SCALAR POTENTIAL

(whose minimum will characterize the classical vacua).

$$V = -\left\{ \frac{1}{2} D^{a\,2} + g \bar{\varphi}^i T^{a\,j}_i \varphi_j D^a + \bar{f}^i f_i + \frac{\partial W}{\partial \varphi_i} f_i + \frac{\partial \bar{W}}{\partial \bar{\varphi}^i} \bar{f}^i \right\}.$$

$$\Rightarrow D^a = -g\, \bar{\varphi}^i T^{a\,j}_i \varphi_j$$

$$f_i = -\frac{\partial \bar{W}}{\partial \bar{\varphi}^i} \quad \& \quad \text{c.c.}$$

$$\Rightarrow V = +\frac{g^2}{2} \left(\bar{\varphi} T^a \varphi \right)^2 + \left| \frac{\partial W}{\partial \varphi} \right|^2.$$

$$\left(\equiv \frac{1}{2} D^2 + \bar{f} f \right) \Big|_{\substack{\text{evaluated} \\ \text{ON-SHELL}}}$$

A susy vacuum must be invariant under Q_α, $\bar{Q}_{\dot\alpha}$: $Q_\alpha|\text{vac}\rangle = \bar{Q}_{\dot\alpha}|\text{vac}\rangle = 0$.

But since $\{Q, \bar{Q}\} \sim P^\mu$ the full 4 momentum, including the vacuum energy, must vanish (more discussion to follow...)

Classically $E_{\text{vac}} = V_{\text{MIN}} \Rightarrow$

Class. Susy vacua $\Rightarrow V_{\text{MIN}} = 0$.

$$V = \underbrace{\frac{g^2}{2}(\varphi^\dagger T^a \varphi)}_{\geq 0} + \underbrace{\left|\frac{\partial W}{\partial \varphi}\right|^2}_{\geq 0} \geq 0.$$

hence $V = 0$ (IF IT EXISTS!) is a <u>susy</u> MIN. (classically)

$$\Rightarrow \begin{cases} \bar\varphi T^a \varphi = 0 & \text{D-flat } \cancel{(\text{FI})} \\[2ex] \dfrac{\partial W}{\partial \varphi} = 0 & \text{F-flat.} \end{cases}$$

WE START WITH THEORIES ADMITTING SUCH SOLUTIONS.

DEFINITION: The space of solutions to $D^a = 0$, $f^i = 0$ modulo gauge transformations is called the CLASSICAL MODULI SPACE of the theory.

Examples:

1) WZ model with one field φ

a) $W = \lambda \varphi \Rightarrow \mathcal{M} = \emptyset$.

b) $W = \lambda \varphi^2 \Rightarrow \mathcal{M} = \{0\}$.

2) WZ model with two fields φ_1, φ_2

$$W = \lambda \varphi_1^2 \varphi_2 \Rightarrow \mathcal{M} = \mathbb{C}.(\varphi_2)$$

3) SQED with $W = 0$

$$D = |\varphi|^2 - |\tilde{\varphi}|^2 = 0$$

First of all $|\varphi|^2 = |\tilde{\varphi}|^2$

$\Rightarrow \quad \varphi = e^{i\alpha}\tilde{\varphi}$ for some $\alpha \in [0, 2\pi]$.

I can always make a gauge transf:

$\tilde{\varphi} \to e^{-i\gamma}\tilde{\varphi} \qquad \varphi \to e^{+i\gamma}\varphi$

$(\gamma \in [0, 2\pi])$

$$e^{+i\gamma}\varphi = e^{i(\alpha-\gamma)}\tilde{\varphi}$$

and choose $\gamma = +\frac{\alpha}{2}$ so that

$\tilde{\varphi} = \varphi$. ($\Rightarrow \varphi$ is enough)

But I could also have chosen

$\gamma = \pi + \frac{\alpha}{2}$ (still $\in [0, 2\pi]$)

$(-\tilde{\varphi} = -\varphi)$.

Hence $\mathcal{M}_{cl} = \mathbb{C}/\mathbb{Z}_2 = \dfrac{\{\varphi \in \mathbb{C}\}}{\{\varphi \sim -\varphi\}}$.

