

$$0 \to \ker \alpha_n \to \mathcal{F}_n \to \ker \alpha_{n+1} \to 0$$

is short exact

Lecture 8: Complex Manifolds IV

Dolbeault's Isomorphism

- Dolbeault's Lemma. Locally (on $\mathbb{C}^n$) if $\bar{\partial}\omega = 0$ then $\omega = \bar{\partial}\eta$. As an example, if $n = 1$ and $\omega = \omega(z, \bar{z})\, d\bar{z}$ then we can choose

$$\eta(z, \bar{z}) = \frac{i}{2\pi} \int \frac{dw \wedge d\bar{w}}{z - w}\, \omega(w, \bar{w}) \qquad \left(\text{recall } \bar{\partial}\frac{1}{z} = \pi\, \delta^2(z, \bar{z})\right)$$

- $\Omega^{p,q}$ smooth (p, q) forms and $\mathcal{A}^p$ holomorphic $(p, 0)$ forms
- With respect to a good cover on M

$$0 \to \mathcal{A}^p \to \Omega^{p,0} \xrightarrow{\bar{\partial}_0} \Omega^{p,1} \xrightarrow{\bar{\partial}_1} \Omega^{p,2} \to \cdots$$

is exact. Equivalent short exact sequences

$$0 \to \mathcal{A}^p \to \Omega^{p,0} \to \ker \bar{\partial}_1 \to 0$$

$$0 \to \ker \bar{\partial}_i \to \Omega^{p,i} \to \ker \bar{\partial}_{i+1} \to 0 \qquad (i \geq 1)$$

- Use

$$H^k\left(\Omega^{p,q}\right) = 0 \qquad \text{for } k \geq 1$$
$$H^0\left(\mathcal{F}\right) = \mathcal{F}\left(M\right)$$

and long exact sequences in cohomology

$$H^q\left(\mathcal{A}^p\right) = H^{q-1}\left(\ker \bar{\partial}_1\right) = H^{q-2}\left(\ker \bar{\partial}_2\right) = \cdots$$
$$= H^1\left(\ker \bar{\partial}_{q-1}\right) = \frac{\ker \bar{\partial}_q}{\operatorname{Im} \bar{\partial}_{q-1}}$$

Therefore

$$H^q\left(\mathcal{A}^p\right) = H^{p,q}_{\bar{\partial}}\left(M\right)$$

Lecture 8: Complex Manifolds VI

- Let L be a holomorphic vector bundle (and the sheaf of holomorphic sections) Then

$$0 \to L \to L \otimes \Omega^{0,0} \xrightarrow{\bar{\partial}_0} L \otimes \Omega^{0,1} \xrightarrow{\bar{\partial}_1} \cdots$$

produces the isomorphism

$$H^q(L) = \frac{\ker \bar{\partial}_q}{\operatorname{Im} \bar{\partial}_{q-1}} \quad \text{closed/exact } (0, q) \text{ forms with values in } L$$

- To obtain an integrable form on M we must integrate against a section of

$$L^\star \otimes K \otimes \Omega^{0,n-q}$$

Serre duality

$$H^q(L) = H^{n-q}(L^\star \otimes K)$$

- Note : the Čech–deRham isomorphism $H^\star(U, \mathbb{R}) \simeq H^\star_{\mathrm{deRham}}(X)$ can be shown as above starting from $0 \to \mathbb{R} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$

Hermitian Metrics

- Definition

$$g_{ab} = g_{\bar{a}\bar{b}} = 0 \qquad \text{(always exists)}$$

- Kähler form

$$\omega \in \Omega^{1,1}(M) \qquad \text{(real form)}$$

$$\omega = i\, g_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}}$$

- Volume form (exercise)

$$\frac{1}{m!}\,\omega^m = \sqrt{\det g_{ij}}\; dx^1 dy^1 \cdots dx^m dy^m$$

- Hermitian connection $\Gamma^a_{\mu b}$ on T_M defined by
  - $\Gamma^a_{\bar{a}b} = 0$ (possible since T_M is holomorphic)

Lecture 8: Complex Manifolds VIII

- metric covariantly constant

$$\partial_a g_{b\bar{c}} - \Gamma^c_{ab} g_{c\bar{c}} = 0$$

  with connection on $\bar{T}_M$

$$\Gamma^{\bar{b}}_{\bar{a}\bar{c}} = \overline{\Gamma^b_{ac}}$$

- Explicit form

$$\Gamma^b_{ac} = g^{b\bar{b}} \partial_a g_{\bar{b}c}$$

- Non–vanishing torsion

$$\Gamma^b_{ac} - \Gamma^b_{ca} = g^{b\bar{b}} \left(\partial_a g_{\bar{b}c} - \partial_c g_{\bar{b}a} \right)$$

- Curvature

$$R^c{}_{da\bar{b}} = -\partial_{\bar{b}} \Gamma^c_{ad} \qquad\qquad R^c{}_{dab} = 0$$

Kähler Manifolds

- Equivalent definitions
  - $d\omega = 0$

Lecture 8: Complex Manifolds IX

- Hermitian and Levi–Civita connections coincide
- Vanishing torsion
- In components

$$\partial_a g_{\bar{b}c} - \partial_c g_{\bar{b}a} \qquad \text{and } c.c.$$

- Curvature is Riemannian and satisfies $R^{\mu}{}_{(\alpha\beta\gamma)} = 0$ so that

$$\begin{aligned} R_{a\bar{b}} &= R^{c}{}_{ac\bar{b}} + R^{\bar{c}}{}_{a\bar{c}\bar{b}} \\ &= R^{c}{}_{ca\bar{b}} + R^{c}{}_{a\bar{b}c} = R^{c}{}_{ca\bar{b}} \end{aligned}$$

