Introduction to BRST Quantization

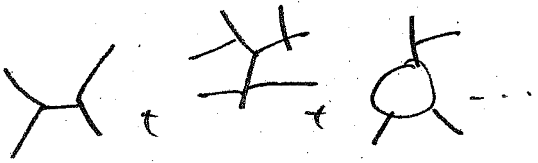
(1) S-matrix, Lagrangians, Observables

We use the LSZ formalism to compute scattering amplitudes.

$$Z[j] = e^{\frac{i}{\hbar} Z_c[j]} = \langle \Omega, T\left(e^{\frac{i}{\hbar}\int d^4x \sum_\alpha \phi_\alpha(x) j^\alpha(x)}\right)\Omega\rangle$$

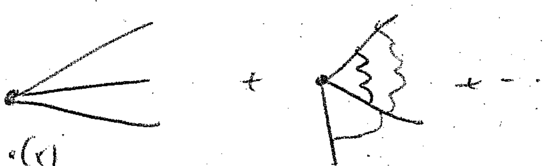
$\phi_\alpha \Rightarrow$ quantized fields.
$j^\alpha \Rightarrow$ sources

Z_c = connected functional



Adding composite operators:

$\mathcal{O}_i(x)$, $\xi^i(x)$.



$\mathcal{O}_i(x)$

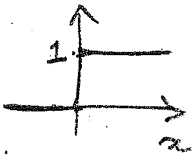
$$Z[j,\xi] = \langle\Omega, T\left(e^{\frac{i}{\hbar}\int d^4x \sum_\alpha j_\alpha \phi^\alpha + \sum_i \xi_i \mathcal{O}^i}\right)\Omega\rangle$$

Ω: Vacuum of the Fock space generated by the quantized fields ϕ^α!

T: Time ordered product:

$$T\phi(x)\phi(y) = \phi(x)\phi(y)\,\theta(x_0-y_0) + \phi(y)\phi(x)\,\theta(y_0-x_0)$$

$\theta(x) =$



$$\left[\partial_0 \phi(x_0), \phi(0)\right] = (2\pi)^3 \delta^3(\vec{x}) \sqrt{p^0}$$

$$\int \frac{d^4p}{(2\pi)^4} \theta(p^0)\, e^{\frac{i}{\hbar}(p_0 x_0 - p_i x_i)}\, \delta(p^2 - m^2) =$$

Green's functions

$$G(x_1 \cdots x_n) = \langle \Omega, T(\phi(x_1) \cdots \phi(x_n))\Omega\rangle$$

$$Z[j] = \prod_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_i \; j(x_1) \cdots j(x_n) \langle \Omega, T(\phi(x_1) \cdots \phi(x_n))\Omega\rangle$$

$$G(x_1 \cdots x_n) = \left(\frac{\hbar}{i}\right)^n \frac{\delta^n}{\delta j(x_1) \cdots \delta j(x_n)} Z[j]\bigg|_{j=0}$$

$$Z[j] = \exp\left(Z_c[j]\right)$$

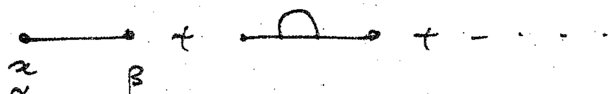
connesso

$$= \langle \Omega, T \exp\left(\frac{i}{\hbar}\int d^4x \; j(x)\phi(x)\right)\Omega\rangle$$

2) Two point function.

$$\left.\frac{\delta^2 Z_{(c)}[j]}{\delta j_\alpha(x)\,\delta j_\beta(0)}\right|_{j=0} \equiv \Delta^{\alpha\beta}(x)$$

Graphically:



(Massive fields, for massless field one needs an IR regularisation mod of scattering states).

$$\Delta^{\alpha\beta} = \Delta^{\alpha\beta}_{(AS)}(x) + R^{\alpha\beta}(x) =$$

$$= \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{e^{\frac{i}{\hbar}p\cdot x}}{m_\lambda^2 - p^2 - i\epsilon} \boxed{\Gamma_\lambda^{\alpha\beta}(p)} + R^{\alpha\beta}(x)$$

1) $\int d^4x\, e^{-ip\cdot x} R^{\alpha\beta}(x)$ has no pole in p^2.

2) $\Gamma_\lambda^{\alpha\beta}$ is defined up to a polynomial vanishing at $p^2 = m_\lambda^2$ (no effect on S-matrix).

$$\left[\phi_{in}^{\alpha(+)}(x), \phi_{in}^{\beta(-)}(0)\right] = \sum_\lambda \int \frac{d^4p}{(2\pi)^4} e^{\frac{i}{\hbar}p\cdot x}\, \theta(p^0)\, \delta(p^2 - m_\lambda^2)\, \Gamma_\lambda^{\alpha\beta}(p)$$

Even though at fixed time, fields referring to various position do commute this is no longer true when we compare them at different times

Scattering of 4 fields
(4-point amplitudes)

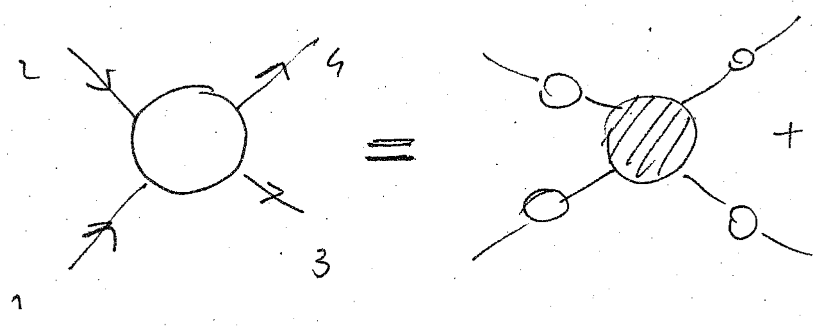
$$\langle \Omega, \mathcal{S} \, \Omega \rangle \Big|_{4\text{ point}} =$$

$$= \prod_{i=1}^{4} \int d^4 x_i \, \phi_{i,u}^{\alpha_i}(x_i) \, k_{\alpha_i \beta_i}(x_i) \frac{\delta}{\delta j_{\beta_i}(x_i)} \, \underbrace{Z[j]}_{\langle 0| \qquad |0\rangle} \Big|_{j=0} =$$

using the cluster decomposition $\langle 0| \phi_{i,u}^{\alpha_1} \cdots \phi_{i,u}^{\alpha_4} |0\rangle = \langle 0| \phi_{i,u}^{\alpha_1} |0\rangle \cdots \langle 0| \phi_{i,u}^{\alpha_4} |0\rangle$

$$= \prod_{i=1}^{4} \underbrace{Z_i^{-1/2} \int d^4 x_i \, e^{i p_i \cdot x_i} \, k_{\alpha_i \beta_i}}_{} \frac{\delta}{\delta j_{\beta_i}(x_i)} \, Z[j] \Big|_{j=0} =$$

LSZ $\quad \langle T \phi(x_1) \cdots \phi(x_4) \rangle$



+ ....

Indeed from canonical quantization:

$$[\varphi(x),\varphi(y)] = \int \frac{d^3k}{(2\pi)^3(2k_0)}\left[e^{-ik(x-y)} - e^{ik(x-y)}\right]$$

$$= \int \frac{d^4k}{(2\pi)^4}\, 2\pi\,\delta(k^2-m^2)\,\theta(k^0)\left[e^{-ik(x-y)} - e^{ik(x-y)}\right]$$

from which it follows eq. above.

In addition: $\Delta^{\alpha\beta}_{(as)}(x)$ satisfies

$$\boxed{K_{\alpha\gamma}\,\Delta^{\gamma\beta}_{(as)}(x) = \delta^{\beta}_{\alpha}\,\delta^4(x)}$$

(see f. example ITZYKSON ZUBER Chap. 1/2)

Then we have a way to defines the S-matrix.

$$\mathcal{S}^{LSZ} = :e^{\frac{1}{\hbar}\int d^4x\, \phi^{\alpha}_{in}(x)\,K_{\alpha\beta}(x)\frac{\delta}{\delta j_\beta(x)}}:\,Z[j]\Big|_{j=0} =$$

$$\boxed{\mathcal{S} = :e^{\Sigma}:\,Z[j]\Big|_{j=0}}$$

$\Longrightarrow$ see example

operator on the Fock space

kinematical information

Dynamical information

Feynman integral

$$Z[j] = \int d\mu\; e^{\frac{i}{\hbar}\int d^4x\,[\phi_\alpha(x) j^\alpha + \Sigma j_i \mathcal{O}^i]}$$

(4

$$d\mu = N \prod_x \mathcal{D}\phi(x)\, e^{\frac{i}{\hbar} S(\phi)}$$

Normalization factor such that. — Action for the fields ϕ.

$$Z[0,0] = 1$$

→ Regularization and Renormalization

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(3) Proper functional (effective action Γ).

$$\boxed{\phi_\alpha[j, \xi, x] = \frac{\delta Z_c}{\delta j^\alpha(x)} - \frac{\delta Z_c}{\delta j^\alpha(0)}\Big|_{j=0}}$$

(local functional: it depends upon x)

and assume that $j = j[\phi, \xi]$ $\exists$ ⇒

$$\Gamma[\phi, \xi] = Z_c[j[\phi, \xi], \xi] - \int d^4x \left(\phi(x) + \frac{\delta Z_c}{\delta j(0)}\Big|_{j=0}\right) j[\phi, \xi]$$

$$\Rightarrow \begin{cases} \dfrac{\delta}{\delta\phi(x)} \Gamma[\phi, \xi] = -\, j[\phi, \xi, x] \quad \text{at } \phi = \phi[j, \xi] \\[2ex] \dfrac{\delta}{\delta\xi(x)} \Gamma[\phi, \xi] = \dfrac{\delta}{\delta\xi} Z_c[j, \xi] \end{cases}$$

$$\Rightarrow \frac{\delta^2}{\delta\phi_\alpha(x)\,\delta\phi_\beta(y)} \Gamma[\phi, \xi]\bigg|_{\phi=\phi[j,\xi]} =$$

$$= - \frac{\delta}{\delta\phi_\alpha(x)} j^\beta[\phi, \xi, y] =$$

(using (*)) $$= - \left[\frac{\delta^2 Z_c}{\delta j^\alpha(x)\,\delta j^\beta(y)}\bigg|_j \right]^{-1}$$

or:

$$\boxed{\frac{\delta^2 \Gamma}{\delta\phi_\alpha \delta\phi_\beta} \frac{\delta^2 Z_c}{\delta j^\beta \delta j^\gamma} = \delta_\alpha^\gamma\, \delta(x-y)}$$

(we use the notation $\Gamma_{\phi_\alpha\phi_\beta}(x)$, $Z_{j^\alpha j^\beta}$).

Wave function (the "full")

Two point Green function

- Perturbation theory:

$$\boxed{\Gamma = \left[S' + \xi_i\, \theta' \right]_{\hbar=0} + O(\hbar)}$$

Γ can be seen as an effective action of ϕ_α and its derivatives:

$$\Gamma = \Lambda + \int d^4x\, \Lambda^{\alpha\beta} \phi_\alpha \phi_\beta + \tilde{\Lambda}^{\alpha\beta}\, \partial_\mu \phi_\alpha\, \partial_\nu \phi_\beta\, g^{\mu\nu} + \dots$$

$$\Lambda^{\alpha\beta} = \Lambda^{(\alpha\beta)} \quad \tilde{\Lambda}^{\alpha\beta} = \tilde{\Lambda}^{\beta\alpha}(\phi), \dots$$

4 Gauge fields and Vector fields

(Covariance and unitarity)
$\Rightarrow$ gauge invariance

(Massive vector fields
(good scattering potentials).)

$$A_\mu^{(in)\alpha}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} \left(\epsilon_{\lambda,\vec{p},\mu}\, a^{(in)\alpha}_{\lambda\vec{p}}\, e^{-ipx} + h.c. \right)$$

annihilation operator.

$\lambda = 1, 2$
(helicity components).

$$\begin{cases} \epsilon_{\lambda\vec{p},\mu}\, g^{\mu\nu}\, \epsilon_{\lambda',\vec{p}\,\nu} = -\delta_{\lambda\lambda'} \\ \epsilon_{\lambda\vec{p},\mu}\, p^\mu = 0 \end{cases}$$

by the fact let us observe that:

$$\boxed{\partial^\mu A_\mu^{(in)\alpha} = 0}$$ (gauge di Landau).

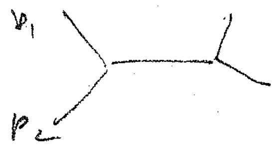
$\Rightarrow$ Only $\left[A_\mu^{(in)\alpha}(x), A_\nu^{(in)\beta}(0) \right] = \int \frac{d^4p}{(2\pi)^4} \theta(p_0)\delta^4(p^2-m^2) \boxed{\Pi_{\mu\nu}^{\alpha\beta}(p)}$

$$\Rightarrow \Delta_{as\,\mu\nu}^{\alpha\beta}(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{m^2 - p^2 - i\epsilon} \underbrace{\left(\frac{p_\mu p_\nu}{m^2} - p^2 g_{\mu\nu} \right) \delta^{\alpha\beta}}$$

let consider the limit $p^2 \to \infty$

$$\tilde{\Delta}^{\alpha\beta}_{\mu\nu}(p) \xrightarrow[p^2\to\infty]{} -\frac{1}{p^2}\frac{p_\mu p_\nu}{m^2}\delta^{\alpha\beta}$$

so it is homogeneous of degree -2.