(Topologically $\mathcal{M}_{cl} \simeq \mathbb{C}$ but the metric is singular).

There is a very usefull theorem that allows us to characterize the $D^a = 0$ solutions modulo gauge trans:

The space of gauge inequivalent sol. to $D^a = 0$ can be parameterized by the set of independent holomorphic gauge invariant monomials in φ^i.

Ex for SQED : $M = \varphi\tilde{\varphi}$.

($\varphi^2\tilde{\varphi}^2 = M^2$ NOT independent
$\bar{\varphi}\varphi$ NOT holomorphic
φ^2 No gauge invariant...).

Proof for $G = U(1)$ φ^i of charge q_i $i=1..N$.
$\{\varphi^i\} = \mathbb{C}^N$ in this space we want to find solutions to

$$q_1|\varphi_1|^2 + q_2|\varphi_2|^2 + \dots q_n|\varphi_n|^2 = 0$$

up to $\varphi_i \simeq e^{iq_i\alpha}\varphi_i$ SAME α!
Call the space $\mathcal{M}_{cl}$

Clearly $\mathcal{M}_{cl} \neq \emptyset$ since $\{\varphi_i \equiv 0\} \in \mathcal{M}_{cl}$.
(~~FI~~)

Other solutions can only occour if some q's have opposite signs. (Also required by anomaly cancellation).

Pick any point $\{\varphi_i^0\} \in \mathbb{C}^N$ and consider

$$\sum_i q_i e^{2q_i C} |\varphi_i^0|^2 = 0.$$

(This is actually the D-term before going to the WZ gauge but we do not need this information).

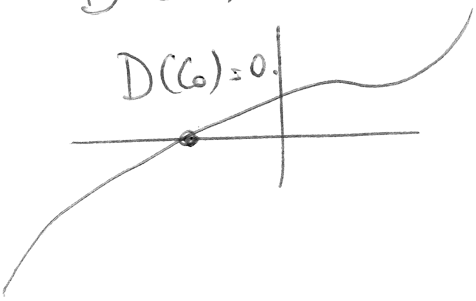
If $\forall \varphi_i^0 \neq 0$ all q_i have same sign there is no solution.

Contrarly, if there are some $\varphi_i^0 \neq 0$ w/ $q_i > 0$ and some $\varphi_j^0 \neq 0$ w/ $q_j < 0$

$$\exists! \, C \text{ s.t. } \underbrace{\sum q_i e^{2q_i C} |\varphi_i^0|^2}_{D(C)} = 0.$$

Proof $D'(c) = \sum 2q_i^2 e^{2q_i c} |\varphi_i|^2 > 0$.

and $D(c) \longrightarrow \pm\infty$ as $c \to \pm\infty$



This means that if I scale

$$\varphi_i \to e^{q_i c} \varphi_i \quad \text{(same } c\text{ !)}$$

I get a sol. of the $\sum q_i |\varphi_i|^2 = 0$.

Hence $\mathcal{M}_{ce} = \{\mathbb{C}^N, D=0\}/G = \mathbb{C}^N/G_{\mathbb{C}}$

where $G_{\mathbb{C}} = \{ e^{i\gamma + c}, \gamma \in [0, 2\pi], c \in \mathbb{R} \}$.

$\mathbb{C}^N/G_{\mathbb{C}}$ is easier to parameterize:

Consider $\varphi_1^{\ell_1} \varphi_2^{\ell_2} \cdots \varphi_N^{\ell_N} \bar{\varphi}_1^{\bar{\ell}_1} \cdots \bar{\varphi}_N^{\bar{\ell}_N}$

$(\ell_i, \bar{\ell}_i = 0, 1, 2 \ldots)$

Invariance under $\varphi_i \to e^{i\alpha q_i}\varphi_i$

$\Rightarrow \sum q_i (\ell_i - \bar{\ell}_i) = 0$

Invariance under $\varphi_i \to e^{c q_i}\varphi$

$\Rightarrow \sum q_i (\ell_i + \bar{\ell}_i) = 0$

$\Rightarrow$ Must be $\sum q_i \ell_i = \sum q_i \bar{\ell}_i = 0$ seperatelly.