- We have

$$c_1\left(T_M\right) = \frac{i}{2\pi} R_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}}$$

- Kähler potential : Given a good cover U_α of M then (Poincaré Lemma and decomposition of forms) one has real functions $\mathcal{K}_\alpha$ on U_α and holomorphic functions $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that

$$\begin{aligned} \omega &= i\, \partial\bar{\partial}\, \mathcal{K}_\alpha && \text{on } U_\alpha \\ \mathcal{K}_\alpha - \mathcal{K}_\beta &= f_{\alpha\beta} + \bar{f}_{\alpha\beta} && \text{on } U_\alpha \cap U_\beta \end{aligned}$$

$\mathbb{C}P^n$

- Homogenous coordinates $(z_0, \cdots, z_n)$ not all zero up to non–vanishing complex rescaling
- Tautological line bundle S in the exact sequence

$$0 \to S \to \mathbb{C}P^n \times \mathbb{C}^{n+1} \to Q \to 0$$

Exercise: Given the open cover $U_i \subset \mathbb{C}P^n$ defined by $z_i \neq 0$ we have transition functions for S

$$g_{ij} = \frac{z_i}{z_j}$$

Moreover

$$T_{\mathbb{C}P^n} = Q \otimes S^\star$$

Lecture 8: Complex Manifolds XI

- Fubini–Study Kähler potential

$$\mathcal{K}_i = \ln \frac{\sum_j z_j \bar{z}_j}{z_i \bar{z}_i}$$
$$\mathcal{K}_i - \mathcal{K}_j = \ln \frac{z_j \bar{z}_j}{z_i \bar{z}_i}$$

- Gives a connection on S

$$A_i = \partial \mathcal{K}_i \qquad\qquad A_i - A_j = d \ln \left(z_i / z_j \right)$$

with

$$x = -c_1 \left(S \right) = -\frac{i}{2\pi} dA_i = \frac{1}{2\pi} \omega$$

- Cohomology of $\mathbb{C}P^n$

$$\begin{aligned} H^{2k} \left(\mathbb{C}P^n, G \right) &= G \text{ for } k = 0, \cdots, n \\ &= 0 \text{ otherwise} \end{aligned}$$

Cohomology H^{2k} generated by x^k

Lecture 9: Hodge Theory I

Hodge Dual

- N real orientable manifold of $\dim_{\mathbb{R}} = n$ with metric g (with s negative eigenvalues) and volume form ϵ
- E^A orthonormal basis of $T^\star N$ with norm $\eta_A = \pm 1$ and with

$$\epsilon = E^1 \wedge \cdots \wedge E^n$$

- Let

$$\omega = E^{A_1} \wedge \cdots \wedge E^{A_k}$$

  If $B_1 \cdots B_{n-k}$ are the complementary indices to $A_1 \cdots A_k$ and π the permutation of $1 \cdots n$ to $A_1 \cdots A_k B_1 \cdots B_{n-k}$ we define

$$\star\omega = \eta_{A_1} \cdots \eta_{A_k} \ (-)^\pi \ E^{B_1} \wedge \cdots \wedge E^{B_{n-k}}$$

- Clearly

$$\star : \Omega^k \to \Omega^{n-k}$$

$$\star^2 = (-)^s \, (-)^{k(n-k)}$$

- If

$$\left\{ \begin{array}{l} \alpha \\ \beta \end{array} \right. = \frac{1}{k!} E^{A_1} \wedge \cdots \wedge E^{A_k} \begin{array}{l} \alpha_{A_1 \cdots A_k} \\ \beta_{A_1 \cdots A_k} \end{array}$$

  we define

$$\alpha \cdot \beta = \frac{1}{k!} \alpha_{A_1 \cdots A_k} \beta^{A_1 \cdots A_k}$$

  Then (exercise)

$$\alpha \wedge \star\beta = \beta \wedge \star\alpha = \alpha \cdot \beta \, \epsilon$$

- In components

$$(\star\alpha)_{B_1 \cdots B_{n-k}} = \frac{1}{k!} \, \alpha_{A_1 \cdots A_k} \, \epsilon^{A_1 \cdots A_k}{}_{B_1 \cdots B_{n-k}}$$

Lecture 9: Hodge Theory III

Laplacian

- Assume N compact and define symmetric form on k–forms (positive definite if $s = 0$)

$$\langle \alpha, \beta \rangle = \int_N \alpha \wedge \star \beta$$

One has

$$\langle d\alpha, \beta \rangle = \int d\alpha \wedge \star\beta = (-)^k \int \alpha \wedge d \star \beta$$
$$= (-)^k (-)^s (-)^{(n-k+1)(k-1)} \int \alpha \wedge \star^2 d \star \beta$$

or

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^\dagger \beta \rangle$$
$$d^\dagger = (-)^{n(k+1)+1+s} \star d \star : \Omega^k \to \Omega^{k-1}$$

Lecture 9: Hodge Theory IV

- Assume $s = 0$ from now on. Define the laplacian

$$\Delta = dd^{\dagger} + d^{\dagger}d : \Omega^k \to \Omega^k$$

Since

$$\langle \alpha, \Delta\alpha \rangle = |d\alpha|^2 + |d^{\dagger}\alpha|^2$$

one has that

$$\Delta\alpha = 0 \qquad \Leftrightarrow \qquad d\alpha = 0 \ , \quad d^{\dagger}\alpha = 0$$

- Consider a cohomology class $[\alpha]$ and assume there is a harmonic representative $\Delta\alpha = 0$. Then
  1. α has minimal norm in the class since

$$|\alpha + d\beta|^2 = |\alpha|^2 + 2\langle d^{\dagger}\alpha, \beta\rangle + |d\beta|^2 = |\alpha|^2 + |d\beta|^2$$