#1 $\Rightarrow$ Produce a cross sections $\sim E^2$ $\qquad \sigma \sim G^2 s$
(as in the case of Fermi theory) $\Rightarrow$ $s = (p_1+p_2)^2$

this induces a violation of S-matrix unitarity.
(in fact from unitarity $S^\dagger S = I$, one derives the optical theorem $\Rightarrow$ from the partial wave decomposition it follows that $\sigma \sim \frac{cost}{s}$

(Then we have a violation of the unitarity $G^2 s \sim \frac{1}{s} \Rightarrow$
$s \sim \frac{1}{G}$)

#2 $\Rightarrow$ One needs a decoupling mechanism for the longitudinal modes:

$$\frac{1}{p^2-m^2}\,\Gamma^{\alpha\beta}_{\mu\nu}(p) = \frac{1}{p^2-m^2}\left(\frac{p_\mu p_\nu}{m^2} - g_{\mu\nu}\right)$$

$$\Rightarrow \frac{p^\mu}{p^2-m^2}\Gamma^{\alpha\beta}_{\mu\nu}(p) = \frac{1}{p^2-m^2}\,\frac{p^2 p_\nu}{m^2} - p_\nu = \frac{p_\nu}{m^2} \neq 0.$$

(this means that the longitudinal modes are not decoupled) let us add a new term to the functional Γ $\Rightarrow$

$$-\int d^4x \, \frac{\xi}{2}\left[\partial_\mu A^{\alpha\mu}\right]\left[\partial_\nu A^{\beta\nu}\right]\delta_{\alpha\beta}$$

$\Rightarrow$

$$\tilde{\Gamma}^{(2)\alpha\beta}_{\mu\nu} = \delta^{\alpha\beta}\left[\left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu}\right)(p^2-m^2)(1+A(p^2)) + \right.$$

$$\left. + \frac{p_\mu p_\nu}{p^2} m^2 (1+B(p^2))\right]$$

$\tilde{\Gamma}^{(2)\alpha\beta}_{\mu\nu}(p^2=0) \sim$ regular $\Rightarrow$ $A(0)=B(0)$

$\tilde{\Gamma}^{(2)\alpha\beta}_{\mu\nu}(p^2=m^2) \sim$ regular $\Rightarrow$ $A(m^2)=B(m^2)$.

Computing the inverse of $\tilde{\Gamma}^{(2)\alpha\beta}_{\mu\nu}$

$\Rightarrow$

$$\tilde{\Delta}^{(2)\alpha\beta}_{\mu\nu} = \delta^{\alpha\beta}\left[\frac{(p_\mu p_\nu/p^2 - g_{\mu\nu})}{(p^2-m^2)(1+A(p^2))} + \frac{p_\mu p_\nu}{p^2(\xi p^2 - m^2(1+B(p^2))}\right]$$

1) it has a pole for $p^2=m^2$.

2) $p \sim \infty$ $\quad \delta^{\alpha\beta}\frac{1}{p^2}$ (degree -2).

3) second pole $p^2 = \frac{m^2}{\xi}(1+B(p^2))$

4) ~~Pro~~ Propagator of a derivative of scalar field but with the wrong sign, this sign is consistent with a QFT with an indefinite metric space:

$$[a_\mu, a^\dagger_\nu] = g_{\mu\nu} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

$\Rightarrow \quad \mu = 0, 1$

$$\begin{cases} [a_0, a_0^+] = 1 \\ [a_1, a_1^+] = -1 \end{cases}$$

Fock space: $(a_0^+)^{n_0} (a_1^+)^{n_1} |0\rangle$, $\begin{cases} a_1|0\rangle = 0 \\ a_0|0\rangle = 0 \end{cases}$

but!

$$\langle 0|0\rangle = 1$$

$$\| a_1^+|0\rangle\|^2 = \langle 0|a_1 a_1^+|0\rangle =$$

$$= \langle 0| a_1^+ a_1 - 1|0\rangle = -\langle 0|0\rangle$$

$$\Rightarrow \quad \| a_1^+|0\rangle\|^2 = -1 \Longleftarrow \underline{\text{negative metric}}$$

(no unitarity).

Let us consider the Lagrangian:

1) $$S = \frac{1}{2}\int d^4x \, (\partial_\mu A^\mu)^2$$

$$P_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = \partial_0 A_\mu \quad \Rightarrow \quad [P_\mu, A_\nu] = \delta^3(x-y)(2\pi)^3\sqrt{2E}$$

$g_{\mu\nu}$

$$\boxed{[a_\mu, a_\nu^+] = i g_{\mu\nu}}$$

2) $$S = \frac{1}{2}\int d^4x \, (\partial^2\varphi)(\partial^2\varphi) = \frac{1}{2}\int d^4x \, \partial_\mu\partial^\mu\varphi \, \partial_\nu\partial^\nu\varphi =$$

$$= \frac{1}{2}\int d^4x \, (\partial_\mu\partial_\nu\varphi)(\partial^\mu\partial^\nu\varphi) \qquad \underline{\underline{A_\mu = \partial_\mu\varphi}} \quad \begin{pmatrix}\text{this system} \\ \text{has ghosts}\end{pmatrix}$$

? eq. di S-D. (free fields)

$$(\Box + \mu^2) \frac{\delta Z_e}{\delta j(x)} = j(x) \qquad \underline{\text{S.D.}}$$

(for free fields)

$$\langle \Omega, T\, \phi(x)\phi(y)\Omega \rangle =$$

$$= \langle \Omega, \phi(x)\phi(y)\Omega \rangle\, \theta(x_0 - y_0) +$$

$$+ \langle \Omega, \phi(y)\phi(x)\Omega \rangle\, \theta(y_0 - x_0)$$

$$\partial_0^x \langle \Omega, T\, \phi(x)\phi(y)\Omega \rangle =$$

$$= \langle \Omega, \partial_0\phi(x)\phi(y)\Omega \rangle\, \theta(x_0 - y_0) +$$

$$+ \langle \Omega, \phi(x)\phi(y)\Omega \rangle\, \delta(x_0 - y_0) +$$

$$+ \langle \Omega, \phi(y)\partial_0\phi(x)\Omega \rangle\, \theta(y_0 - x_0) +$$

$$- \langle \Omega, \phi(y)\phi(x)\Omega \rangle\, \delta(y_0 - x_0) =$$

$$= \langle \Omega, T(\partial_0\phi(x)\phi(y))\Omega \rangle + \langle \Omega, \underbrace{[\phi(x), \phi(y)]}_{\text{but this is a c-number (contact terms)}}\Omega \rangle$$

Then we need to prove

$$\mathcal{S}_{(\xi)} = \text{S-matrix of the theory with the new term } \frac{\xi}{2}\int d^4x\, (\partial_\mu A^\mu)^2 \text{ (with good UV behaviour)}$$

$$\lim_{\xi\to 0} \mathcal{S}_{(\xi)} = \mathcal{S}_{(\text{original})}$$

if we can prove that $\boxed{\partial_\xi \mathcal{S}_{(\xi)} = 0}$

then there are no problem with $\mathcal{S}_{(\text{original})}$ and the de longitudinal states are decoupled.

$$\boxed{:\partial_\mu \frac{\delta Z_c}{\delta j_\mu}: = \partial_\mu A^{\alpha\mu}_{(in)}(x)}$$

Proof:

the decoupling means:

$$\boxed{:\partial_\mu \frac{\delta Z_c}{\delta j^\alpha_\mu(x)}: = \partial_\mu A^{\alpha\mu}_{(in)}} \quad :\partial_\mu A^{\alpha\mu}(x)\Big|_{\substack{\phi = \frac{\partial Z_c}{\partial j} \\ j = K\phi_{in}}} : = \partial_\mu A^{\alpha\mu}_{(in)}(x) \quad \boxed{\text{WARD IDENTITY}}$$

(there is no interaction since the quantum equation reduces to the classical equation)

equation of motion: $\dfrac{\delta\left(S - \frac{\xi}{2}\int d^4x\, (\partial_\mu A^\mu)^2\right)}{\delta A_\mu} =$

$$= +\xi\,\partial_\mu \partial_\nu A^\nu + \frac{\delta S}{\delta A_\mu} \Longrightarrow$$

vedi Retro

$$\boxed{:\left(\xi\,\partial_\mu \partial_\nu A^\nu + \frac{\delta \Gamma}{\delta A_\mu}\right)\Big|_{\phi = \frac{\delta Z_c}{\delta j},\ j = K\phi_{(in)}} : = 0}$$

eq of Schwinger-Dyson

In the subspace such that:

$$\boxed{\partial_\mu A^{\alpha\mu}_{(1,\mu)}(x) = 0}$$

$$\Rightarrow :\ \partial^\mu \underbrace{\frac{\delta\Gamma}{\delta A^\alpha_\mu(x)}}_{I^{\alpha\mu}(x)} \Bigg|_{\begin{cases}\phi = \frac{\delta Z^c}{\delta j} \\ j = k\phi_{1,\mu}\end{cases}} := 0$$

$\Leftarrow$ the longitudinal modes are decoupled.

Conserved current

Noether theorem: if $\exists$ a set of infinitesimal transformations leaving the action Γ invariant.

classically $\delta S = \int d^4x\, \delta\mathcal{L} = 0 \Rightarrow \delta\mathcal{L} = d(*j)$

or $\delta\Gamma = \int d^4x\, P^i_\alpha(x) \frac{\delta\Gamma}{\delta\phi^i(x)} = 0$.

$$\Rightarrow \begin{cases} \boxed{P^{i\alpha}(x) \frac{\delta\Gamma}{\delta\phi^i(x)} = \partial_\mu I^{\mu\alpha} = \partial_\mu \frac{\delta\Gamma}{\delta A^\alpha_\mu}} \\ \delta\phi^i = P^i_\alpha\, \epsilon^\alpha \end{cases}$$

ϵ^α: infinitesimal parameter

$P^i_\alpha(\phi, x)$: polynomials of field, their derivatives and eventually non-local operators.

$$\Gamma \to \Gamma - \frac{\xi}{2}\int d\xi\ \partial_\mu A^{\mu\beta}\partial_\nu A^\nu_\beta$$

applying

$$P_\alpha^i \frac{\delta\Gamma}{\delta\phi^i} - \xi\,\partial_\mu A^{\beta\mu}\partial_\nu P_\alpha^{\beta\nu} = 0$$

Using field equations: $:\frac{\delta\Gamma}{\delta A^{\mu\alpha}} + \xi\,\partial_\mu\partial_\nu A^{\alpha\nu}: = 0$

$$: P_\alpha^{\beta\nu}(-\xi\partial_\nu\partial_\rho A_\beta^\rho) - \xi\partial_\mu A^{\beta\mu}\partial_\rho P^\rho_{\beta\alpha}: =$$

$$=: -\xi\left(P_\alpha^{\beta\nu}\partial_\nu + \partial_\rho P_\alpha^{\rho\beta}\right)\partial^\mu A_{\beta\mu}: = 0$$

this implies that on the sub space where

$$\partial^\mu A_{\mu\beta\,(as)} = 0$$

$$\Rightarrow\ :\partial^\mu A_{\mu\beta}: = 0 \quad \left(\text{if } \ker\left(P_\alpha^{\beta\nu}\partial_\nu + \partial_\rho P_\alpha^{\rho\beta}\right) \neq 0\right)$$

this means that the decoupling of long. modes is independent from ξ.

Then we had:

$$\left(\partial_\mu \frac{\delta}{\delta A^\alpha_\mu} - P^i_a \frac{\delta}{\delta \phi_i}\right)\Gamma \equiv X_\alpha(x)\,\Gamma = 0 \qquad (13)$$

(invariance of the action).

$$\phi_i = (A^{\alpha\mu}, \varphi^a)$$

$$\delta\phi_i = P_i^\alpha \Lambda_\alpha = \begin{cases} \delta A^{\alpha\mu} = \partial^\mu \Lambda^\alpha + P^{\alpha\mu}_\beta \Lambda^\beta \\ \delta\varphi^a = P^a_\beta \Lambda^\beta \end{cases}$$

$$\left(\underbrace{\partial_\mu \frac{\delta}{\delta A^\alpha_\mu} - P^\alpha_{\mu\beta} \frac{\delta}{\delta A^\beta_\mu}} - P^\alpha_a \frac{\delta}{\delta \varphi^a}\right)\Gamma = 0$$

$$\left(\nabla_\mu \frac{\delta}{\delta A^\alpha_\mu} - P^\alpha_a \frac{\delta}{\delta \varphi^a}\right)\Gamma = 0.$$

We do not assume any other condition. $\Rightarrow$

$$[X_\alpha(x), X_\beta(y)] = \int d^4z\, F_{\alpha\beta}{}^\gamma(x,y;z)\, X_\gamma(z).$$

These are vector field in the space of Jet bundles

$$X_\alpha(x) = X^i_\alpha \frac{\partial}{\partial\phi^i} + \partial_\mu X^i_\alpha \frac{\partial}{(\partial\partial_\mu\phi^i)} + \partial_\mu\partial X^i_\alpha \frac{\partial}{\partial(\partial_\mu\partial\phi^i)} + \cdots$$

due to the Frobenius theorem the

comm. relations $\left[X_\alpha, X_\beta\right] = \int F_{\alpha\beta}{}^{\gamma} X_\gamma$.

are sufficient and necessary for the system to be integrable.