$\Rightarrow$ Only Holomorphic invariant (and their c.c.).

This is true for ANY gauge group and if there is a $W(\varphi) \neq 0$? must also require $\frac{\partial W}{\partial \varphi_i} = 0$

We are now ready to look at a more interesting theory displaying a very rich set of dynamical properties, Namely $G = SU(N_c)$ SQCD w/ N_f FLAVORS.

NOTE: 1 flavor = 1 PAIR $Q, \tilde{Q}$ of chiral superfields transforming in the $\square$ and $\bar{\square}$ of $SU(N_c)$.

Sometimes it will be useful to turn on a superpotential

$$W = m_f^{f'} \tilde{Q}^c_{f'} Q^f_c$$

but at the beginning we consider $W = 0$.

We start with the CLASSICAL THEORY.

We can write:

$$Q^f_c = \underbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}_{\longleftrightarrow\, N_f} \updownarrow N_c$$

$$\tilde{Q}^c_f = \underbrace{\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}}_{\longleftrightarrow\, N_c} \updownarrow N_f$$

Recall that if $[t^A, t^B] = i f^{ABC} t^C$ are the generators of $SU(N_c)$ in $\square$ then $\tilde{t}^A = -t^{A*} \equiv -t^{AT}$ generates $\bar{\square}$

$$D^A = Q^\dagger t^A Q + \tilde{Q}^\dagger \tilde{t}^A \tilde{Q} =$$

$$= t^{A\,c}{}_{c'} \underbrace{\left(Q^{\dagger c'}_f Q^f_c - \tilde{Q}^{c'}_f \tilde{Q}^{\dagger f}_c \right)}_{\text{The only matrix } \perp \text{ to all } t^A\text{'s must be } \propto\, \delta^{c'}_c}$$

$$\Rightarrow \left(Q^{\dagger c'}_f Q^f_c - \tilde{Q}^{c'}_f \tilde{Q}^{\dagger f}_c \right) = \mathcal{N} \delta^{c'}_c$$

$$\Rightarrow QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} = \frac{1}{N_c} \mathbb{1}\, \mathrm{tr}\left(QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} \right)$$

Start considering $N_f = 1$. $N_c \geq 2$ arbitrary

$$Q = \begin{pmatrix} \\ \\ \end{pmatrix} \qquad \tilde{Q} = \overline{}$$

I can rotate $Q = \begin{pmatrix} a \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\tilde{Q} = (b_1 \cdots b_{N_c})$

$$\begin{pmatrix} |a|^2 & 0 & \cdots & 0 \\ & & 0 & \\ & & & \end{pmatrix} - \begin{pmatrix} |b_1|^2 & \bar{b}_1 b_2 & \cdots & \bar{b}_1 b_{N_c} \\ \vdots & & & \\ b_1 \bar{b}_{N_c} & & \cdots & |b_{N_c}|^2 \end{pmatrix} =$$

$$= \frac{1}{N_c} \left(|a|^2 - |b_1|^2 - |b_2|^2 \cdots - |b_{N_c}|^2 \right) \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

- Only one b_i can be $\neq 0$ otherwise there would be off diag terms $b_i \bar{b}_j$.
- Whatever it is it must be such that the trace vanishes: $|a|^2 = |b_i|^2$.
- But then it can only be $b_1 \neq 0$.

$$\mathcal{M} = \{ \tilde{Q} Q \} \simeq \mathbb{C}.$$

Makes sense since $SU(N_c)$ is broken down to $SU(N_c - 1)$ giving rise to

$(N_c^2 - 1) - ((N_c - 1)^2 - 1) = 2N_c - 1$ massive vector superfields

Each vector superfield "eats" a whole chiral superfield!
$\Rightarrow$ there is only $2N_c - (2N_c - 1) = 1$ left.

Suppose now there are $N_f < N_c$ flavors.