  2. α is unique since

$$d^{\dagger}(\alpha + d\beta) = d^{\dagger}d\beta = 0 \qquad \to \qquad |d\beta|^2 = \langle d^{\dagger}d\beta, \beta\rangle = 0$$

Lecture 9: Hodge Theory V

- In coordinates

$$(d^\dagger \alpha)_{A_1 \cdots A_{k-1}} = -\nabla^A \alpha_{A A_1 \cdots A_{k-1}}$$
$$(\Delta \alpha)_{A_1 \cdots A_k} = -\nabla_A \nabla^A \alpha_{A_1 \cdots A_k}$$

Hodge Theorem

- Let $\mathcal{H}^p \subset \Omega^p$ the harmonic forms. Then
  - $\dim \mathcal{H}^p < \infty$ and therefore the orthogonal projection $P : \Omega^p \to \mathcal{H}^p$ is well defined
  - There is a unique Green operator

$$G : \Omega^p \to \Omega^p$$

such that $G\mathcal{H}^p = 0$, it commutes with d and $d^\dagger$and

$$1 = P + \Delta G$$

Lecture 9: Hodge Theory VI

- Corollary 1: Since

$$\alpha = P\alpha + d(d^{\dagger} G\alpha) + d^{\dagger}\,(dG\alpha)$$

we obtain the orthogonal decomposition

$$\Omega^p = \underbrace{\mathcal{H}^p \oplus d\Omega^{p-1}}_{\text{closed forms}} \oplus d^{\dagger}\Omega^{p+1}$$

and the isomorphism

$$H^p_{\text{deRham}} = \mathcal{H}^p$$

- Corollary 2: If α is harmonic so is $\star\alpha$. Since $\star$ is invertible we recover Poincaré duality

$$\star : \mathcal{H}^p \to \mathcal{H}^{n-p}$$

Complex Version

- Let N complex, compact with hermitian metric g, Kähler form ω and $\dim_{\mathbb{C}} N = n$. Then

$$\star : \Omega^{p,q} \longrightarrow \Omega^{n-q,n-p}$$
$$\star^2 = (-)^{p+q}$$

- Hermitian product on $\Omega^{p,q}$

$$\langle \alpha, \beta \rangle = \int_N \alpha \wedge \star \bar{\beta}$$

Since $\bar{\partial} = d$ on $\Omega^{n,k}$ we have

$$\left\langle \bar{\partial}\alpha, \beta \right\rangle = \langle \alpha, \bar{\partial}^\dagger \beta \rangle$$
$$\bar{\partial}^\dagger = - \star \partial \star : \Omega^{p,q} \longrightarrow \Omega^{p,q-1}$$

Lecture 9: Hodge Theory VIII

- Laplacian and harmonic forms

$$\Delta_{\bar\partial} = \bar\partial^\dagger\bar\partial + \bar\partial\bar\partial^\dagger$$
$$\mathcal{H}^{p,q}_{\bar\partial} \subset \Omega^{p,q}$$

Hodge decomposition

$$\Omega^{p,q} = \mathcal{H}^{p,q}_{\bar\partial} \oplus \bar\partial\Omega^{p,q-1} \oplus \bar\partial^\dagger\Omega^{p,q+1}$$
$$H^{p,q}_{\bar\partial} = \mathcal{H}^{p,q}_{\bar\partial}$$

- Isomorphism

$$\mathcal{H}^{p,q}_{\bar\partial} \overset{\text{Hodge dual } \star}{=} \mathcal{H}^{n-q,n-p}_{\partial} \overset{\text{complex conjugation}}{=} \mathcal{H}^{n-p,n-q}_{\bar\partial}$$

Lecture 9: Hodge Theory IX

- If N is Kähler then

$$\Delta_{\bar{\partial}} = \Delta_{\partial} = \frac{1}{2}\Delta$$

  Implies Kähler decomposition

$$\mathcal{H}^k = \bigoplus_{p+q=k} \ \mathcal{H}_{\bar{\partial}}^{p,q}$$
$$\mathcal{H}_{\bar{\partial}}^{p,q} = \mathcal{H}_{\bar{\partial}}^{q,p}$$

Introduction to Hypersurfaces in $\mathbb{C}P^n$

- Let $p\,(z_0, \cdots, z_n)$ a homogeneous polynomial of degree d and assume that

$$M \subset P^n \qquad \text{(omit } \mathbb{C} \text{ from now on)}$$

  defined by $p = 0$ is a complex manifold without singularities
- Tangent bundle

$$T_{P^n}|_M = T_M \oplus N_M \qquad \text{(normal line bundle } N_M\text{)}$$
$$c(T_M) = c(T_{P^n})|_M \ / \ c(N_M)$$

Lecture 9: Hodge Theory X

- Use the exact sequences

$$0 \to S \to \mathbb{C}^{n+1} \to Q \to 0$$
$$0 \to \mathbb{C} \to S^{\star n+1} \to T_{P^n} = Q \otimes S^\star \to 0$$

to get

$$c(T_M) = c(S^\star)^{n+1} = (1+x)^{n+1}$$

- On $U_i \subset \mathbb{C}P^n$ defined by $z_i \neq 0$ define

$$p_i = p\left(\frac{z_0}{z_i}, \cdots, \frac{z_n}{z_i}\right) \qquad \frac{p_j}{p_i} = \left(\frac{z_i}{z_j}\right)^d$$