Using the condition on polynomiality for the action Γ. (approximately a local functional $\Gamma = \int d^4x\, \mathcal{L}(x)$)

$$\begin{cases} P_\alpha^{\beta\mu}(x) = C_{\alpha\gamma}^{\beta} A^{\gamma\mu}(x) \\ P_\alpha^{a}(x) = V_\alpha^{a} + t_\alpha^{ab} \varphi^b(x) \end{cases}$$

Let us assume for the moment that $V_\alpha^a = 0$.

$$X_\alpha(x) = \partial_\mu \frac{\delta}{\delta A_\mu^\alpha} - C_{\alpha\gamma}^{\beta} A^{\gamma} \frac{\delta}{\delta A_\mu^\beta} - t_\alpha^{ab} \varphi_b \frac{\delta}{\delta \varphi^a}$$

$\Rightarrow$

$$\left[X_\alpha^{(x)}, X_\beta^{(y)}\right] = -\left[\partial_\mu \frac{\delta}{\delta A^\alpha_{\mu(x)}}, C_{\beta\delta}^{\gamma} A^\delta_{(y)} \frac{\delta}{\delta A^\gamma_{(y)}}\right] + \begin{pmatrix} x \leftrightarrow y \\ \alpha \leftrightarrow \beta \end{pmatrix} +$$

$$+ \left[C_{\alpha\alpha'}^{\alpha''} A_\mu^{\alpha'} \frac{\delta}{\delta A_\mu^{\alpha''}}, C_{\beta\beta'}^{\beta''} A_\mu^{\beta'} \frac{\delta}{\delta A_\mu^{\beta''}}\right] +$$

$$+ \left[t_\alpha^{ab} \varphi_b \frac{\delta}{\delta \varphi^b}, t_\beta^{rs} \varphi_r \frac{\delta}{\delta \varphi_s}\right]$$

$$\left[\partial_\mu^x \frac{\delta}{\delta A_\mu^\alpha(x)},\ C^\gamma_{\beta\delta} A_\mu^\delta(y) \frac{\delta}{\delta A_\nu^\gamma(y)}\right] =$$

$$= \partial_\mu^x \left[C^\gamma_{\beta\delta}\, \delta^4(x-y)\, g_{\mu\nu}\, \delta_\alpha^\delta \frac{\delta}{\delta A_\nu^\gamma(y)}\right] =$$

$$= C^\gamma_{\beta\alpha}\left[\partial_\mu^x \delta^4(x-y) \frac{\delta}{\delta A_\mu^\gamma(x)} + \delta^4(x-y)\, \partial_\mu^x \frac{\delta}{\delta A_\mu^\gamma(x)}\right] +$$

$$- C^\gamma_{\alpha\beta}\left[\partial_\mu^y \delta^4(y-x) \frac{\delta}{\delta A_\mu^\gamma(x)} + \delta^4(y-x)\, \partial_\mu^x \frac{\delta}{\delta A_\mu^\gamma(x)}\right] =$$

$$= \partial_\mu^x \delta^4(x-y)\left(C^\gamma_{\alpha\beta} + C^\gamma_{\beta\alpha}\right) \frac{\delta}{\delta A_\mu^\gamma(x)} + \delta^4(x-y)\left(C^\gamma_{\beta\alpha} - C^\gamma_{\alpha\beta}\right) \partial_\mu^x \frac{\delta}{\delta A_\mu^\gamma(x)}.$$

$$\Rightarrow \quad C^\gamma_{\alpha\beta} + C^\gamma_{\beta\alpha} = 0$$

$$\delta A_\mu = \nabla_\mu \lambda = \partial_\mu \lambda + \Gamma \lambda = \partial_\mu \lambda^\gamma + C^\alpha_{\beta\gamma} A_\mu^\beta \lambda^\gamma$$

$$[\delta_1, \delta_2] A_\mu = \delta_1 \left[\cancel{\partial_\mu \lambda_2} + \Gamma \lambda_2\right] - \delta_2 \left[\cancel{\partial_\mu \lambda_1} + \Gamma \lambda_1\right] =$$

$$= C^\alpha_{\beta\gamma}\left(\partial_\mu \lambda_1^\beta \lambda_2^\gamma + C^\beta_{\rho\sigma} A_\mu^\rho \lambda_1^\sigma \lambda_2^\gamma\right) +$$

$$- C^\alpha_{\beta\gamma}\left(\partial_\mu \lambda_2^\beta \lambda_1^\gamma + C^\beta_{\rho\sigma} A_\mu^\rho \lambda_2^\sigma \lambda_1^\gamma\right) =$$

$$= C^\alpha_{\beta\gamma}\left(\partial_\mu \lambda_1^\beta \lambda_2^\gamma - \partial_\mu \lambda_2^\beta \lambda_1^\gamma\right) + \left(C^\alpha_{\beta\gamma} C^\beta_{\rho\sigma} \left(\lambda_1^\sigma \lambda_2^\sigma - \lambda_2^\sigma \lambda_1^\gamma\right)\right)$$

$$= C^\alpha_{\beta\gamma}\left(\partial_\mu (\lambda_1^\beta \lambda_2^\gamma) - (\lambda_1^\beta \partial_\mu \lambda_2^\gamma + \lambda_1^\gamma \partial_\mu \lambda_2^\beta)\right)$$

(1) – (2) $\Rightarrow$

$$= - \partial_\mu \delta^4(x-y) \left(C_{\beta\alpha}^{\gamma} + C_{\alpha\beta}^{\gamma} \right) \frac{\delta}{\delta A_\mu^\gamma} + \delta^4(x-y) \left(C_{\alpha\beta}^{\gamma} \partial_\mu \frac{\delta}{\delta A_\mu^\gamma} \right)$$

This term is not of the type present in the W.I.

$$\Rightarrow \boxed{C_{\beta\alpha}^{\gamma} + C_{\alpha\beta}^{\gamma} = 0}$$

$\Rightarrow$

$$F_{\alpha\beta}^{\gamma}(x,y;z) = \delta^4(x-z)\, \delta^4(y-z)\, C_{\alpha\beta}^{\gamma}$$

and therefore:

$$\boxed{\left[X_\alpha(x), X_\beta(y) \right] = C_{\alpha\beta}^{\gamma}\, \delta^4(x-y)\, X_\gamma(x)}$$

and inserting the other terms:

$$C_{\alpha\beta}^{\gamma} C_{\gamma\delta}^{\eta} + C_{\beta\delta}^{\gamma} C_{\gamma\alpha}^{\eta} + C_{\delta\alpha}^{\gamma} C_{\gamma\beta}^{\eta} = 0$$

(Jacobi identity).

and from the fourth term:

$$\left[t_\alpha, t_\beta \right] = C_{\alpha\beta}^{\gamma}\, t_\gamma .$$

$C_{\alpha\beta}^{\gamma}$: structure constants $\qquad$ t_α: representation.

Now we relax the condition. $V^a_a = 0$. (16

$$\Rightarrow X_\alpha(x) \longrightarrow X_\alpha(x) - V^a_\alpha \frac{\delta}{\delta\varphi^a}$$

and for the commutator:

$$[X_\alpha, X_\beta] = \cdots + \left[V^a_\alpha \frac{\delta}{\delta\varphi^a}, t^{cd}_\beta \varphi^d \frac{\delta}{\delta\varphi^c}\right] - (\alpha \rightleftarrows \beta)$$

$$\Rightarrow \quad t_\beta V_\alpha - t_\alpha V_\beta = C^\gamma_{\alpha\beta} V_\gamma$$

This eq has a cohomological meaning:

Let consider a reps of G:

$$[t_\alpha, t_\beta] = C^\gamma_{\alpha\beta} t_\gamma$$

and we shift $t_\alpha \to t_\alpha + \epsilon V_\alpha \qquad \epsilon^2 \simeq 0$

$$[t_\alpha + \epsilon V_\alpha, t_\beta + \epsilon V_\beta] = C^\gamma_{\alpha\beta}(t_\gamma + \epsilon V_\gamma)$$

$\Rightarrow$ at first order in ϵ

$$\Rightarrow \quad [t_\alpha V_\beta - t_\beta V_\alpha] = C^\gamma_{\alpha\beta} V_\gamma.$$

By purely algebraic means for a <u>semisimple group</u>

$$G = \prod_s G_s \leftarrow \underline{\text{simple factors}}.$$

for each simple factor $(t_{\alpha_s}, t_{\beta_s}) = k_{\alpha_s\beta_s}$ (Killing Cartan form).

$$\Rightarrow \quad \boxed{v^a_\alpha = t^{ab}_\alpha\, v_\beta}$$

Constant parameter.

$$\begin{cases} t_\alpha = i\tau_\alpha & \tau_\alpha = \text{hermitian matrices.} \\ \mathrm{Tr}(\tau_\alpha \tau_\beta) = \delta_{\alpha\beta} \end{cases}$$

$$\begin{cases} \delta A^a_\mu = \partial_\mu \Lambda^a - \Lambda^\beta A^\gamma_\mu f^{\ \alpha}_{\beta\gamma} \\ \delta \varphi^a = i \Lambda^\alpha t^{ab}_\alpha (\varphi^b + v^b) \end{cases}$$

→ Different story for non-semisimple group.

→ COV. DERIVATIVES AND TRS.

$$\delta\left(\partial_\mu \varphi - i A^\alpha_\mu \tau_\alpha (\varphi + v)\right) =$$

$$= \partial_\mu\left(i\Lambda^\alpha t_\alpha (\varphi+v)\right) - i\left(\partial_\mu \Lambda^\alpha - \Lambda^\beta A^\gamma_\mu f^{\ \alpha}_{\beta\gamma}\right)\tau_\alpha(\varphi+v)$$

$$- i A^\alpha_\mu \tau_\alpha \left(i\Lambda^\beta \tau_\beta (\varphi+v)\right) =$$

$$= i\Lambda^\alpha t_\alpha \left[\partial_\mu \varphi - i\Lambda^\beta \tau_\beta (\varphi+v)\right].$$

$$\Rightarrow \quad \boxed{D_\mu(\varphi + v) \equiv \partial_\mu \varphi - i A^\alpha_\mu \tau_\alpha (\varphi + v)}$$

is the covariant derivative.

Since there is an inv. scalar product.

$$\begin{cases} (\varphi, \varphi) = \mathrm{Tr}(\varphi^2) \\ (\varphi+v, \varphi+v) = \mathrm{Tr}((\varphi+v)^2) \end{cases}$$

$$\Gamma_s = \int d^4x \left[\left(D_\mu(\varphi+v), D^\mu(\varphi+v) \right) - \frac{\lambda}{4!} \left[(\varphi+v, \varphi+v) - (v,v) \right]^2 \right)$$

λ is dimensionless.

and $\boxed{\left.\frac{\partial \Gamma_s}{\partial \varphi}\right|_{\varphi=0} = 0}$

$$[D_\mu, D_\nu](\varphi+v) = [\partial_\mu - i A_\mu^\alpha \tau_\alpha, \partial_\nu - i A_\nu^\beta \tau_\beta](\varphi+v) =$$

$$= -i\tau_\alpha \left(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + A_\mu^\beta A_\nu^\gamma f_{\beta\gamma}^\alpha \right)(\varphi+v).$$

$$= -i\tau_\alpha G_{\mu\nu}^\alpha (\varphi+v) = G_{\mu\nu}(\varphi+v).$$

$$\delta G_{\mu\nu} = i[\Lambda^\beta \tau_\beta, G_{\mu\nu}].$$

$$\Rightarrow \quad \Gamma_g = -k \int d^4x \, \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu})$$

$$\Gamma_{\theta} = \alpha \int d^4x \, \mathrm{Tr}(G_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}).$$

Fermionization of the Frobenius theorem:

$$\left[X_\alpha(x),\, X_\beta(y)\right] = \int d^4z\, C_{\alpha\beta}^{\gamma}(x,y;z)\, X_\gamma(z)$$

$$\delta \equiv \int d^4x\; c^\alpha(x)\, X_\alpha(x) + \cdots$$

$$\delta^2 = \int d^4x \int d^4y\; c^\alpha(x)\, c^\beta(y)\, X_\alpha(x)\, X_\beta(y) =$$

$$= \int d^4x \int d^4y\; \frac{1}{2}\, c^\alpha(x)\, c^\beta(y) \left[X_\alpha(x), X_\beta(y)\right] =$$

$$= \int d^4x \int d^4y\; \frac{1}{2}\, c^\alpha(x)\, c^\beta(y) \int C_{\alpha\beta}^{\gamma}(x,y;z)\, X_\gamma(z)\, d^4z =$$

$$= \int d^4z\, \frac{1}{2}\left[\int d^4x\, d^4y\;\, c^\alpha(x)\, c^\beta(y)\, C_{\alpha\beta}^{\gamma}(x,y;z)\right] X_\gamma(z) \neq 0.$$

1) Solution adding an additional piece to BRST operator:

$$\delta \equiv \int d^4x \left[c^\alpha(x)\, X_\alpha(x) + \left[\frac{1}{2}\int d^4y\, d^4z\;\, c^\alpha(y)\, c^\beta(z)\, C_{\alpha\beta}^{\gamma}(y,z;x)\right] \frac{\delta}{\delta c^\gamma(x)}\right]$$

2) Require that

$$\int d^4x\, d^4y\;\, c^\alpha(x)\, c^\beta(y)\, C_{\alpha\beta}^{\gamma}(x,y;z) = 0$$

Summary

Covariance ⊕ Unitarity

$\Downarrow$

Decoupling of long modes of theta & gss symmetry (invariance)

$\Downarrow$

Ward Identities $\Rightarrow$ $\boxed{X_\alpha(x)\Gamma = 0}$

$\Downarrow$

By F. Heiorn

$[X_\alpha, X_\beta] = C_{\alpha\beta}^{\gamma} X_\gamma \Rightarrow$ we complete X_α cohomology $\Rightarrow$ Lie algebra

$\Downarrow$

Construction of an action

$$\boxed{X_\alpha(x)\, S' = 0}$$

Renormalisation

Another example

Point particle.