I can use $SU(N_c)$ to rotate

$$Q = \begin{pmatrix} x & & & & \\ & x & & & \\ & & x & & \\ & & & \ddots & \\ & & & & x \\ & & 0 & & \end{pmatrix} \begin{array}{l} \}N_f \\ \\ \}N_c - N_f \end{array}$$

$$\underbrace{}_{N_f}.$$

Let $\tilde{Q} = \left(\underset{N_f}{X} \;\vdots\; \underset{N_c - N_f}{Y} \right) \Big| N_f$

$$QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} = \left(\begin{array}{c|c} \cdots & -X^\dagger Y \\ \hline -Y^\dagger X & \cdots \end{array} \right)$$

Assume for simplicity X invertible

$\Rightarrow Y=0 \Rightarrow SU(N_c)$ broken to $SU(N_c-N_f) \Rightarrow$ # of uneaten chiral fields =

$$\underbrace{2N_f\cdot N_c}_{\text{Tot } \# Q,\tilde{Q}} - \underbrace{\Big(\underbrace{(N_c^2-1)}_{\dim SU(N_c)} - \big(\underbrace{(N_c-N_f)^2-1}_{\dim SU(N_c-N_f)}\big) \Big)}_{\text{massive bosons.}} = N_f^2$$

$$\Rightarrow \mathcal{M} = \{\tilde{Q}^c_f Q^{f'}_c\} \simeq \mathbb{C}^{N_f^2}.$$

For $N_f \geq N_c$ the gauge group is (generically) totally broken.

$\Rightarrow$ # uneaten chiral fields =

$$= 2N_f N_c - (N_c^2-1) < N_f^2$$

$$(= \text{ for } N_f = N_c+1)$$

An OVERCOMPLETE set of holomorphic invariants is:

$$M^{f}_{f'} = \tilde{Q}^{c}_{f'} Q^{f}_{c}$$

$$B^{f_1 \cdots f_{N_c}} = \epsilon^{c_1 \cdots c_{N_c}} Q^{f_1}_{c_1} \cdots Q^{f_{N_c}}_{c_{N_c}}$$

$$\tilde{B}_{f'_1 \cdots f'_{N_c}} = \epsilon_{c_1 \cdots c_{N_c}} \tilde{Q}^{c_1}_{f'_1} \cdots \tilde{Q}^{c_{N_c}}_{f'_{N_c}}$$

Together with an OVERCOMPLETE set of constraints:

$$B^{f_1 \cdots f_{N_c}} \tilde{B}_{f'_1 \cdots f'_{N_c}} = M^{[f_1}_{f'_1} \cdots M^{f_{N_c}]}_{f'_{N_c}}$$

$$B^{[f_1 \cdots f_{N_c}} M^{f]}_{f'} = 0$$

$$\tilde{B}_{[f'_1 \cdots f'_{N_c}} M^{f}_{f']} = 0.$$

QUANTUM CORRECTIONS: RECALL:

SQCD with $G = SU(N_c)$ and N_f flavors.

	G
Q	$\square$
$\tilde{Q}$	$\bar{\square}$

Remember: $\mu \frac{d}{dg} = -\frac{\beta_1}{16\pi^2} g^3 + \mathcal{O}(g^5)$.

$$\beta_1 = \frac{11}{6} T(\text{Adj}) - \frac{1}{3} \sum_{\psi\,\text{Weyl}} T(\psi) - \frac{1}{12} \sum_{\phi\,\text{Real}} T(\phi).$$

$\text{tr}(t^a_R t^b_R) = T(R)\,\delta^{ab}$. $T(\square) = 1$ by def.

$T(\bar{R}) = T(R)$

in $\mathcal{N}=1$ SYM we have

1 gaugino $\lambda \in \text{Adj}$

N_f Weyl & N_f $\mathbb{C}$ bosons $\in \square$.

N_f Weyl & N_f $\mathbb{C}$ bosons $\in \bar{\square}$

$$\beta_1 = \left(\frac{11}{6} - \frac{1}{3}\right) T(\text{Adj}) - 2N_f \left(\frac{1}{3} + \frac{2}{12}\right) T(\square)$$

$$\frac{3}{2} \cdot 2N_c - 2N_f \frac{1}{2} = \underline{(3N_c - N_f)}$$

must be > 0 for A.F.