Sections of S^d are given by

$$f_i p_i = f_j p_j$$

and define functions on P^n which vanish on M and therefore

$$S^d|_M = N_M^\star$$

- Therefore (omitting $|_M$)

$$c(N_M) = 1 + dx$$
$$c(T_M) = (1+x)^{n+1} / (1+dx)$$

and

$$c_1(T_M) = (n+1-d)x$$
$$c_{n-1}(T_M) = \frac{1}{d^2}\left[(1-d)^{n+1} - 1 + (n+1)d\right]\ x^{n-1}$$

- Euler characteristics (true for Kähler manifolds as we shall see)

$$\chi(M) = \int_M c_{n-1}(T_M) = \frac{1}{d}\left[(1-d)^{n+1} - 1 + (n+1)d\right]$$

- We have used

$$\int_M x^{n-1} = d \int_{P^n} x^n = d$$

This can be shown using the connection for S^d

$$A_i = -d \ln p_i$$

with curvature a δ function on M such that

$$\int_M \mu = \frac{i}{2\pi} \int_{P^n} dA_i \wedge \mu$$
$$\frac{i}{2\pi} dA_i = d \cdot x + \text{coboundary}$$

Variation of complex structure

- Complex structure

$$z^\mu_\alpha = f^\mu_{\alpha\beta}\left(z_\beta\right)$$

- Variation is

$$\Delta f^{\mu}_{\alpha\beta} \quad \text{holomorphic vectors on } U_\alpha \cap U_\beta$$

such that

$$\delta\left(\Delta f\right) = 0$$

modulo holomorphic reparameterization of the z_α given by $z^{\mu}_{\alpha} \longrightarrow z^{\mu}_{\alpha} + \epsilon^{\mu}_{\alpha}$ or

$$\delta\epsilon$$

Therefore

$$H^1\left(T_M\right) \quad \overset{\text{Serre}}{=} \quad H^{n-1}\left(T^{\star}_{M} \otimes K\right)$$

- When $n = 1$ then $K = T^{\star}_{M}$ and

$$H^1\left(T_M\right) = H^0(K^2) = K^2\left(M\right) \qquad \text{quadratic differentials}$$

- If K is trivial

$$H^1\left(T_M\right) = H^{1,n-1}_{\bar{\partial}}\left(M\right)$$

Lecture 10: Elliptic Operators I

Definition

- $E, F \longrightarrow N$ complex vector bundles with N compact and oriented
- Orded D differential operator

$$A : \Gamma(E) \rightarrow \Gamma(F)$$

- Given local coordinates x^i and trivializations of E, F

$$A = \sum_{0 \leq k \leq D} A^{i_1 \cdots i_k}(x)\ \partial_{i_1} \cdots \partial_{i_k}$$

with $A^{i_1 \cdots i_k}(x)$ matricies $\dim E \times \dim F$

- Maximal symbol

$$A^{i_1 \cdots i_D}(x) \in \Gamma\left(\mathrm{Sym}^D\ T \otimes \mathrm{Hom}(E, F)\right)$$

A elliptic if

$$A^{i_1 \cdots i_D}(x)\ p_{i_1} \cdots p_{i_D}$$

invertible when p_i is real and non–vanishing

Lecture 10: Elliptic Operators II

- Basic fact : If A is elliptic $\ker A$ and $\operatorname{coker} A = \Gamma(F) / \operatorname{im} A$ are finite dimensional. Define

$$\operatorname{index}(A) = \dim \ker A - \dim \operatorname{coker} A$$

Hodge Theory

- Hermitian form on sections

$$\langle \hat{s}, s \rangle_{E,F} = \int_N h_{E,F}(\hat{s}, s)\ \epsilon$$

with $h_{E,F}$ hermitian metrics on E, F and fixed volume form ϵ

- Integrating by parts construct adjoint

$$A^\dagger : \Gamma(F) \to \Gamma(E)$$
$$\langle \hat{s}, As \rangle_F = \langle A^\dagger \hat{s}, s \rangle_E$$

If A is order D and elliptic so is $A^\dagger$

Lecture 10: Elliptic Operators III

- Elliptic, selfadjoint and positive Laplacians

$$\Box_E = A^\dagger A \qquad\qquad \Box_F = AA^\dagger$$

- Hodge Theorem : Let $\Gamma_\lambda(E) \subset \Gamma(E)$ the eigenspace of $\Box_E$ with eigenvalue $\lambda \geq 0$. Then
  - $\dim \Gamma_\lambda(E) < \infty$ with discrete spectrum
  - $L_2(E) = \oplus_\lambda \Gamma_\lambda(E)$
  - $1_{\Gamma(E)} = P_E + \Box_E G_E$ with

$$P_E : \Gamma(E) \to \Gamma_0(E)$$
$$G_E : \Gamma(E) \to \Gamma(E)$$

    orthogonal projection and Green operator with $G_E\Gamma_0(E) = 0$ and $\Box_E G_E = G_E \Box_E$

  and similarly for F

- Basic consequences
  - $\Gamma_0(E) = \ker A$
  - $\Gamma_0(F) = \operatorname{coker} A$

Lecture 10: Elliptic Operators IV

- $A : \Gamma_\lambda(E) \to \Gamma_\lambda(F)$ isomorphism for $\lambda > 0$

The second point follows from

$$s = P_F s + A(A^\dagger G_F s)$$

The third from

$$A^\dagger A s = \lambda s$$

for $s \in \Gamma_\lambda(E)$. Applying A we get

$$AA^\dagger (As) = \lambda (As)$$

so that $As \in \Gamma_\lambda(F)$. Also $As = 0$ implies $s = 0$ so that A is injective. Finally for $s \in \Gamma_\lambda(F)$ we have $s = A\left(A^\dagger G_F s\right)$ and A is surjective