$$S = k\int d\tau \sqrt{\dot{x}^2} \qquad \dot{x}^\mu = \frac{d}{d\tau}x^\mu \qquad \mu = 0,\dots,d-1$$

$$\Rightarrow \int d\tau\left(\frac{1}{e}\dot{x}^2 + \Lambda e\right) = \quad \text{by computing the eq. for } e$$

$$-\frac{1}{e^2}\dot{x}^2 + \Lambda = 0$$

$$= \int d\tau\left(\frac{\sqrt{\Lambda}}{\cancel{\sqrt{\dot{x}^2}}}\sqrt{\dot{x}^2} + \frac{\Lambda}{\cancel{\sqrt{\Lambda}}}\sqrt{\dot{x}^2}\right) \qquad e^2\Lambda = \dot{x}^2$$

$$e = \frac{1}{\sqrt{\Lambda}}\sqrt{\dot{x}^2}$$

$$= 2\sqrt{\Lambda}\int d\tau \sqrt{\dot{x}^2}$$

$$2\sqrt{\Lambda} = k \qquad \Lambda = \left(\frac{k}{2}\right)^2$$

$$S_{(e)} = \int d\tau\left(\frac{1}{e}\dot{x}^2 + \Lambda e\right) \rightarrow$$

$$P_{(e,P)} \rightarrow \int d\tau\left(P_\mu \dot{x}^\mu + \frac{e}{2}P_\mu P_\nu g^{\mu\nu} + \Lambda e\right) =$$

$$\int d\tau \left(P_\mu \dot{x}^\mu + \frac{e}{2}\left(P_\mu P_\nu g^{\mu\nu} + 2\Lambda\right)\right)$$

let us set $\Lambda = 0$ (massless particle)

eq for e: $\boxed{P^2 = 0}$

eq for $\dot{x}^\mu + e P_\nu g^{\mu\nu} = 0$

$$\Rightarrow P_\mu = -g_{\mu\nu}\frac{1}{e}\dot{x}^\nu$$

$$P^{(\Lambda=0)}_{(e,P)} = \int d\tau \left(P_\mu \dot{x}^\mu + \frac{e}{2}P^2\right)$$

This has the following symmetry:

$$\boxed{\begin{array}{ll} \delta x^\mu = \lambda P^\mu & \delta P_\mu = 0 \\ \delta e = -\dot{\lambda} & \end{array}}$$

$$\int d\tau \left[P_\mu (\lambda \dot{P}^\mu)^{\circ} + (-\dot{\lambda}) \frac{P^2}{2} \right] = 0.$$

$$-\dot{P}_\mu P^\mu \lambda = -\frac{1}{2}(\dot{P^2})\lambda \overset{\text{b. I.b.p.}}{=} +\frac{1}{2} P^2 \dot{\lambda}$$

why do we need that:

let us quantize the model. ~~but~~

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = P_\mu .$$

$$[P_\mu, x^\nu] = i\hbar \delta_\mu^\nu$$

but: $P_\mu = -\frac{1}{e} g_{\mu\nu} \dot{x}^\nu \implies [\dot{x}^\mu, x^\nu] = i\hbar g^{\mu\nu}$

~~and~~ non-def -ive Hilbert space.

W.I.

$$\boxed{-\partial_\tau \frac{\delta}{\delta e(\tau)} + P^\mu \frac{\delta}{\delta X^\mu} = X}$$

It is abelian sym.: $[X(x), X(y)] = 0.$

<u>BRST sym</u>:

$$s = \int d\tau\, c(\tau) X(\tau)$$

$$\begin{cases} se = -\dot{c} \\ sX^\mu = eP^\mu \\ sP_\mu = 0 \\ s\bar{c} = b \quad sb = 0 \end{cases}$$

$$S_{(g.f)} = S^{(\Lambda=0)}_{(e,P)} + \int s[\bar{c}\, (e-1)]$$

$$= S^{(\Lambda=0)}_{(e,P)} + \int (b(e-1) - \bar{c}\dot{c})\, d\tau$$

$$\Rightarrow = S^{(\Lambda=0)}_{(1,P)} = \int d\tau \left(P_\mu \dot{x}^\mu + \frac{1}{2}P^2 - \bar{c}\dot{c}\right)$$

Gauge symmetry vs Constraints

Suppose that we have a dynamical system: $S(\phi)$.

$\oplus$ Constraints : $F_\alpha(\phi) \approx 0$.

(they are first class: $\{F_\alpha(\phi), F_\beta(\phi)\} = C_{\alpha\beta}^{\ \ \gamma} F_\gamma(\phi)$)

then they generate a gauge symmetry:

$$\delta\phi_i = \left\{ \int d^q x \, \epsilon^\alpha(x) F_\alpha(\phi), \phi_i \right\}$$

then we can:

1) Use the constraint $F_\alpha(\phi) \approx 0$ to eliminate $\# \alpha$ ϕ_i

2) Use the gauge symmetry to set some of field to zero ($\#\alpha$!)

In total we have eliminated 2α d.o.f. from the original system.

Example: Maxwell: A_μ 4 dof's, by the Gauge cond. & Φ) gauge modes (their unphysical!)

$$A_\mu, \; c, \bar{c}$$

$$+4 \quad -2 = 2 \quad \text{physical degrees of freedom}$$

(24

Einstein: (4d) $g_{(\mu\nu)}$ 10 dof. $\rightarrow \begin{cases} \Lambda_\mu & \text{4 gauge modes} \\ \rho^\mu & \text{4 constraints} \end{cases}$

$\Rightarrow 10 - 8 = 2$ dof.

2-form: $B_{[\mu\nu]}$ 6 dof.

$$\delta B_{[\mu\nu]} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$

but this symmetry is redundant.

~~so~~ $\delta \Lambda_\mu = \partial_\mu \tau$.

So we have:

$$B_{\mu\nu} \longrightarrow 6$$

$$\rho^\mu \quad \Lambda_\mu \longrightarrow -8$$

$$\sigma \quad \eta \quad \tau \longrightarrow +3$$

1 dof

Indeed in 4d a 2 form is dual to a scalar

$$\begin{cases} H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} \\ (*H)^\sigma = \epsilon^{\mu\nu\rho\sigma} \partial_\mu B_{\nu\rho} = \partial^\sigma \rho \end{cases}$$

$\boxed{\text{Gauge fixing}}$

- Functional approach.

$$S + \int d^4x \, \mathrm{tr}(\rho\, \partial^\mu A_\mu) \rightarrow$$

variation under gauge transf.

$$\int d^4x \, \mathrm{tr}\left(\rho\, \partial^\mu \nabla_\mu \Lambda\right) =$$

$$= \int d^4x \quad \rho_\alpha \partial^\mu \left(\partial_\mu \Lambda^\alpha - C^\alpha_{\beta\gamma} A^\beta_\mu \Lambda^\gamma\right)$$

interaction term

Abelian case (free field)

Non-Abelian (non-decoupl.)

$\Lambda^\gamma \rightarrow$ become a field.

$\rho_\gamma \rightarrow$ conjugate field.

Introduction of a Grassmann algebra.

$$\begin{cases} \{\theta_i, \bar\theta_j\} = 0 \\ \{\theta_i, \theta_j\} = 0 \\ \{\bar\theta_i, \bar\theta_j\} = 0 \end{cases}$$

$$\tfrac{1}{2}\left(\partial_{\theta_i}\theta_j + \partial_j \theta_i\right) = \delta_{ij}$$

$$\int d\theta_k \, F(\theta, \bar\theta) = \partial_{\theta_k} F(\theta, \bar\theta)$$

Given a generic matrix M

$$\int d\theta_1 \ldots d\theta_n \, d\bar{\theta}_1 \ldots d\bar{\theta}_n \, e^{\bar{\theta}_i M^{ij} \theta_j} =$$

$$= \int \prod_i d\theta_i \prod_j d\bar{\theta}_j \frac{(\bar{\theta} M \theta)^n}{n!} = \det M$$

On the other hand:

$$\int \prod_i dz_i d\bar{z}_i \, e^{-\bar{z}_i M z_j} = \frac{1}{\det M}.$$

$$\Rightarrow \int \prod_i d\theta_i d\bar{\theta}_j dz_i d\bar{z}_j \, e^{-\bar{z} M z - \bar{\theta} M \theta} = 1$$

$\Rightarrow$ Functional integrals

$$\int \prod_x d\rho_\alpha(x) \, d\Lambda^\alpha(x) \, d\bar{c}_\alpha \, dc^\alpha \, e^{\int d^4x \left[\mathrm{tr}(\rho\, \partial_\mu \nabla^\mu \Lambda + \bar{c}\, \partial_\mu \nabla^\mu c \right]} = 1$$

we assume that

$\partial_\mu \nabla^\mu$ be negative def.

(However this not always possible $\Rightarrow$ Gribov copies)

Slavnov-Taylor id.

$$s\,(d\mu_c) = 0$$ (it depends on the regularization procedure)

$$\int d\mu_c\, e^{-s\int d^4x\, \mathrm{Tr}\left[\bar{c}\,(\partial^\mu A_\mu + \alpha\rho)\right]}\, s\,\Xi = 0.$$

Indeed:

$$\int d\mu_c\, s\left[e^{-s\int d^4x\, \mathrm{Tr}\left[\bar{c}\,(\partial^\mu A_\mu + \alpha\rho)\right]}\, \Xi \right] = 0$$

$$\Downarrow$$

$$Z[j,\xi] = \int d\mu_c\, e^{-s\int d^4x\, \mathrm{Tr}\left[\bar{c}\,(\partial^\mu A_\mu + \alpha\rho)\right] + \xi^i s\,\Xi_i}$$

$$\Rightarrow \boxed{\partial_{\xi_i} Z[j,\xi] = 0}$$

In addition: if $s\,\mathcal{O} = 0$. $\quad \eta_k \mathcal{O}^k$

$$\Rightarrow \partial_{\eta_k}\partial_{\xi_i} Z[j,\xi,\eta] = 0.$$

$$\partial_\xi \int d\mu_c\, e^{-s\int d^4x\,[\,\dots\,]} = \int d\mu_c\, s\int d^4x\, \mathrm{tr}[\bar{c}\rho]\, e^{-s[\;]} = 0.$$

Let us recall some facts

1) X_α : diff. operator $= R_\alpha^I(\phi)\frac{\delta}{\delta\phi^I} =$

$$= \partial_\mu \frac{\delta}{\delta A_\mu^\alpha} - P_\alpha{}^\beta{}_\mu \frac{\delta}{\delta A_\mu^\beta} - P_\alpha^i \frac{\delta}{\delta \varphi^i}$$

$$[X_\alpha, X_\beta] = C_{\alpha\beta}^\gamma\, \delta^4(x-y)\, X_\gamma$$

2) c^α : ghosts fields.

$\boxed{\text{FADDEEV - POPOV}}$

3)

$$\delta_{inv}(\eta-\bar\eta) = \underbrace{\delta(\eta-\bar\eta)\det(X_I\eta^I)}_{\text{invariant Delta Dirac function.}} =$$

$$= \int \prod_\alpha dp_\alpha\, e^{i p_\alpha(\eta-\bar\eta)^\alpha} \int \prod_\alpha dc^\alpha d\bar c_\alpha\, e^{-\bar c_\alpha (X_I \eta^I) c}$$

$$= \int \prod dp_\alpha dc^\alpha d\bar c\; e^{i(p_\alpha(\eta-\bar\eta)^\alpha - \bar c (X_I\eta^I) c}$$

$$\boxed{d\mu_c = d\mu \prod dp\, dc\, d\bar c}$$

Simple example: $p, \bar c, c, \eta$ $\quad s\eta = c(X_I\eta^I)$ $\quad s\bar c = p$ $\quad \Rightarrow$

$sc = 0$ $\quad sp = 0$

$$\Rightarrow \int dc_I d\bar{c}_I dp_I d\eta_I \, e^{\delta[i\bar{c}(\eta-\bar{\eta})]} F(\eta)$$

$$= \int dc\, d\bar{c}\, dp\, d\eta_I \, e^{ip(\eta-\bar{\eta}) - i\bar{c}X_I\eta^I c} F(\eta)$$

$$= \int dp_I d\eta_I \, e^{ip(\eta-\bar{\eta})} \int dc\, d\bar{c} \, e^{-i\bar{c}X_I\eta^I c} F(\eta)$$

$$= \int d\eta_I \, \delta(\eta-\bar{\eta}) \det(X_I\eta^I)$$

BRST for SM

Spectrum

W_μ^i $\quad i = 1, 2, 3$ $\quad SU(2)$

B_μ $\quad U(1)$

G_μ^a $\quad a = 1 \dots 8$ $\quad SU(3)$

$\begin{cases} L_L, e_R & L_L = \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} \\ U_L, D_R & \end{cases}$

$(U_R \;\; D_L)$.

Higgs doublet.

What is the coupling with B_μ?

Normally:

$$B_\mu j^\mu_Y \qquad j^\mu_Y = \text{hyper charge}$$

at the quantum level (only by BRST)

$$\downarrow$$

$$B_\mu \left(j^\mu_Y + \alpha_1 j^\mu_B + \alpha_2 j^\mu_e + \alpha_3 j^\mu_\tau + \alpha_4 j^\mu_\mu \right)$$

We need a further condition.

let us consider two elements of the action:

$$\int F \wedge * F \qquad , \int F \wedge F$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\int d^4x \, F^{\alpha}_{\mu\nu} F^{\mu\nu}_{\alpha} \qquad\qquad \int d^4x \, F^{\alpha}_{\mu\nu} F_{\rho\sigma\alpha} \epsilon^{\mu\nu\rho\sigma}$$

$$s(F \wedge * F) = 0 \qquad \text{but} \qquad s(F \wedge F) = 0.$$

but

$$F \wedge F = d(\underbrace{A dA + \tfrac{2}{3} A^3}_{\text{Chern - Simons}})$$

then we have

$$s\left(AdA + \tfrac{2}{3} A^3\right) = d\omega_2^1$$

$$s\omega_2^1 + d\omega_1^2 = 0$$

$$s\omega_1^2 + d\omega_0^3 = 0$$

$$s\omega_0^3 = 0$$

$$\boxed{\omega_0^3 = C^{\alpha\beta\gamma} \frac{C_\alpha C_\beta C_\gamma}{3!}}$$

In the same way we can study the anomalies.