Solution : $\dfrac{1}{g^2(\mu)} = - \dfrac{\beta_1}{8\pi^2} \log \dfrac{\Lambda}{\mu}$

$\Lambda = \mu \, e^{-\frac{8\pi^2}{\beta_1 g^2(\mu)}}$ INDEPENDENT on μ.
(can set $\mu = \mu_0$ UV scale)

is the energy where the theory becomes strongly coupled.

Now remember that the complexified gauge coupling is $\tau = \dfrac{\Theta}{2\pi} + i \dfrac{4\pi}{g^2}$

and Θ is <u>NOT</u> RENORMALIZED.

- I can write $\Lambda = \mu \, e^{\frac{2\pi i \tau}{\beta_1}} \in \mathbb{C}$.

- PERTURBATIVE CORRECTIONS are parameterized by $g^{2n} \sim \dfrac{1}{(\log \Lambda)^n}$

- NON PERTURBATIVE CORRECTIONS are param. by Λ^n (eg $e^{-\frac{8\pi^2}{g^2} + i\Theta} \sim \Lambda^{\beta_1}$ INSTANTONS)

- CLASSICAL LIMIT: $\Lambda \to 0$!

ANOMALIES.

Let us focus on SQCD for the moment although the discussion can be generalized.

SQCD contains the fermions

ψ , $\tilde{\psi}$ and λ^a ← gaugino

Out of which I can construct the following currents:

GAUGE

$$\bar{\psi}\,\bar{\sigma}^\mu t^A \psi + \bar{\tilde{\psi}}\,\bar{\sigma}^\mu \tilde{t}^A \tilde{\psi} + i f^{ABC}\,\bar{\lambda}^B \bar{\sigma}^\mu \lambda^C = J^{A}_{\mu}$$

$t_R^A \equiv t^A,\ \tilde{t}^A$ or $i f^{ABC}$

for $R = \square,\ \bar{\square}$ or. Adj

FLAVOR.

$$\bar{\psi}\,\bar{\sigma}^\mu \psi = J^\mu_\psi : U(1)_\psi$$

$$\bar{\psi}\,\bar{\sigma}^\mu \tau^a \psi = J^{a\mu}_\psi : SU(N_f)_\psi$$

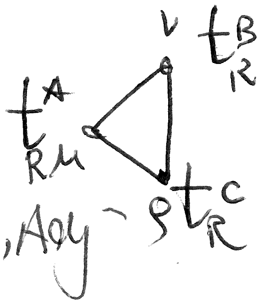
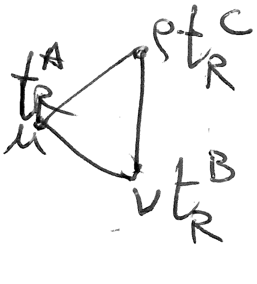
$$\bar{\tilde{\psi}}\,\bar{\sigma}^\mu \tilde{\psi} = J^\mu_{\tilde{\psi}} : U(1)_{\tilde{\psi}}$$

$$\bar{\tilde{\psi}}\,\bar{\sigma}^\mu \tilde{\tau}^{\tilde{a}} \psi = J^{\tilde{a}\mu}_{\tilde{\psi}} : SU(N_f)_{\tilde{\psi}}$$

$$\bar{\lambda}^A \bar{\sigma}^\mu \lambda^A = J^\mu_\lambda : U(1)_\lambda$$

Consider first

$$\langle 0 | T\left(J^{A\mu}(x_1)\, J^{B\nu}(x_2)\, J^{C\rho}(x_3)\right) | 0 \rangle$$

Computed via $\sum_{R = \square, \bar{\square}, Adj}$  + 

$$\partial_\mu \langle \cdots \rangle \propto \sum_R \mathrm{tr}\left(t_R^A \{ t_R^B, t_R^C \}\right)$$

and the theory is consisten ONLY if

$$\sum_R \mathrm{tr}\left(t_R^A \{ t_R^B, t_R^C \}\right) = 0 .$$

For any group and any irrep R

$$\mathrm{tr}\left(t_R^A \{ t_R^B, t_R^C \}\right) = \mathcal{A}(R) . d^{ABC}$$

and $\mathcal{A}(\bar{R}) = - \mathcal{A}(R)$

$\therefore$ Only $\mathbb{C}$-irreps of $SU(N)$ have the product $\mathcal{A}(R) . d^{ABC} \neq 0$.