Lecture 10: Elliptic Operators V

Heat Kernel and Seeley Formula

- The trace (for $\Box_{E,F}$)

$$\mathrm{Tr}(e^{-t\,\Box}) = \textstyle\sum_\lambda e^{-\lambda t}\ \dim\Gamma_\lambda$$

  converges for $t > 0$

- Asymptotic exansion for $t \to 0$

$$\mathrm{Tr}(e^{-t\,\Box}) \sim \sum_{k \geq -n} t^{\frac{k}{2D}} \int_M \mu_k\,(\Box)$$

  with μ_k built canonically from the coefficients of $\Box$

- Index

$$\begin{aligned}\mathrm{index}\,(A) &= \mathrm{Tr}(e^{-t\,\Box_E}) - \mathrm{Tr}(e^{-t\,\Box_F}) \\ &= \int_M \mu_0\,(\Box_E) - \int_M \mu_0\,(\Box_F)\end{aligned}$$

Lecture 10: Elliptic Operators VI

- Locally $\Box$ is

$$\sum_{0\leq k\leq 2D} \Box^{i_1 \ldots i_k}(x)\ \partial_{i_1}\cdots\partial_{i_k}$$

  with $\Box^{i_1\ldots i_k}$ matricies $m\times m$ with $m = \dim E = \dim F$

- Fix p_i and define symbol of a differential operator a as

$$\sigma(a) = e^{-ipx}\ a\ e^{ipx}$$
$$\sigma(ab) = \sigma(a)\,\sigma(b)$$

  Obtain by replacing

$$\partial_i \longrightarrow \partial_i + ip_i$$

Lecture 10: Elliptic Operators VII

- Define

$$\begin{aligned}\sigma &= \sigma\left(\Box - \lambda\right) \\ &= \underbrace{\sigma_0 + \sigma_1 + \cdots + \sigma_{2D-1}}_{\rho} + \left(\sigma_{2D} - \lambda\right)\end{aligned}$$

where σ_ℓ is of order p^ℓ (with $\lambda \sim p^{2D}$). We have in particular

$$\begin{aligned}\sigma_0 &= \Box \\ \sigma_{2D} &= \text{maximal invertible symbol of } \Box\end{aligned}$$

- Assume from now on

$$\sigma_{2D} = a\left(x, p\right) \cdot \mathbf{1}_{m \times m}$$

Lecture 10: Elliptic Operators VIII

- Define

$$\begin{aligned}\hat{\sigma} = \sigma\left(\frac{1}{\Box - \lambda}\right) &= \frac{1}{a - \lambda + \rho} \\ &= \frac{1}{a-\lambda} - \frac{1}{a-\lambda}\rho\frac{1}{a-\lambda} + \cdots \\ &= \hat{\sigma}_{-2D} + \hat{\sigma}_{-2D+1} + \cdots\end{aligned}$$

  where $\hat{\sigma}_\ell$ is of order p^ℓ

- Acting with derivatives of ρ on the terms $1/\left(a-\lambda\right)$ one gets

$$\hat{\sigma}_\ell = \sum_s \frac{(-)^s}{(a-\lambda)^{s+1}}\ \hat{\sigma}_\ell^s \qquad\qquad (\ell \geq -2D)$$

  where $\hat{\sigma}_\ell^s$ is polynomial in the p_i's of order

$$\ell + 2D\left(s+1\right) \geq 0$$

  and polynomial in the coefficients of $\Box$ and their derivatives

Lecture 10: Elliptic Operators IX

- Consider

$$\begin{aligned}\langle x|e^{-t\Box}|x\rangle &= -\int_\Gamma \frac{d\lambda}{2\pi i}\; e^{-\lambda t}\;\langle x|\frac{1}{\Box-\lambda}|x\rangle \\ &= -\int_\Gamma \frac{d\lambda}{2\pi i}\;\int \frac{d^n p}{(2\pi)^n}\; e^{-\lambda t}\;\langle x|\frac{1}{\Box-\lambda}|p\rangle\langle p|x\rangle \\ &= -\int_\Gamma \frac{d\lambda}{2\pi i}\;\int \frac{d^n p}{(2\pi)^n}\; e^{-\lambda t}\;\;\hat{\sigma}(x,p)\end{aligned}$$

  where $\hat{\sigma}(x,p)$ is the symbol $\hat{\sigma}$ without derivatives (acting on the constant function 1) and Γ is the path circling the positive real λ axis

- Use

$$-\int_\Gamma \frac{d\lambda}{2\pi i}\;\frac{1}{(a-\lambda)^{s+1}}e^{-\lambda t} \;=\; (-)^s\;\; e^{-at}\;\frac{t^s}{s!}$$

  to get

$$\langle x|e^{-t\Box}|x\rangle \sim \sum_{\ell,s}\int \frac{d^n p}{(2\pi)^n}\; e^{-at}\;\frac{t^s}{s!}\;\hat{\sigma}^s_\ell(x,p)$$

- Write the $d^n p$ integral as

$$d^n p = \frac{1}{2D} \frac{d\eta}{\eta} \eta^{\frac{n}{2D}} \; d\Omega_p$$

where $\eta^{1/2D}$ is the radial variable ($p^{2D} \sim \eta$). We then get

$$\langle x | e^{-t\Box} | x \rangle \sim \frac{(2\pi)^{-n}}{2D} \sum_{\ell, s} \int d\Omega_p \; \hat{\sigma}_\ell^s \, (x, p) \cdot$$
$$\cdot \int \frac{d\eta}{\eta} \eta^{\frac{n}{2D}} \quad e^{-\eta a t} \; \frac{t^s}{s!} \; \eta^{\frac{\ell + 2D}{2D} + s}$$