Bottom line

$$\boxed{H^{(0)}(s|d) \Rightarrow \text{local observables}}$$

All gauge invariant observables are element of the $\rightarrow$ BRST cohomology.

$$H_{eff} = \sum_i c_i O_i$$

$\mathcal{A} = A + C$ then we have:

$$\delta' \mathcal{A} = (\partial + d)(A + C) = \partial A + dA + \partial C + dC$$

but we have to recall that we can also introduce: $\mathcal{A} \wedge \mathcal{A}$ (as a wedge product of forms)

$$\delta' \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \partial A + dA + \partial C + dC + (A + C) \wedge (A + C) =$$
$$= (dA + A \wedge A) + (\partial A + dC + C \wedge A + A \wedge C) + (\partial C + C \wedge C) = \mathcal{F}$$

$\boxed{\text{total degree is } 2}$

if we infer that:

$$\mathcal{F} = F_0^{(2)} + \psi_1^1 + \phi_2^0$$

and we set $\psi_1^1 = \phi_2^0 = 0 \Rightarrow$ $\boxed{\text{TOPOLOGICAL FIELD THEORIES if } \psi_1^1, \phi_2^0 \neq 0.}$

$$\begin{cases} dA + A \wedge A = F \\ \partial A = - dC - C \wedge A - A \wedge C \\ \partial C = - C \wedge C. \end{cases}$$

which is equivalent to:

$$\begin{cases} F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - C^\alpha_{\beta\gamma} A^\beta_\mu A^\gamma_\nu \\ \partial A^\alpha_\mu = (\nabla_\mu C)^\alpha = \partial_\mu C^\alpha - C^\alpha_{\beta\gamma} C^\beta A^\gamma_\mu \\ \partial C^\alpha = - \frac{1}{2} C^\alpha_{\beta\gamma} C^\beta C^\gamma \end{cases}$$

why this contraction is useful:

$$A = \int d^4x \, \mathcal{L}(A, \varphi, c).$$

$$sA = 0 \Rightarrow \int d^4x \, s\mathcal{L}(A, \varphi, c) = 0.$$

$$\Rightarrow s\mathcal{L}^0_4(A, \varphi, c) = d\Lambda^1_3 \quad \text{then we apply } s.$$

$$0 = s^2 \mathcal{L}^0_4 = s\,d\Lambda^1_3 = -d\,s\Lambda^1_3 \Rightarrow$$

$$s\Lambda^1_3 + d\Lambda^2_2 = 0 \quad \text{(by Poincaré lemma).}$$

$$s\Lambda^2_2 + d\Lambda^3_1 = 0$$

$$s\Lambda^3_1 + d\Lambda^4_0 = 0$$

$$\boxed{s\Lambda^4_0 = 0}$$

this is the highest form that I can construct (analogous to a top form).

and then I can compute $H^4_0(s)$.

In particular to compute $\boxed{H(S) \simeq H(s \,|\, H(d))}$

$\boxed{\text{How to translate everything in the form language:}}$

$A^\alpha_\mu \longrightarrow A^\alpha_\mu dx^\mu = A^\alpha$ (1-form with ghost number 0).

c^α (0-form with ghost number +1).

φ^i (0-form w/ ghost # 0)

BRST cohomology

$$(A_\mu^\alpha, \varphi^i, c^\alpha, \bar{c}_\alpha, \rho_\alpha)$$

$$\begin{cases} s A_\mu^\alpha = (\nabla_\mu c)^\alpha \\ s c^\alpha = \frac{1}{2} C^\alpha{}_{\beta\gamma} c^\beta c^\gamma \\ s\varphi^i = c^\alpha t_\alpha (\varphi + v)^i \end{cases} \qquad \begin{aligned} s\bar{c}_\alpha &= \rho_\alpha \\ s\rho_\alpha &= 0. \end{aligned}$$

1) Contractible pairs $\qquad H(s_{red})$

$$\boxed{s\bar{c}_\alpha = \rho_\alpha \quad s\rho_\alpha = 0} \qquad s_{red} = \rho_\alpha \frac{\delta}{\delta \bar{c}_\alpha}$$

Let us introduce the following operators

$$N_{\bar{c}} = \bar{c}_\alpha \frac{\delta}{\delta \bar{c}_\alpha} \qquad N_\rho = \rho_\alpha \frac{\delta}{\delta \rho_\alpha}$$

they counts the number of $\bar{c}, \rho$'s in a function $F(\bar{c}, \rho)$.

$$\begin{cases} N_{\bar{c}} \left(\bar{c}_{\alpha_1} \cdots \bar{c}_{\alpha_n}\right) = n \left(\bar{c}_{\alpha_1} \cdots \bar{c}_{\alpha_n}\right). \\ N_\varphi \left(\rho_{\alpha_1} \cdots \rho_{\alpha_n}\right) = n \left(\rho_{\alpha_1} \cdots \rho_{\alpha_n}\right). \end{cases}$$

Now:

$$\left[s_{red}, \bar{c}_\alpha \frac{\delta}{\delta \rho_\alpha}\right\} = \rho_\alpha \frac{\delta}{\delta \rho_\alpha} + \bar{c}_\alpha \frac{\delta}{\delta \bar{c}_\alpha} = $$

$$= N_\rho + N_{\bar{c}}$$

Suppose that

$$s_{red} F(\rho, \bar{c}) = 0.$$

$$\Rightarrow \left[\bar{c}_\alpha \frac{\delta}{\delta \rho_\alpha} s + s\left(\bar{c}_\alpha \frac{\delta}{\delta \rho_\alpha}\right)\right] F(\rho, \bar{c}) = (N_{\bar{c}} + N_\rho) F(\rho, \bar{c})$$

$$\Rightarrow \quad s\left[\bar{c}_\alpha \frac{\delta}{\delta \rho_\alpha} F\right] = (N_{\bar{c}} + N_\rho) F(\rho, \bar{c}) =$$

$$F = F_0 + \rho F_1 + \bar{c} F_1' + \dots$$

$$= (n_c + n_\rho)(F).$$

if $n_c, n_\rho \neq 0 \Rightarrow$ $$F = \underbrace{\frac{1}{\mu_c + \mu_\rho} s\left[\bar{c}_\alpha \frac{\delta}{\delta \rho_\alpha} F\right]}_{\text{namely this is BRST trivial.}}$$

if $\mu_c = \mu_\rho = 0$ $\Rightarrow$ then we cannot conclude the same result!

$$H(s_{red}) = F_0 \text{ (the rest of the fields)}.$$

$$H(s) = H(\hat{s} + s_{red}) \approx H(\hat{s}, \text{ space with } \bar{c}, \rho).$$

$$\hat{s} \to s \qquad \boxed{H(s)}$$

2) <u>Ghost number</u>

We assign the following ghost number:

$$\begin{cases} \#(c_\alpha) = +1 \\ \#(A_\mu^\alpha, \rho_\alpha, \varphi^i) = 0 \\ \#(\bar{c}^\alpha) = -1 \end{cases} \qquad \#(s) = +1$$

If we denote by Ω^{n} the space of local function of $A_{\mu}^{\alpha}, c^{\alpha}, \varphi^{i}$ with ghost number n.

$$s : \Omega^{n} \to \Omega^{n+1}$$

this is similar to usual differential d.

$$s \longleftrightarrow d$$

$$\# \text{number} \longleftrightarrow \text{form degree.}$$

$$\{c^{\alpha}, c^{\beta}\} = 0 \longleftrightarrow dx^{\mu} \wedge dx^{\nu} + dx^{\nu} \wedge dx^{\mu} = 0.$$

$$H(s) = \longleftrightarrow H(d) \text{ (de Rham).}$$

$$= \bigoplus_{n=0}^{N} H^{(n)}(s) \qquad H(d) = \bigoplus_{p=0}^{N} H^{(p)}(d).$$

N = dim of M.

N = dim of gauge group.

$$c^{\alpha_1} \cdots c^{\alpha_n} = 0$$

if $n >$ dim of the group.

3) local cohomology

but we can also define a new differential

$$\tilde{s} = s + d$$

by requiring that $\tilde{s}^{2} = 0 \Rightarrow$

$$s^{2} = 0$$
$$d^{2} = 0$$
$$\{s, d\} = 0.$$

then the space has a double filtration

$$\Omega = \bigoplus_{u,p} \Omega^{p}_{u}$$

p → ghost number.

u → form degree

$$\begin{cases} s : \Omega^{p}_{u} \longrightarrow \Omega^{p+1}_{u} \\ d : \Omega^{p}_{u} \longrightarrow \Omega^{p}_{u+1} \end{cases}$$

$$\tilde{s} : \Omega \to \Omega.$$

Now we can study a generalized form of cohomology $H(\tilde{s}, \Omega)$.

$$H = \{\tilde{s}\omega = 0\} / \{\omega = \tilde{s}\eta\}$$

this means: $(s+d)\omega = 0$. $\omega = \omega^{p}_{0} + \omega^{p-1}_{1} + \cdots + \omega^{0}_{p}$

total degree $= p$.

$$(s+d)(\omega^{p}_{0} + \omega^{p-1}_{1} + \cdots + \omega^{0}_{p}) = 0.$$

$$s\omega^{p}_{0} + d\omega^{p}_{0} + s\omega^{p-1}_{1} + d\omega^{p-1}_{1} + \cdots + s\omega^{0}_{p} + d\omega^{0}_{p} = 0$$

$$\Rightarrow \begin{cases} s\omega^{p}_{0} = 0 \\ s\omega^{p-1}_{1} + d\omega^{p}_{0} = 0 \\ s\omega^{p-2}_{2} + d\omega^{p-1}_{1} = 0 \\ \vdots \\ d\omega^{0}_{p} = 0 \end{cases}$$

Descent equations

STI ids:

$$s S' = \int d^4x \left[(\nabla_\mu c)^\alpha \frac{\delta S}{\delta A_\mu^\alpha} + \frac{1}{2} C^\alpha_{\beta\gamma} c^\beta c^\gamma \frac{\delta S}{\delta c^\alpha} + c^\alpha t_\alpha(\varphi + v) \frac{\delta S}{\delta \varphi} + \bar{\rho}_\alpha \frac{\delta S}{\delta \bar{c}^\alpha} \right] = 0$$

Composite operators

$$\Rightarrow \int d^4x \left[\frac{\delta S'}{\delta A^{\mu *}_\alpha} \frac{\delta S'}{\delta A^\alpha_\mu} + \frac{\delta S}{\delta c^*_\alpha} \frac{\delta S}{\delta c^\alpha} + \frac{\delta S}{\delta \varphi^*_i} \frac{\delta S}{\delta \varphi^i} + \rho_\alpha \frac{\delta S'}{\delta \bar{c}^\alpha} \right] = 0$$

Non-linear equation $\Rightarrow$ we extend this system to at the quantum level:

$$\begin{cases} \Rightarrow \dfrac{\delta \Gamma}{\delta \rho_\alpha} = \partial^\mu A^\alpha_\mu + \xi \rho^\alpha \\[2ex] \displaystyle\int d^4x \left[\frac{\delta \Gamma}{\delta A^{*\alpha}_\mu} \frac{\delta \Gamma}{\delta A^\mu_\alpha} + \frac{\delta \Gamma}{\delta c^*_\alpha} \frac{\delta \Gamma}{\delta c^\alpha} + \frac{\delta \Gamma}{\delta \varphi^*_i} \frac{\delta \Gamma}{\delta \varphi^i} + \rho^\alpha \frac{\delta \Gamma}{\delta \bar{c}_\alpha} \right] = 0 \end{cases}$$

$$\mathcal{S}(\Gamma) = 0$$

Contributing from the com. relation:

$$\mathcal{S}_\Gamma \left(\frac{\delta \Gamma}{\delta \rho_\alpha} - (\partial A + \xi \rho) \right) - \frac{\delta}{\delta \rho_\alpha} \mathcal{S}(\Gamma) =$$

$$= \frac{\delta \Gamma}{\delta \bar{c}_\alpha} - \partial^\mu \frac{\delta \Gamma}{\delta A^{*}_{\mu\alpha}} = 0 \quad \Leftarrow \text{Ghost equation.}$$

Let us simplify the computation.

$$\Gamma \to \hat{\Gamma} = \Gamma - \int d^4x \left(\rho_\alpha \partial^\mu A_\mu^\alpha + \frac{\xi}{2} \rho^2 \right)$$

$$A_\mu^{*\alpha} \to A_\mu^{*\alpha} - \partial_\mu \bar{c}^\alpha$$

$$\Rightarrow \begin{cases} \dfrac{\delta \hat{\Gamma}}{\delta \rho_\alpha} = 0 \qquad \dfrac{\delta \hat{\Gamma}}{\delta \bar{c}^\alpha} = 0 \\[2ex] \displaystyle\int d^4x \left[\frac{\delta \hat{\Gamma}}{\delta \hat{A}_\mu^{*\alpha}} \frac{\delta \hat{\Gamma}}{\delta A^\mu_\alpha} + \frac{\delta \hat{\Gamma}}{\delta c^*_\alpha} \frac{\delta \hat{\Gamma}}{\delta c^\alpha} + \frac{\delta \hat{\Gamma}}{\delta \varphi^*_i} \frac{\delta \hat{\Gamma}}{\delta \varphi^i} \right] = 0 \end{cases}$$

This equation has a nice form and it can be put in the following reps:

$$(A,B) = \frac{\delta A}{\delta \Phi^*_I} \frac{\delta B}{\delta \Phi^I} + \frac{\delta A}{\delta \Phi^I} \frac{\delta B}{\delta \Phi^*_I} \quad \Rightarrow \boxed{(\Gamma,\Gamma) = 0}$$

MASTER EQUATION.

$$\Phi^I = (A_\mu^\alpha, c^\alpha, \varphi^i)$$

$$\Phi^*_I = (A^{*\mu}_\alpha, c^*_\alpha, \varphi^*_i)$$

which has some interesting properties.

$$(A,(B,C)) + (B,(C,A)) + (C,(A,B)) = 0.$$

and in particular: $\boxed{A = B = C = \Gamma}$

$$(\Gamma,(\Gamma,\Gamma)) = 0 \quad , \quad \boxed{S_\Gamma' = (\Gamma, \cdot)}$$

Note on renormalization

1) INGREDIENTS

- $\underbrace{\text{Regularization} \oplus \text{Renormalization}}_{\text{Q.A.P.}}$ $\quad \Gamma \to \underbrace{\Gamma^{(1)}}_{\text{finite}}$

Given a Ward id. operator $W_{(x)}$ such that the action is invariant

$$\boxed{W^I_{(x)} S = 0}$$

$S = S$ (finite number of parameters).