For SQCD

$$\sum \mathrm{tr}\left(t_R^A \{ t_R^B, t_R^C \}\right) \propto \left(N_f \mathcal{A}(\square) + N_f \underset{-\mathcal{A}(\square)}{\underset{\parallel}{\mathcal{A}(\bar{\square})}} + \underset{0}{\underset{\parallel}{\mathcal{A}(\mathrm{Adj})}} \right) = 0$$

So the theory is consistent.

(I could write more explicitly:

$$T^A \otimes \underset{\text{flavor}}{\underset{\uparrow}{\mathbb{1}}} , \quad \tilde{T}^A \otimes \mathbb{1} , \quad i f^A \otimes 1 .$$

1 Flavor + 2 Gauge

$$\partial_\mu \langle 0| T(J^{\mu a}_\psi(x_1)\, J^{\nu B}(x_2)\, J^{\rho C}(x_3))|0\rangle$$

$$\propto \mathrm{tr}\left((\mathbb{1}\otimes\tau^a)\,\{t^B, t^C\}\otimes \mathbb{1}\right) =$$

$$= \underbrace{\mathrm{tr}\,\tau^a}_{=0}\, 2\,\mathrm{tr}\, t^B t^C = 0$$

only ψ

So $SU(N_f)$ is conserved.

However the ABELIAN FLAVOR:

$$\partial_\mu \langle 0| T(J^{\mu}_\psi(x)\, J^{\nu B}(x_2)\, J^{\rho C}(x_3))|0\rangle$$

$$\propto \mathrm{tr}\,\mathbb{1}\cdot \mathrm{tr}\, t^B t^C = N_f\, T(\square)\,\delta^{BC}$$

Now $T(\square) = T(\bar{\square}) = 1 \quad T(\mathrm{Adj}) = 2N_c$.

out of $J^\mu_\psi \quad J^\mu_{\tilde\psi} \quad J^\mu_\lambda$ there are 2 conserved currents:

$$J^\mu_B = J^\mu_\psi - J^\mu_{\tilde\psi}$$

$$J^\mu_R = J^\mu_\lambda - \frac{N_c}{N_f}\left(J^\mu_\psi + J^\mu_{\tilde\psi}\right).$$

2 flavor - 1 Gauge trivial ($tr(t^A)=0$)
3 flavor ('t Hooft anomaly).
The current is conserved but the anomalies must match if the theory flows from UV -> IR.

What kind of symmetry rotates λ?
It cannot be a "flavor" symmetry acting on V as a whole since $V = V^\dagger$.
It is a R-SYMMETRY : $\theta_\alpha \to e^{i\beta}\theta_\alpha$
$\bar\theta^{\dot\alpha} \to e^{-i\beta}\bar\theta^{\dot\alpha}$

$$V \ni \underset{0}{A_\mu \theta\sigma^\mu\bar\theta} \,,\quad \underset{1}{\lambda\theta\bar\theta^2} \,,\quad \underset{0}{D\theta^2\bar\theta^2}$$

$$Q \ni \underset{q}{Q} \,,\quad \underset{q-1}{\psi\theta} \,,\quad \underset{q-2}{f\theta^2}$$

For us $q-1 = -\frac{N_c}{N_f} \Rightarrow q = 1 - \frac{N_c}{N_f}$

	$SU(N_f)$	$\widetilde{SU(N_f)}$	$U(1)_B$	$U(1)_R$	Dim
Q	$\square$	/	1	$1-N_c/N_f$	1
$\tilde{Q}$	/	$\overline{\square}$	-1	$1-N_c/N_f$	1
M	$\square$	$\overline{\square}$	0	$2(1-N_c/N_f)$	2
B	$\left.\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right. N_c$	/	N_c	$N_c-N_c^2/N_f$	N_c
$\tilde{B}$	/	$\overline{\left.\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right.} N_c$	$-N_c$	$N_c-N_c^2/N_f$	N_c

Under $SU(N_f) \times \widetilde{SU}(N_f)$:
$\quad\quad\quad g \quad\quad\quad \tilde{g}$

$$M \to g M \tilde{g}^\dagger$$

So note that $\mathrm{tr}(M^p)$ is NOT invariant.
but $\det M$ is. $R(\det M) = 2(N_f - N_c)$
$[\det M] = 2N_f$.