- Define

$$k = -2D - \ell - n \geq -n$$

Lecture 10: Elliptic Operators XI

- Integrate on η to get

$$\langle x | e^{-t\Box} | x \rangle \sim \sum_{k \geq -n} t^{\frac{k}{2D}} \; \mu_k \left(\Box \right)$$

$$\mu_k \left(\Box \right) = \frac{\left(2\pi \right)^{-n}}{2D} \sum_{s \geq \frac{k+n}{2D}} \int d\Omega_p \; \cdot$$

$$\cdot \frac{\Gamma \left(s - \frac{k}{2D} \right)}{s!} \; \left[a \left(x, p \right) \right]^{-s + \frac{k}{2D}} \; \hat{\sigma}_\ell^s \left(x, p \right)$$

- Invariance of trace under $\Box \to \lambda \Box$, $t \to \lambda^{-1} t$ implies

$$\mu_k \left(\lambda \Box \right) = \lambda^{\frac{k}{2D}} \; \mu_k \left(\Box \right)$$

Gilkey's Argument

- Let $\mu\left(g\right)$ be an n–form built from the metric g_{ab}, its inverse g^{ab} and derivatives of the metric up to a finite order. Going to normal coordinates it is constructed terms of the form

$$\nabla_{a_1}\cdots\nabla_{a_n}R_{b_1\cdots b_4} \qquad (\star)$$

- Assume that $\mu\left(g\right)$ is of weight k

$$\mu(\lambda^2 g) = \lambda^k \mu\left(g\right)$$

- Consider a monomial with r terms of the form $(\star)$ with a total of d covariant derivatives. Out of the $4r+d$ indices $4r+d-q$ are contracted with g^{ab} and the other q are antisymmetrized. Since the weights of $R_{b_1\cdots b_4}$ and g_{ab} are 2 we have

$$k = q - 2r - d$$

Lecture 11: Hirzebruch Signature Theorem II

- In $R_{b_1 \cdots b_4}$ at most two indices can be antisymmetrized otherwise one gets a vanishing contribution. Therefore

$$q \leq 2r + d$$
$$k \leq 0$$

- Consider the case $k = 0$. Then $d = 0$. This follows from the fact that $(\star)$ vanishes also if we antisymmetize two b indices and one a index due to the Bianchi identity. This implies $q = 2r$
- Using

$$R_{abcd} = R_{cdab} = -R_{abdc}$$
$$R_{a[bc]d} = \frac{1}{2} R_{adbc}$$

every monomial is built from terms of the form

$$R_{i_1 i_2 [j_1 j_2} R_{i_2 i_3 j_3 j_4} \cdots R_{i_r i_1 j_{q-1} j_1]} \qquad (j \text{ indices antisymmetrized})$$

Lecture 11: Hirzebruch Signature Theorem III

- Therefore $\mu(g)$ is built from Pontrjagin classes when $k = 0$
- More generally, let $E \longrightarrow N$ be a complex vector bundle with g a metric on N and h a metric on the fibers on E with ∇_m a connection on E such that h is covariantly constant
- Let $\mu(g, h)$ have weights

$$\mu(\lambda^2 g, h) = \lambda^k \mu(g, h)$$
$$\mu(g, \lambda^2 h) = \lambda^\ell \mu(g, h)$$

  By similar arguments

  - $\mu = 0$ if $k > 0$ or if $\ell \neq 0$
  - μ built from $p(TN)$ and $c(E)$ if $k = \ell = 0$

Lecture 11: Hirzebruch Signature Theorem IV

The Signature Theorem

- N compact oriented Riemannian manifold of dimension $n = 4k$
- Elliptic selfadjoint operator

$$A = d + d^\dagger : \Omega \to \Omega$$
$$\Omega = \bigoplus_p \Omega^p$$
$$\Box = A^2 = dd^\dagger + d^\dagger d = \Delta_{\text{Hodge}}$$

- Involution

$$\tau : \Omega^p \to \Omega^{4k-p}$$
$$\tau = i^{p(p-1)+2k} \star$$
$$\tau^2 = 1$$
$$A\tau + \tau A = 0$$

Lecture 11: Hirzebruch Signature Theorem V

- Decomposition

$$\begin{aligned} \Omega =& \Omega_+ \oplus \Omega_- \qquad\qquad (\tau \text{ on } \Omega_\pm \text{ is } \pm 1) \\ A :& \Omega_+ \to \Omega_- \\ A^\dagger = A :& \Omega_- \to \Omega_+ \end{aligned}$$

- Index of A

$$\ker A = \text{Harmonic forms } \cap \ \Omega_+$$
$$\ker A^\dagger = \text{Harmonic forms } \cap \ \Omega_-$$

- By Poincaré duality, for $p \neq 2k$

$$\mathcal{H}^p \oplus \mathcal{H}^{4k-p} = \left(\frac{1-\tau}{2}\right) \mathcal{H}^p \oplus \left(\frac{1+\tau}{2}\right) \mathcal{H}^p$$

The two spaces on the right have the same dimension and do not contribute to the index

Lecture 11: Hirzebruch Signature Theorem VI

- Also

$$\mathcal{H}^{2k} = \mathcal{H}^{2k}_{+} \oplus \mathcal{H}^{2k}_{-}$$
$$\operatorname{ind}(A) = \dim \mathcal{H}^{2k}_{+} - \dim \mathcal{H}^{2k}_{-}$$