Then at 1-loop we have:

$$\boxed{W^I_{(x)} \Gamma^{(1)} = \underbrace{\Delta^{(1),I}_{(x)}}_{\substack{\text{This is a}\\ \text{local operator} \Rightarrow\\ \text{expandible in terms of}\\ \text{local fields.}}} + O(\hbar \Delta^{(1)})}$$

1) Quantum numbers of $\Delta^{(1),I}_{(x)}$ = quantum number of $W_{(x)}\Gamma^{(1)}$.

2) Symmetry properties of $\Delta^{(1),I}_{(x)}$

this means that we have to renormalize them together the rest of the theory: the best way to do it is to add the following sources to the action:

$$S'_0 + \int d^4x\, s\left[\bar{c}_\alpha \left(\partial^\mu A^\alpha_\mu + \xi\rho^\alpha\right)\right] + \int s\left[A^{*\mu}_\alpha A^\alpha_\mu + c^*_\alpha c^\alpha + \varphi^*_i \varphi^i\right]$$

$$= S_0 + S_{gf+FP} + \int d^4x \left[A^{*\mu}_\alpha (\nabla_\mu c)^\alpha + \frac{c^*_\alpha}{2} C^\alpha_{\beta\gamma} c^\beta c^\gamma + (\dots)\right]$$

ANTIFIELDS

		# ghost	# antifield	statistics
$A^{*\mu}_\alpha$	=	-1	$+1$	$-$
c^*_α	=	-2	$+1$	$+$
φ^*_i	=	-1	$+1$	$-$

$$Z[j,\xi] \longrightarrow Z[j,\xi,A^*,c^*] \longrightarrow$$

$$\rightarrow \boxed{\Gamma[A,c,\bar{c},\rho,\varphi,A^*,\varphi^*,c^*]} =$$

$$= S' + \underbrace{O(\hbar)}_{\text{radiative corrections}}.$$

$$\frac{\delta S}{\delta \rho_\alpha} = \underbrace{\partial^\mu A^\alpha_\mu + \xi \rho^\alpha}_{\text{this is linear in quantum fields}}$$

GAUGE FIXING

We have said that

$$\left[W^I_{(x)}, W^J_{(y)}\right] = C^{IJ}_k \, \delta^4(x-y) \, W^k_{(x)}.$$

$\Rightarrow$ Acting on $\Gamma^{(1)}$.

$$\boxed{W^I_{(x)} \Delta^J_{(y)} - W^J_{(y)} \Delta^I_{(x)} = C^{IJ}_k \, \delta^4(x-y) \, \Delta^k_{(y)}}$$

Wess-Zumino consistency conditions

Using BRST.

$$s = \int d^4x \left(c_I W^I_{(x)} + C^{IJ}_k c_I c_J \frac{\delta}{\delta c_k} \right).$$

$$\Delta = \int d^4x \; c_I \Delta^I$$

then we have: $[W, W] = \cdots \Rightarrow \boxed{s^2 = 0}$

$W \Gamma = \Delta \Rightarrow \boxed{s \Gamma = \Delta}$

consistency condition

$$\boxed{s \Delta^{(1)} = 0}$$

then we compute the BRST cohomology:

$$H^{(1)}(s \mid \# \text{q.numbers}).$$

$$\Rightarrow \Delta^{(1)} = \mathcal{A}^{(1)} + s\,\Xi^{(1)}$$

Anomaly

Counter terms

and then we have:

$$s\,\Gamma^{(1)} = \mathcal{A}^{(1)} + s\,\Xi^{(1)} + O(\hbar\Delta^{(1)}) \Rightarrow$$

$$s(\underbrace{\Gamma^{(1)} - \Xi^{(1)}}) = \boxed{\mathcal{A}^{(1)}}$$ if this is not absent we cannot conclude the Ren. process.

$\Pi^{(1)}$ = renormalized functional.

Computation for $\mathcal{A}^{(1)}$

q.#: ghost number + 1
dimension 5
integrated. $A = \int d^4x\, \omega_4^1$

$$s\mathcal{A}^{(1)} = 0 \Rightarrow s\omega_5^0 + d\omega_4^1 = 0$$

$\vdots$

$$s\,\omega_0^5 = 0$$

$$\omega_0^5 = d^{axy} f_x^{\ rs} f_y^{\ tu}\, C_a C_r C_s C_t C_u / 5!$$

and

$$\omega_4^1 = d^{abc} \quad C_a\, dA_b \wedge dA_c + F^{abcd}\, C_a\, A_b \wedge A_c \wedge A_d$$

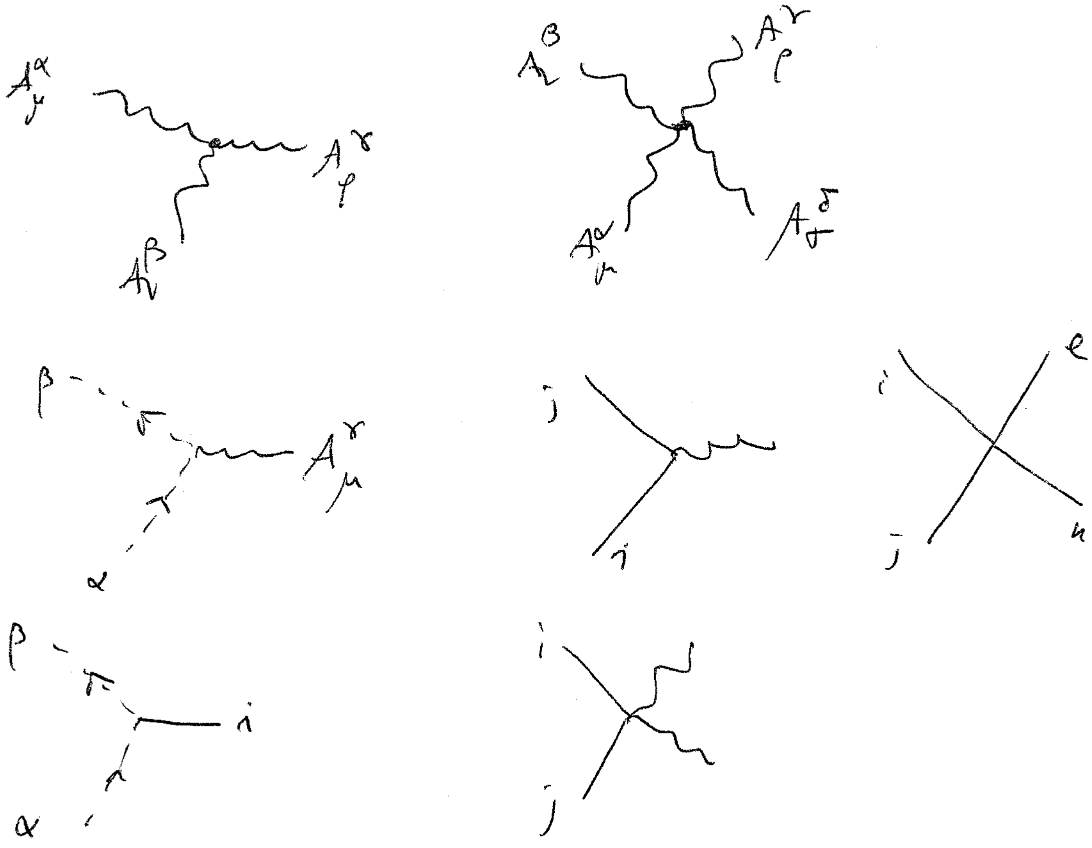
$$\omega_5^0 = d^{abc} \quad A_a \wedge dA_b \wedge dA_c + \ldots$$

Introduction of Antifields

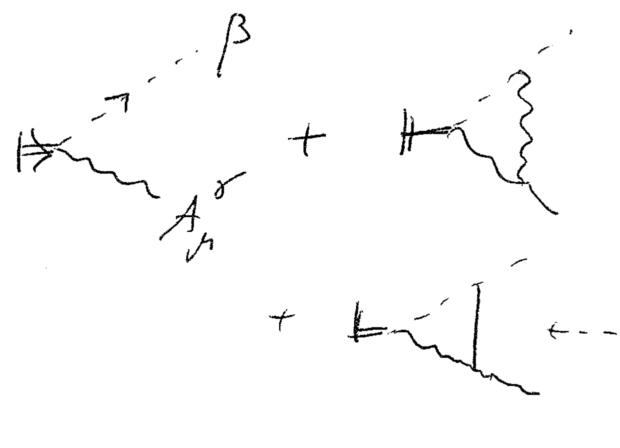
$$\left\{\begin{aligned} sA^a_\mu &= (\nabla_\mu C)^\alpha \\ sC^\alpha &= \tfrac{1}{2} C^\alpha_{\beta\gamma} C^\beta C^\gamma \\ s\phi_i &= C^\alpha\, t_\alpha(\phi+v)_i \end{aligned}\right\}$$ These are composite operators.

$$s\bar{C}_\alpha = \rho_\alpha \qquad s\rho_a = 0$$

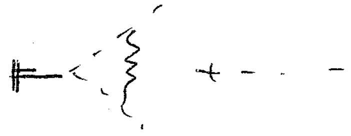
Let us recall the vertices (Feynman) of YM:



$$(\nabla_\mu C)^\alpha = \partial_\mu C^\alpha - \underbrace{C^\alpha_{\beta\gamma} C^\beta A^\gamma_\mu}_{\text{composite operator}}$$



$$C^\alpha_{\beta\gamma}\frac{C^\beta C^\gamma}{2}$$



Correlation functions and gauge independence

$$\langle \Omega, \prod_{i=1}^{N} \mathcal{O}^i(x_i)\, \Omega \rangle = \prod_{i=1}^{N} \frac{\delta^N}{\delta \eta_i} Z[j,\eta]\Big|_{j=\eta=0}$$

this is an operator on the Fock space acting on the vacuum $|\Omega\rangle$.

At the level of Fock space operators:

$$s\,\mathcal{O}^i(x_i) \equiv [\,Q, \mathcal{O}^i(x_i)]$$

BRST charge

Since the action S is invariant under the BRST symmetry we can construct the Nöether current.

$$sS = 0 \Rightarrow \int d^4x\, s\mathcal{L} = 0 \Rightarrow$$

$$s\mathcal{L} = d(*j)$$

j^{BRST} = Nöether current $\rightarrow$

$$Q^{BRST} = \int_{x_0 = t} d^3x\, j_0^{BRST}, \quad Q^2 = \{Q, Q\} = 0$$

$$\Rightarrow \quad \langle \Omega, \prod_{i=1}^{N} \mathcal{O}^i(x_i) \left[Q^{BRST}, \Lambda \right] \Omega \rangle =$$

$$= \langle \Omega, \mathcal{O}^1(x_1) \cdots \mathcal{O}^N(x_N) \left(Q^{BRST} \Lambda - \Lambda \underbrace{\cancel{Q^{BRST}}}_{0 \text{ (Invariant vacuum)}} \right) \Omega \rangle =$$

$$= \langle \Omega, \mathcal{O}^1(x_1) \cdots \mathcal{O}^N(x_N) \, Q^{BRST} \Lambda | \Omega \rangle =$$

(using $[Q, \mathcal{O}^i] = 0$)

$$= \langle \Omega, \mathcal{O}^1(x_1) \cdots \mathcal{O}^{N-1}(x_{N-1}) \, Q^{BRST} \mathcal{O}^N(x_N) \Lambda \Omega \rangle$$

$$\vdots$$

$$= \langle \Omega, Q^{BRST} \prod_{i=1}^{N} \mathcal{O}^i(x_i) \Lambda \Omega \rangle = 0$$

Any insertion of BRST trivial operator in Green's function between gauge invariant op's $[Q^{BRST}, \mathcal{O}^i] = 0 \;\; \forall i$ <u>vanishes</u>.

$$\partial_\xi \langle \Omega, \prod_{i=1}^{N} \mathcal{O}^i(x_i) \Omega \rangle =$$

$$= \langle \Omega, \prod_{i=1}^{N} \mathcal{O}^i(x_i) \left[Q^{BRST}, \partial_\xi \Psi \right] \Omega \rangle = 0$$

Note that we have used:

$$\partial_\xi \mathcal{O}^i(x_i) = 0$$

$$\partial_\xi | \Omega \rangle = \left[Q^{BRST}, \partial_\xi \Psi \right] | \Omega \rangle =$$

$$= Q^{BRST} \left(\partial_\xi \Psi | \Omega \rangle \right)$$

$Q^{BRST}|\Omega\rangle = 0$ and $(Q^{BRST})^\dagger = Q^{BRST}$

$\Rightarrow \langle\Omega|Q^{BRST} = 0$.