$\Rightarrow W = \text{const} \times \Lambda^x (\det M)^y$ such that

$$[W] = x + 2N_f y = 3$$
$$R(W) = 2(N_f - N_c) y = 2$$

$$\Rightarrow \quad x = \frac{3N_c - N_f}{N_c - N_f}, \quad y = -\frac{1}{N_c - N_f}.$$

$$\Rightarrow W_{\text{non pert}} = \text{const} \cdot \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}}$$

NON PERTURBATIVE ! and $W \to 0$ as $\Lambda \to 0$
OK ! for $N_f < N_c$.

For $N_f \geq N_c$ $W_{\text{non pert}} = 0$.

We can already obtain our first exact quantum result!

Consider SQCD w/ $0 < N_f < N_c$.

We saw that $\mathcal{M} = \{ M^f_{f'} \} \simeq \mathbb{C}^{N_f^2}$

are the massless modes left over generically. They will have an effective IR action just like the pions of QCD.

SUSY $\Rightarrow$ $\int d^2\theta\, d^2\bar\theta\, K(M, \bar M) + \left(\int d\theta^2 W(M) + cc \right)$

K is a complicated function but W can be written explicitly up to an overall const.

The point is that W must OBEY ALL SYMMETRIES of the UV theory and have $[W] = 3$, $R(W) = 2$

Note: $[\theta_\alpha] = -\frac{1}{2} = [\bar\theta^{\dot\alpha}]$ $\quad R(\theta_\alpha) = 1 = -R(\bar\theta^{\dot\alpha})$

$\int d\theta^2 \theta^2 = 1 \Rightarrow [d\theta^2] = +1 \quad R(d\theta^2) = -2$

$\int d\bar\theta^2 \bar\theta^2 = 1 \Rightarrow [d\bar\theta^2] = +1 \quad R(d\bar\theta^2) = +2.$

- For $N_f = N_c - 1$ $\quad W \sim \dfrac{\Lambda^{2N_c+1}}{\det M}$

But $\beta_1 = 3N_c - N_f = 2N_c + 1$ as well.

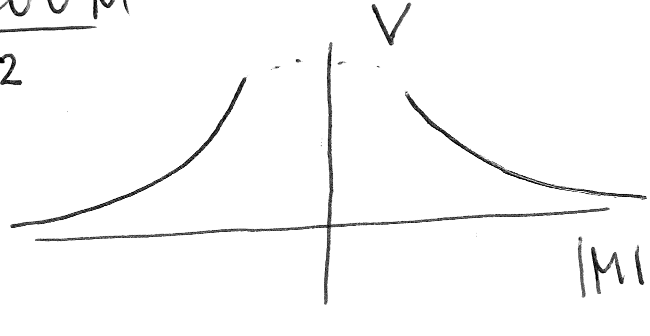
$$\Rightarrow W \sim \frac{\mu^{2N_c+1} \, e^{-\frac{8\pi}{g^2}}}{\det M} \qquad \text{Instanton effect.}$$

$\Rightarrow$ Can be computed and shown to be $\neq 0$.

- The const. is $\neq 0$ $\forall\ 0 < N_f < N_c$ (Can be shown by turning on the masses)

- This means that the theory has <u>NO VACUUM</u> since

$$V = \left| \frac{\partial W}{\partial M^{f}_{f'}} \right|^2$$



To go forward we must discuss the (strong!) restrictions that holomorphy puts on W in general

<u>Theorem</u>: The superpotential in the Wilsonian effective action receives NO perturbative corrections.

This is easier to prove for a WZ model w/ a field ϕ since the theory is IR free and it has the same d.o.f. as we go $\to$ IR.