- Note that for $n = 4k$ we have $\tau|_{\Omega^{n/2}} = \star$. In the case $n/2$ odd then $\star^2 = -1$ and $\tau|_{\Omega^{n/2}} = \pm i\star$. In this case complex conjugation exchanges $\mathcal{H}^{2k}_{+}$ and $\mathcal{H}^{2k}_{-}$ and the index vanishes
- Nondegenerate (by Poincaré) bilinear form $\cdot$ on $H^{2k}(N, \mathbb{R})$ given by

$$[\alpha] \cdot [\beta] = \int_N \alpha \wedge \beta$$

- Signature of $\cdot$ denoted by $\operatorname{sign}(N)$. Since on $\mathcal{H}^{2k}_{\pm}$

$$\pm \alpha \cdot \alpha = \int \alpha \wedge \star\alpha = |\alpha|^2 \geq 0$$

we have that

$$\operatorname{sign}(N) = \operatorname{ind}(A)$$

Lecture 11: Hirzebruch Signature Theorem VII

- Since under $g \to \lambda^2 g$ one has $\Box \to \lambda^{-2}\Box$ by Gilkey one has

$$\text{sign}\,(N) = \int_N f_k\,(p_1, \cdots, p_k)$$

with f_k a polynomial in the Pontryagin classes p_i

- To fix f_k it suffices to consider the spaces

$$P_{2k_1} \times \cdots \times P_{2k_r} \qquad\qquad (\textstyle\sum_i k_i = k)$$

using

$$\text{sign}\,(M \times N) = \text{sign}\,(M)\,\text{sign}\,(N)$$
$$\text{sign}\,(P_{2n}) = 1$$

Lecture 11: Hirzebruch Signature Theorem VIII

- Note that
$$\int_{P_{2q}} \mathrm{L}(T_{P_{2q}}) = 1$$
since
$$\begin{aligned}\mathrm{L}(T_{P_{2q}}) &= \mathrm{L}\left(S^\star\right)^{2q+1} = \left(\frac{x}{\tanh x}\right)^{2q+1} \\ &= \cdots + x^{2q} + \cdots\end{aligned}$$
and that $\mathrm{L}\left(M \times N\right) = \mathrm{L}\left(M\right)\mathrm{L}\left(N\right)$
- Therefore
$$\text{sign}\left(N\right) = \int_N \mathrm{L}\left(N\right)$$

Lecture 11: Hirzebruch Signature Theorem IX

- Notation : For a complex manifold we denote with $\mathrm{L}(N) = \mathrm{L}(T_N)$. Since L is even in the x_i it actually is only a function of the Pontrjagin classes

$$p_k(TN) = \sum_{i_1 < \cdots < i_k} x_{i_1}^2 \cdots x_{i_k}^2$$

For a general manifold we define $\mathrm{L}(N)$ using its expression in terms of the Pontrjagin classes, which starts as

$$\mathrm{L}(TN) = 1 + \frac{1}{3} p_1 + \frac{1}{45}\left(7p_2 - p_1^2\right) + \cdots$$

The same comments apply to $\widehat{\mathrm{A}}(TN)$

Lecture 12: General Index Theorem I

General Index Theorem

- N compact orientable of even dimension $\dim_{\mathbb{R}} N = n$
- $E_q \to N$ complex vector bundles
- Complex

$$0 \to \Gamma\left(E_0\right) \xrightarrow{d_0} \Gamma\left(E_1\right) \xrightarrow{d_1} \cdots \to \Gamma\left(E_m\right) \to 0$$
$$d_{i+1} \circ d_i = 0$$

- The maximal symbols $\sigma_i\left(p\right)$ of d_i give maps

$$0 \to E_0 \xrightarrow{\sigma_0} E_1 \xrightarrow{\sigma_1} \cdots \to E_m \to 0$$

Complex is elliptic if the above is exact

Lecture 12: General Index Theorem II

- Like before

$$A : \Gamma\left(\bigoplus_{i \text{ even}} E_i\right) \to \Gamma\left(\bigoplus_{i \text{ odd}} E_i\right)$$
$$A = \sum_{i \text{ even}} d_i + d_i^\dagger$$
$$A^\dagger = \sum_{i \text{ odd}} d_i + d_i^\dagger$$

- Index Theorem

$$\text{ind}(d) = \sum_{i=0}^{m} (-)^i \ \dim \frac{\ker d_i}{\text{im}\, d_{i-1}}$$
$$= (-)^{\frac{n}{2}} \int_N \textstyle\sum_i (-)^i \, \mathrm{ch}\,(E_i) \wedge \dfrac{\mathrm{Td}\,(TN_{\mathbb{C}})}{\mathrm{e}\,(TN)}$$

Lecture 12: General Index Theorem III

Hirzebruch–Riemann–Roch

- N complex manifold of $\dim_{\mathbb{C}} N = n$
- Elliptic complex (exercise)

$$0 \to \Omega_V^{0,0} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega_V^{0,n} \to 0$$

with

$$\Omega_V^{0,q} = \Gamma\left(E_q\right)$$
$$E_q = V \otimes \bigwedge^q \bar{T}_N^\star$$
V holomorphic vector bundle

- Use splitting principle

$$T_N = \bigoplus_i L_i$$
$$\bigwedge^q \bar{T}_N^\star = \bigoplus_{i_1 < \cdots < i_q} \left(\bar{L}_{i_1}^\star \otimes \cdots \otimes \bar{L}_{i_q}^\star\right)$$

Lecture 12: General Index Theorem IV

- Obtain

$$c\left(L_i\right) = 1 + x_i$$
$$\mathrm{ch}\left(E_q\right) = \mathrm{ch}\left(V\right) \sum_{i_1 < \cdots < i_q} e^{x_{i_1} + \cdots + x_{i_q}}$$

and

$$\sum_q \left(-\right)^q \mathrm{ch}\left(E_q\right) = \mathrm{ch}\left(V\right) \prod_i \left(1 - e^{x_i}\right)$$