<u>Hamiltonian form</u> (Quantum <u>mechanics</u>)
(1+0 dimension)

Given a set of 1st class constraints:

$$\{\Phi_I, \Phi_J\} = C_{IJ}{}^k \Phi_k$$

$\Phi_I(\varphi) \approx 0$, φ = fields of the theory

$\{\cdot,\cdot\}$ = Poisson brackets

$$\boxed{Q^{BRST} = \left[c^I \Phi_I + C_{IJ}{}^k \frac{c^I c^J}{2} b_k\right]}$$

$$\{c^I, b_k\}_+ = \delta^I_k$$ (New Poisson bracket)

$$Q^2 = \{c^I\Phi_I + \tfrac{1}{2}C_{IJ}{}^k c^I c^J b_k,$$
$$c^L\Phi_L + \tfrac{1}{2}C_{LM}{}^N c^L c^M b_N\} =$$

$$= c^I c^L \{\Phi_I, \Phi_L\} + \frac{\Phi_I}{2}\{c^I, C_{LM}{}^N c^L c^M b_N\} + \cdots = 0$$

$$\tfrac{1}{4} C_{IJ}{}^k C_{LM}{}^N \{c^I c^J b_k, c^L c^M b_N\} = 0$$

$$\begin{cases} Q\varphi = c^I\{\Phi_I,\varphi\} \\ Q c^I = C^I_{Jk}\,\dfrac{c^J c^k}{2} \\ Q\Phi_I = c^J C_{JI}{}^k\,\Phi_k \\ Q b_k = \Phi_k + C_{kI}{}^J c^I b_J \end{cases}$$

$$Q^2 c^I = \frac{C^I_{Jk}}{2}\left(C^{J}_{LM}\, c^L c^M c^k - c^J C^k_{LM}\, c^L c^M\right) =$$

$$= C^I_{J[k} C^J_{LM]}\, c^L c^M c^k = 0 \quad \text{by } \underline{\text{JACOBI}}$$

$$Q^2\Phi_I = C^J_{LM}\frac{c^L c^M}{2} C_{JI}{}^k\,\Phi_k +$$

$$+\, c^J C_{JI}{}^k\, c^R C_{Rk}{}^S\,\Phi_S = 0 \quad \text{by JACOBI}$$

$$Q^2 b_k = \cancel{c^J C_{Jk}{}^L}\,\Phi_L + C_{kI}{}^J\,\frac{1}{2}C^I_{LM}\, c^L c^M b_J +$$

$$-\,C_{kI}{}^J c^I\underbrace{\left(\cancel{\Phi_J} + C_{JR}{}^S c^R b_S\right)}_{} = 0$$

by antisym $c^I c^k$
and using JACOBI ID

1) ABJ Anomaly

1) Solution for $s\omega_4^1 + d\omega_3^2 = 0$.

2) Antifield formalism (BV)

1) Antibracket
2) Master equation and solutions
3) YM
4) 2-form. (Reducibility)
5) Open algebras.

Anomaly for gauge theories

$$(\Gamma,\Gamma) = \Delta^{(1)} + O(\hbar\Delta^{(1)})$$

Expanding this at one-loop $\quad \Gamma = S + \Gamma^{(1)} + O(\hbar\Gamma^{(1)})$

$$\left(S+\Gamma^{(1)}, S+\Gamma^{(1)}\right) + O(\hbar\Omega^{(1)}) =$$

$$= (S,S) + 2\,(S,\Gamma^{(1)}) + O(\hbar\hat{\Omega}^{(1)})$$

Now at tree level: $(S,S)=0$.

$\Rightarrow$ at one-loop: $\boxed{2\,(S,\Gamma^{(1)}) = \Delta^{(1)}}$

$$\big((S,A),S\big) + \underbrace{\big((A,S),S\big)} + \big((S,S),A\big) = 0 \qquad \text{JACOBI}$$

$$2\,\big((S,A),S\big) + \big((S,S),A\big) = 0$$

Now if $(S,S)=0 \quad \Rightarrow \big((S,A),S\big)=0$.

$$s_S A = (S,A) \qquad s_S^2 A = \big(S,(S,A)\big) = 0$$

so this implies $\boxed{s_S^2 = 0}$

then acting on ⊛ $\quad 2\,\big(S,(S,\Gamma^{(1)})\big) = \boxed{(S,\Delta^{(1)}) = 0}$

$$\Rightarrow \quad s_{S'} \Delta^{(1)} = 0 \quad \text{if } A^{(1)} = s_S\, \Xi^{(1)} + \mathcal{A}^{(1)}$$

$$\Rightarrow \quad \mathbf{\Gamma}^{(1)} = \Gamma^{(1)} - \Xi^{(1)} \qquad \boxed{s\, \mathbf{\Gamma}^{(1)} = \mathcal{A}^{(1)}}$$

Computation of the anomaly $\mathcal{A}^{(1)}$

Quantum numbers (Using $[A] = +1$, $[c] = 0$, $[A^*] = 3$, $[c^*] = 4$)

1) $[\Gamma^{(1)}] = 4$, $[s] = 0$ $\Rightarrow$ $[\mathcal{A}^{(1)}] = 4$

2) ghost number:

$\#[\Gamma^{(1)}] = 0$, $\#[s] = +1$ $\Rightarrow$ $\#[\mathcal{A}^{(1)}] = 1$

3) By using the QAP:

$$\mathcal{A}^{(1)} = \underbrace{\int d^4x\, a(x)}_{\text{dimension } 5} = \int \omega^{1}_{4}$$

ghost number (1), dimension (4)

4) Positive antifields for $\Gamma^{(1)}$ $\Rightarrow$

$$\omega^1_4 = \hat{\omega}^1_4 + \Big[\overset{3}{A^{*\alpha}_\mu}\, \underset{+2}{\hat{\omega}^\mu_\alpha} + \overset{4}{c^{*\alpha}}\, \underset{+3}{\hat{\omega}_\alpha} \Big]$$

after some computations we get

$$\omega_4^1 = \hat{\omega}_4^1[A,c] + s_S[M]$$

Now we have to recall that

$$s_S \int \hat{\omega}_4^1[A,c] = \int \underbrace{s}_{\text{the original BRST}} \hat{\omega}_4^1[A,c] = 0$$

$$\Rightarrow \quad s\hat{\omega}_4^1 + d\omega_3^2 = 0$$

$$s\omega_3^2 + d\omega_2^3 = 0$$

$$s\omega_2^3 + d\omega_1^4 = 0$$

$$s\omega_1^4 + d\omega_0^5 = 0$$

$$s\omega_0^5 = 0 \qquad \Rightarrow H(s)$$

the solution is:

$$\omega_0^5 = \frac{1}{5!} d^{\alpha\beta\gamma} f_\beta^{\ \mu\nu} f_\gamma^{\ \tau\sigma} c_\alpha c_\mu c_\nu c_\tau c_\sigma =$$

$$= \frac{\mathrm{Tr}[c^5]}{5!}$$

$$\omega_4^1 = \frac{1}{3} c^\alpha d\left[d_{\alpha\beta\gamma} A^\beta \wedge dA^\gamma + \frac{1}{4} d_{\alpha\beta\gamma} f^\gamma_{\ \tau\sigma} A^\beta \wedge A^\tau \wedge A^\sigma\right]$$

which can be written as:

$$\omega_4^1 = \frac{\Gamma^{(1)}}{3} d^4x\, \epsilon^{\mu\nu\rho\sigma} \Big[d_{\alpha\beta\gamma}\, \partial_\mu A_\nu^\beta\, \partial_\rho A_\sigma^\gamma +$$

$$+ \frac{1}{4} d_{\alpha\beta\gamma} f^\gamma_{\ \tau\omega}\, \partial_\mu A_\nu^\beta A_\rho^\tau A_\sigma^\omega \Big]$$

this is known as Adler–Bardeen–Jackiw anomaly

$\Gamma^{(1)}$ = is one-loop contribution

\# is independent from the gauge parameter ξ

\# is independent from the masses of the particles

\# it exists at one-loop.
it is removed from one-loop,
the Adler-Bardeen theorem $\Rightarrow$
no anomaly at higher loops.

\# see Luca's lectures.

ANTIBRACKET

Field content:

	$\underbrace{A^\alpha_\mu,\ c^\alpha,\ \varphi_i}_{\Phi_A}$	$\underbrace{A^{*\mu}_\alpha,\ c^*_\alpha,\ \varphi^{*i}}_{\Phi^{*A}}$
ghost #	0	-1 -2 -1
anti-field #	0	$+1$ $+1$ $+1$
statistics	$+$ $-$ $+$	$-$ $+$ $-$

From the ST:

$$\boxed{\frac{x\overleftarrow{\partial}}{\partial\Phi^A_i}\frac{\overrightarrow{\partial}y}{\partial\Phi^{*A}} - \frac{x\overleftarrow{\partial}}{\partial\Phi^*_A}\frac{\overrightarrow{\partial}y}{\partial\phi^A}}$$

$$\boxed{(x,y) = \frac{\partial_r X}{\partial\Phi^A}\frac{\partial_e Y}{\partial\Phi^*_A} - \frac{\partial_r X}{\partial\Phi^*_A}\frac{\partial_e Y}{\partial\Phi^A}}$$

We remove the Integers $\Rightarrow$ index notation (Compact)

$$\epsilon_x = \pm 0 \quad \text{if bosonic}$$
$$= 1 \quad \text{if fermionic}$$

$$(y,x) = -(-1)^{(\epsilon_x+1)(\epsilon_y+1)}(x,y)$$

(in the case of Γ: $\epsilon_\Gamma = 0$

$$(\Gamma,\Gamma) = -(-1)^1(\Gamma,\Gamma) = (\Gamma,\Gamma) \neq 0.$$)

$$((x,y),z) + (-1)^{(\epsilon_x+1)(\epsilon_z+\epsilon_y)}((y,z),x) + (-1)^{(\epsilon_z+1)(\epsilon_x+\epsilon_y)}((z,x),y) = 0$$

$$gh[(x,y)] = gh(x) + gh(y) + 1$$

$$\epsilon_{(x,y)} = \epsilon_{(x)} + \epsilon_{(y)} + 1 \pmod 2$$

if $X = Y = B$ (boson) $\qquad (B,B) = 2 \dfrac{\partial_r B}{\partial \phi_A} \dfrac{\partial_e B}{\partial \phi^*_A}$.

$X = Y = F$ (fermion) $\qquad (F,F) = 0$.

(X,Y) is a graded derivation:

$$(X, YZ) = (X,Y)Z + (-1)^{\epsilon_Y \epsilon_Z} (X,Z) Y$$

$$(XY, Z) = X(Y,Z) + (-1)^{\epsilon_X \epsilon_Y} Y(X,Z).$$

Symplectic structure:

$$(X,Y) = \frac{\partial_r X}{\partial z_a} \omega^{ab} \frac{\partial_e X}{\partial z_b} \qquad \omega^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^+_B & 0 \end{pmatrix}.$$

$$z_a = \{\phi_A, \phi^{*A}\}$$

$\Rightarrow$ Analogous to Poisson bracket:

1) $$(\phi^A, \phi^*_B) = \sum_C \frac{\partial_r \phi^A}{\partial \phi_C} \frac{\partial_e \phi^*_B}{\partial \phi^{*C}} + \frac{\partial_r \phi^*_B}{\partial \phi_C} \frac{\partial_e \phi^A}{\partial \phi^*_C}$$

$$= \sum_C \delta^A_C \delta^C_B = \delta^A_B.$$

2) INFINITESIMAL CANON. TRS:

$$\begin{cases} \phi_A' = \phi_A + \epsilon\,(\phi_A, F) + o(\epsilon^2) \\ \phi'^{*A} = \phi^{*A} + \epsilon\,(\phi^*_A, F) + \cdots \end{cases}$$

Then we have:

$$(\phi_A', \phi'^{*B}) = (\phi_A + \epsilon\,(\phi_A, F), \phi^{*B} + \epsilon\,(\phi^{*B}, F)) + o(\epsilon^2)$$

$$= (\phi_A, \phi^{*B}) + \epsilon\,(\phi_A, (\phi^{*B}, F)) +$$

$$+ \epsilon\,((\phi_A, F), \phi^{*B}) + o(\epsilon^2) =$$

$$= \delta_A{}^B + \epsilon\,(F, (\phi^{*B}, \phi_A)) + o(\epsilon^2) =$$

$$= \delta_A^B + \epsilon\,(F, \underbrace{\delta^B_A}_{0}) + o(\epsilon^2) = \delta_A^B + o(\epsilon^2)$$

3) TRS. under the antibracket:

$$\delta G = \epsilon\,(G, F) + o(\epsilon^2).$$

4) Master eq: S = classical action.

$$\boxed{(S, S) = 0}$$

notice that

$$(S, \phi_A) = \frac{\partial_r S}{\delta \phi^*_C} \frac{\partial_\ell \phi_A}{\partial \phi^C} - \underbrace{\frac{\partial_r \phi_A}{\partial \phi^{*C}}}_{0} \frac{\partial_\ell S}{\partial \phi_C} =$$

$$= \frac{\partial_r S}{\partial \phi^*_A}$$

Solution of Master equation

$$S = S_0[\phi] + \Phi^*_A R^A + \Phi^*_A \Phi^*_B R^{AB} + \dots$$

antighost number: $0 \quad +1 \quad +2 \quad + \dots$

ghost # $R^A = -\#(\phi^*_A) \Rightarrow$

$R^{AB} = -(\phi^*_A) + -(\phi^*_B)$

$\vdots$

$$(S,S) = \frac{\delta S}{\delta \Phi^*_A} \frac{\delta S}{\delta \Phi^A} =$$

$$= \left(R^A + \Phi^*_B R^{AB} + \dots\right)\left(\frac{\delta S_0}{\delta \Phi_A} + \Phi^*_C \frac{\delta R^C}{\delta \Phi_A} + \Phi^*_C \Phi^*_B \frac{\delta R^{CB}}{\delta \Phi_A} + \dots\right)$$

$$= R^A \frac{\delta S_0}{\delta \Phi_A} + \left(\Phi^*_B R^{AB} \frac{\delta S_0}{\delta \Phi_A} + \Phi^*_C R^A \frac{\delta R^C}{\delta \Phi_A}\right) +$$

$$+ \Phi^*_B \Phi^*_C \left(R^{AB} \frac{\delta R^C}{\delta \Phi_A} + R^A \frac{\delta R^{CB}}{\delta \Phi_A} + R^{ABC} \frac{\delta S_0}{\delta \Phi_A}\right) + \dots$$

and: $(S, \phi^{*A}) = \dfrac{\partial_r S}{\partial \phi^A}$ = <u>equation of motion</u>

$(S, \phi_A) = \dfrac{\partial_r S}{\delta \phi_A^*}$ = <u>symmetries</u>.

then we have:

$$\delta \phi_A = (\phi_A, S) = \frac{\partial_r S}{\delta \phi_A^*}.$$

$$\delta \phi^{*A} = (\phi^{*A}, S) = \frac{\partial_r S}{\partial \phi^A}$$

$$\delta^2 \phi_A = ((\phi_A, S), S) = \frac{1}{2}((S, S), \phi_A) = 0.$$

using the JACOBI id

$$\delta^2 \phi^{*A} = ((\phi^{*A}, S), S) = \frac{1}{2}((S, S), \phi^{*A}) = 0.$$

4) Notice that: (this is true in general).

$$\delta_S G = (G, S)$$

$$\delta_S^2 G = ((G, S), S) + (G, \underbrace{\delta_S S}_{0}) = \frac{1}{2}((S, S), G) = 0.$$

but $\delta_S S = (S, S) = 0$

with a little more algebra we get:

$$\boxed{\delta^2 \Phi^{*A} = 0}$$

$$\delta\left(\frac{\delta S_0}{\delta \phi_A} + \Phi^*_B \partial_A R^B\right) = \left(R_B + \Phi^*_C \Lambda^{CB}_{\ \ B}\right) \partial_B \frac{\delta S_0}{\delta \phi_A} +$$

$$+ \left(\frac{\delta S_0}{\delta \Phi_B} + \Phi^*_C \Lambda^{C\,r}_{\ \ B}\right) \partial_A R^B +$$

$$+ \Phi^*_B \left(R_C + \Phi^*_D \Lambda^D_C\right) \partial_C \partial_A R^B$$

Reducibility

$B_{\mu\nu}$ two forms:

$$\delta B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu \,, \quad H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$$

Gauge invariant F.S.

Action:

$$S_0 = \frac{1}{2} \int d^4x \; H_{\mu\nu\rho} H^{\mu\nu\rho}$$

But the transformation rules of $\delta B_{\mu\nu}$ has zero modes:

$$\delta(\delta B_{\mu\nu}) = \partial_\mu (\delta\lambda_\nu) - \partial_\nu (\delta \lambda_\mu) = 0$$

if $\boxed{\delta\lambda_\nu = \partial_\nu \eta}$

So we have the BRST transformations:

$$\begin{cases} s B_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu \\ s c_\mu = \partial_\mu c \\ s c = 0 \end{cases}$$

$$\boxed{s^2 B_{\mu\nu} = s^2 c_\mu = s^2 c = 0.}$$

So we have to modify the action as follows:

$$S' = \frac{1}{2} H_{\mu\nu\rho} H^{\mu\nu\rho} + B^*_{\mu\nu} s B^{\mu\nu} + C^*_\mu s C^\mu$$

Master equation $(S, S) = 0$.

$$\delta B_{\mu\nu} = s B_{\mu\nu} \qquad \delta B^{*\mu\nu} = -\partial^\rho H^{\mu\nu}{}_\rho$$

$$\delta C_\mu = s C_\mu . \qquad \delta c^{*\mu} = -2 \partial^\rho B^{*\mu}{}_\rho$$

$$\delta c = 0 \qquad \delta c^* = \partial^\mu c^*_\mu$$

So we immediately see that:

$$\begin{cases} \delta^2 B^{*\mu\nu} = -\partial^\rho \delta H^{\mu\nu}_\rho = -\partial^\rho (s H^{\mu\nu}_\rho) = 0 \\ \delta^2 c^{*\mu} = -2 \partial^\rho (-\partial^\sigma H_{\sigma\rho}{}^\nu) = 0 \quad \text{by antisymmetry of } B_{\mu\nu}. \\ \delta^2 c^* = \partial^\mu (-2 \partial^\rho B^*_{\mu\rho}) = 0 \end{cases}$$

So we have that

$$\boxed{(S, S) = 0}$$

Let us set:

$$\Phi_A = (A^\alpha_\mu, c^\alpha) \qquad \Phi^{*A} = (A^{*\mu}_\alpha, c^*_\alpha)$$

$$R^A = \left(\underset{+1}{R^\mu_\alpha}, \underset{+2}{R_\alpha}\right) = \left(\underset{+1}{R^\mu_{\alpha\beta} c^\beta}, \underset{+2}{R_{\alpha\beta\gamma} c^\beta c^\gamma}\right) =$$

$$= \left((\nabla^\mu c)_\alpha, \; C_{\alpha\beta\gamma} \frac{c^\beta c^\gamma}{2}\right)$$

$$S' = \int d^4x \left[\underbrace{\frac{1}{2} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha}_{S_0} + A^{*\alpha}_\mu (\nabla^\mu c)_\alpha + c^*_\alpha C^\alpha_{\beta\gamma} \frac{c^\beta c^\gamma}{2}\right]$$

$$\int d^4x \left((\nabla^\mu c)_\alpha \frac{\delta S_0}{\delta A^\alpha_\mu} + \frac{1}{2} C^\alpha_{\alpha\beta\gamma} c^\beta c^\gamma \frac{\delta S_0}{\delta c^\alpha} \right) +$$

$$+ \int d^4x \; A^{*\alpha}_\mu \underbrace{\left[(\nabla^\nu c)_\beta \frac{\delta (\nabla^\mu c)_\alpha}{\delta A^\beta_\nu} + \frac{1}{2} \left(C_{\beta\gamma\sigma} c^\gamma c^\sigma \right) \frac{\delta (\nabla^\mu c)}{\delta c^\beta} \right]}_{0} +$$

$$+ \int d^4x \; c^*_\alpha \left[(\nabla^\nu c)_\beta \underbrace{\frac{\delta \left(C^\alpha_{\rho\sigma} \frac{c^\rho c^\sigma}{2} \right)}{\delta A^\beta_\nu}}_{0} + \underbrace{\frac{1}{2} \left(C_{\beta\gamma\sigma} c^\gamma c^\sigma \right)}_{\text{by JACOBI identity} \to 0} \frac{\delta \left(C^\alpha_{\rho\sigma} \frac{c^\rho c^\sigma}{2} \right)}{\delta c^\beta} \right]$$

0

by using IbP and antisymmetry of $c^\alpha c^\beta = -c^\beta c^\alpha$.

0

Gauge invariance of the action.

this can also be verified as follow:

$$\delta \phi_A = \frac{\delta S}{\delta \phi^{*A}}:$$

$$\boxed{\delta A_\mu^\alpha = (\nabla_\mu c)^\alpha \qquad \delta c^\alpha = \frac{1}{2} C^\alpha_{\beta\gamma}\, c^\beta c^\gamma.}$$

$$\delta \phi^{*A} = \frac{\delta S}{\delta \phi_A}:$$

$$\boxed{\begin{aligned} \delta A^{*\mu}_\alpha &= -\nabla_\nu F^{\mu\nu}_\alpha - C_{\alpha\beta\gamma} A^{*\beta}_\mu c^\gamma \\ \delta c^*_\alpha &= -\nabla_\mu A^{*\mu}_\alpha + c^*_\beta C^\beta_{\alpha\gamma} c^\gamma \end{aligned}}$$

and then: $\delta^2 A_\mu^\alpha = 0 \qquad \delta^2 c^\alpha = 0.$

$$\left\{\begin{aligned} \delta^2 A^{*\mu}_\alpha &= -\delta\left(\nabla_\nu F^{\mu\nu}_\alpha\right) - C_{\alpha\beta\gamma}\left[-\nabla_\nu F_\mu^{\nu\beta} - C_{\beta\gamma\delta} A_\mu^{*\gamma} c^\delta\right] c^\gamma \\ &\quad - C_{\alpha\beta\gamma}\left(\nabla_\nu F^{\mu\nu}_\beta c^\gamma\right) \\ &\quad + C_{\alpha\beta\gamma} A_\mu^{*\beta}\left(\frac{1}{2} C^\gamma_{\delta\rho} c^\delta c^\delta\right) = 0 \\ \delta^2 c^*_\alpha &= 0 \end{aligned}\right.$$

On-shell-Closed Algebras (Open-algebras).

$$s\Phi_A = R_A(\Phi) \quad \text{and} \quad s S_0' = 0$$

$$s^2 \Phi_A = s R_A(\Phi) \propto \Lambda_{AB} \frac{\delta S_0}{\delta \Phi_B}$$ = equation of motion (Λ^{AB} = constants).

Then we modify the action as follow:

$$\boxed{S' = S_0' + \Phi^*_A R^A(\Phi) + \frac{1}{2} \Phi^*_A \Phi^*_B \Lambda^{AB}}$$

$$\delta \Phi_A = \frac{\delta S'}{\delta \Phi^*_A} = R_A(\Phi) + \Phi^*_B \Lambda^{AB} = s\Phi_A + \Phi^*_B \Lambda^{AB}$$

$$\delta \Phi^{*A} = \frac{\delta S'}{\delta \Phi^A} = \frac{\delta S_0}{\delta \Phi^A} + \Phi^*_B \frac{\partial}{\partial \Phi^A} R^B(\Phi)$$

Then we have:

$$\delta^2 \Phi_A = \delta R_A + \delta \Phi^*_B \Lambda^{AB} =$$

$$= \cancel{\Lambda_{AB} \frac{\delta S_0}{\delta \Phi_B}} + \left(\cancel{\frac{\delta S_0}{\delta \Phi_B}} + \Phi^*_C \frac{\partial}{\partial \Phi^A} R^C \right) \Lambda^{BA} =$$

$$+ \Phi^*_B \Lambda^{BC} \frac{\delta R_A}{\delta \phi^C}$$

$$=$$

$$\Rightarrow \quad \frac{\delta R_A}{\delta \phi^C} \Lambda^{CB} + \frac{\delta R^B}{\delta \phi^C} \Lambda^{CA} = 0$$ which follows from the invariance of eq. of motion.

Another example

$$s\varphi_i = \omega_{ij}\varphi_j \qquad S_0^1 = -\frac{1}{2}\varphi_i\varphi^i$$

obviously $\boxed{s S_0^1 = 0}$

Fixing $\varphi_i = (\varphi_0, 0, \ldots, 0)$ leave a $O(n-1)$ symmetry

then we consider such a reducible system

by adding new ghost γ_{ij}

$$s\omega_{ij} = \omega_{ik}\omega_j{}^k - \gamma_{ij} \quad \text{— } O(n-1) \text{ no takes.}$$

$$s^2\omega_{ij} = (\omega_{il}\omega_k{}^l - \gamma_{ik})\omega_j{}^k +$$

$$-\omega_{ik}(\omega_{jm}\omega^{mk} - \gamma_j{}^k) - s\gamma_{ij} \Rightarrow$$

$$s\gamma_{ij} = \omega_{ik}\gamma_j{}^k - \gamma_{ik}\omega_j{}^k$$

and $s^2\gamma_{ij} = (\omega_{il}\omega^l_k - \cancel{\gamma_{ik}})\gamma_j{}^k - \omega_{ik}(\omega_{jm}\gamma^{km} - \gamma_{jm}\omega^{m}{}_j)$

$$-(\omega_{im}\gamma_k{}^m - \gamma_{in}\omega_k{}^n)\omega_j{}^k - (\gamma_{ik})(\omega_{jm}\omega^{mk} - \cancel{\gamma_j{}^k})$$

$$= \omega_{il}\omega^l_k\gamma_j{}^k - \omega_{ik}\cancel{\omega_{jm}}\gamma^{km} + \omega_{ik}\gamma_{jm}\omega^{mk}$$

$$- \cancel{\omega_{in}\gamma_k{}^n}\omega_j{}^k + \gamma_{in}\omega^n_k\omega_j{}^k + \gamma_{in}\omega_k{}^n\omega_j{}^k = 0.$$

However:

$$\partial^2 \varphi_i = (\cancel{\omega_{im}\omega_j{}^m} - \gamma_{ij})\varphi_j - \omega_{ij}(\cancel{\omega_{jk}\varphi_k}) =$$

$$= -\gamma_{ij}\varphi_j \neq 0. \quad \text{but}$$

$$= -\gamma_{ij}\frac{\delta S_0}{\delta\varphi_j}$$ proportional the the equation of motion.

So we introduce the anti fields φ^*_i, ω^*_{ij}, γ^*_{ij} $\Rightarrow$

$$\boxed{S = S_0 + \varphi^*_i \partial\varphi^i + \omega^*_{ij}\partial\omega^{ij} + \gamma^*_{ij}\partial\gamma^{ij} + \frac{\varphi^*_i\varphi^*_j}{2}\gamma^{ij}}$$

So we have

$$\begin{cases} \delta\varphi_i = \dfrac{\delta S}{\delta\varphi^*_i} = \partial\varphi_i + \varphi^*_j\gamma^{ij} \\ \delta\omega_{ij} = \dfrac{\delta S}{\delta\omega^*_{ij}} = \partial\omega_{ij} \\ \delta\gamma_{ij} = \dfrac{\delta S}{\delta\gamma^*_{ij}} = \partial\gamma_{ij} \end{cases}$$

$$\begin{cases} \delta\varphi^*_i = \dfrac{\delta S}{\delta\varphi_i} = -\varphi_i + \varphi^*_j\omega_{ji} \\ \delta\omega^*_{ij} = \dfrac{\delta S}{\delta\omega_{ij}} = \varphi^*_{[i}\varphi_{j]} + \omega^*_{[ik}\omega^k{}_{j]} + \gamma^*_{[ik}\gamma^k{}_{j]} \\ \delta\gamma^*_{ij} = -\omega_{ij} + \gamma^*_{[ik}\omega^k{}_{j]} + \varphi^*_{[i}\varphi^*_{j]} \end{cases}$$