- Todd class

$$\mathrm{Td}\left(TN_{\mathbb{C}}\right) = \mathrm{Td}\left(T_N \oplus \bar{T}_N\right) = \mathrm{Td}\left(T_N\right) \mathrm{Td}\left(\bar{T}_N\right)$$
$$= \mathrm{Td}\left(T_N\right) \prod_i \frac{-x_i}{\left(1 - e^{x_i}\right)}$$

- Note that

$$\left(T_N\right)_{\mathbb{R}} = TN$$
$$\prod_i x_i = c_n\left(T_N\right) = e\left(TN\right)$$

Lecture 12: General Index Theorem V

- Therefore

$$\text{ind}\left(\bar{\partial}_V\right) = \int \mathrm{ch}\left(V\right)\ \mathrm{Td}\left(T_N\right)$$
$$= h^0\left(V\right) - h^1\left(V\right) + \cdots$$

where $h^q\left(V\right) = \dim_{\mathbb{C}}\ H^q\left(V\right)$. Recall also Serre duality

$$h^q\left(V\right) = h^{n-q}\left(V^\star \otimes K\right) \qquad \left(\ K = \bigwedge^n T_N^\star\ \right)$$

Riemann–Roch

- N Riemann surface of genus g $(n = 1)$
- Use $x/\left(1 - e^{-x}\right) = x/2 + \cdots$ and

$$\text{ind}\left(\bar{\partial}\right) = \frac{1}{2}\int c_1\left(T_N\right) = h^{0,0} - h^{0,1}$$
$$= 1 - g$$

We use that N is Käher and $h^{0,1} = h^{1,0} = g$

Lecture 12: General Index Theorem VI

- Given a holomorphic line bundle L

$$\begin{aligned} \text{ind}\left(\bar{\partial}_L\right) &= h^0\left(L\right) - h^1\left(L\right) \\ &= h^0\left(L\right) - h^0\left(K \otimes L^{\star}\right) \end{aligned}$$

- Index theorem

$$\begin{aligned} \text{ind}\left(\bar{\partial}_L\right) &= \int_N \left(1 + \frac{1}{2} c_1\left(T_N\right)\right)\left(1 + c_1\left(L\right)\right) \\ &= 1 - g + \deg\left(L\right) \end{aligned}$$

- Degree

$$\begin{aligned} \deg\left(L\right) &= \int_N c_1\left(L\right) \\ &= \#\text{ of zeros } - \#\text{ of poles of meromorphic sections} \end{aligned}$$

Lecture 12: General Index Theorem VII

- To show use connection

$$A_\alpha = -d \ln s_\alpha$$
$$s_\alpha = g_{\alpha\beta} s_\beta \text{ meromorphic section}$$

When

$$s_\alpha = z^n$$
$$A_\alpha = -n \frac{dz}{z}$$
$$-\frac{1}{2\pi i} F_\alpha = n\, \delta(z, \bar{z}) \;\; \frac{i}{2} dz \wedge d\bar{z}$$

- Clearly

$$h^0(L) > 0 \qquad \Rightarrow \qquad \deg(L) \geq 0$$

and

$$h^0(\mathbb{C}) = 1 \qquad \deg(\mathbb{C}) = 0$$
$$h^0(K) = g \qquad \deg(K) = 2g - 2$$

Lecture 12: General Index Theorem VIII

- Finally

$$h^1(T_N) = 3g - 3 + h^0(T_N)$$
$$h^0(T_N) = 0 \text{ for } g > 1 \text{ since } \deg(T_N) = 2 - 2g$$

Twisted Hirzebruch Signature Index

- N real manifold of $\dim_{\mathbb{R}} N = 4k$
- Given complex vector bundle V look at V–valued q–forms

$$\Omega_V^q = \Gamma\left(\wedge^q (TN^\star)_{\mathbb{C}} \otimes V\right)$$
$$D : \Omega_V^q \to \Omega_V^{q+1}$$
$$D = d + \text{connection on } V$$

and

$$A : \Omega_+ \to \Omega_-$$
$$A = D + D^\dagger$$

Lecture 12: General Index Theorem IX

- Index theorem

$$\text{ind}\,(A) = 2^{n/2} \int \mathrm{ch}\,(V)\ \ \mathcal{L}\,(TN_{\mathbb{C}})$$

$$\mathcal{L}\,(E) = \prod_i \frac{x_i/2}{\tanh x_i/2}$$

Dirac Index

- N real manifold of even dimension n with fixed metric and spin structure
- V complex vector bundle with connection
- $S_\pm$ positive and negative chirality spinor bundles
- Elliptic complex

$$\not{D} : \Gamma\,(S_+ \otimes V) \to: \Gamma\,(S_- \otimes V)$$

with adjoint

$$\not{D} : \Gamma\,(S_- \otimes V) \to: \Gamma\,(S_+ \otimes V)$$

- Index theorem

$$\begin{aligned} \text{ind}\,(\not{D}) &= \#\text{ of zeros od } \not{D} \text{ with positive chirality} \\ &\quad - \#\text{ of zeros od } \not{D} \text{ with negative chirality} \\ &= \int_N \mathrm{ch}\,(V)\ \widehat{\mathrm{A}}\,(TN) \end{aligned}$$

with

$$\widehat{\mathrm{A}}\,(TN) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right) + \cdots$$

Euler Index

- deRham complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \longrightarrow 0$$

has index the Euler characteristic

$$\begin{aligned} \chi(N) &= \operatorname{ind}(d) = h^0 - h^1 + \cdots \\ &= \int_N e(TN) \qquad\qquad (n \text{ even}) \end{aligned}$$

and 0 for n odd by Poincaré duality