Consider $W = m\phi^2 + \lambda\phi^3$

TRICK: Think of m and λ as vevs of the scalar comp. of a chiral superf. as well:

$$m + \sqrt{2}\,\psi_m\theta + \dots$$
$$\lambda + \sqrt{2}\,\psi_\lambda\theta + \dots$$

Then: a) W_{eff} must be holomorphic in λ, m as well as ϕ.
(cannot depend on λ^*, m^*).

b) $\exists$ can assign

$$R(\phi) = 1 \,, \quad R(m) = 0 \,, \quad R(\lambda) = -1$$

(and of course $[\phi] = 1 \quad [m] = 1 \quad [\lambda] = 0$)

Consider : $\lambda^x m^y \phi^z \begin{cases} y + z = 3 \\ -x + z = 2 \end{cases}$

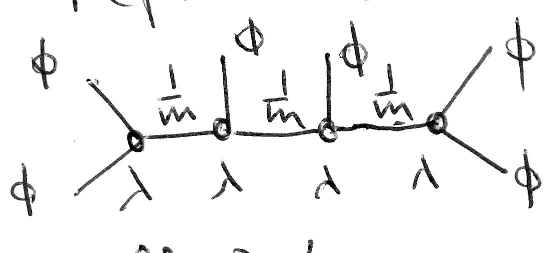
$$\Rightarrow x = p \,, \quad y = 1 - p \,, \quad z = p + 2 \,.$$

$$\lambda^p m^{1-p} \phi^{p+2} = \begin{cases} m \phi^2 & p = 0 \\ \lambda \phi^3 & p = 1 \\ \lambda^2 \frac{1}{m} \phi^4 & p = 2 \\ \vdots & \end{cases}$$

($p<0$ not allowed since $\lambda \to 0$ must be a free theory).

$p = 0, 1$ reproduce the classical W.

$p > 1$:



1P. REDUCIBLE $\notin W_{eff}$.

eg $p = 4$

So in general:

$$W_{exact} = W_{tree} + \cancel{W_{pert}} + W_{non\ pert}$$

makes sense only for theories well defined in the UV.

Applying this to SQCD I can turn on a mass term: for $N_f = N_c - 1$

$$W = tr(mM) + \frac{\Lambda^{2N_c+1}}{\det M}$$

Assuming $\det m \neq 0$:

$$\frac{\partial W}{\partial M^i_j} = m^j_i - {M^{-1}}^j_i \frac{\Lambda^{2N_c-1}}{\det M} = 0.$$

// $\frac{\partial}{\partial M^i_j}(\det M) = \det M \cdot {M^{-1}}^j_i = \text{Minor}(M)^j_i$ //

$$\Rightarrow M m = \frac{\Lambda^{2N_c-1}}{\det M} \mathbb{1}_{N_f \times N_f}$$

$$\Rightarrow \det M \cdot \det m = \left(\frac{\Lambda^{2N_c-1}}{\det M}\right)^{N_c-1}$$

$$\Rightarrow (\det M)^{N_c} = \frac{\Lambda^{(2N_c+1)(N_c-1)}}{\det m}$$

$$\Rightarrow M = m^{-1} . (\det m)^{\frac{1}{N_c}} . \Lambda^{\frac{2N_c+1}{N_c}}.$$

The $1/N_c$ in the $\det m$ means that for m non degenerate there are N_c susy vacua $\left(\mathcal{M}_q = \{N_c \text{ pts.}\}\right)$ and this is CONSISTENT with the computation of the WITTEN INDEX:

$$\operatorname{tr} (-1)^F = \frac{1}{2} T(Adj) = \frac{1}{2} . 2N_c = N_c$$

$N_f = N_c$.

We will see that $W_{\text{non pert}} = 0$

$\mathcal{M}_q \neq \mathcal{M}_{cl}$. $(\equiv \{\det M - B\tilde{B} = 0\})$.

(Sometimes one writes $\mathcal{M}_q$ by introducing W with a lagr. multiplier)

The basic point is that going from classical $\to$ quantum introduces an EXTRA PARAM. Λ.

and the eq. for $\mathcal{M}_q$ is generalized to an eq. containing Λ and obeying the same symmetries (non anomalous).

Note that for $N_f = N_c$

$R(B) = R(\tilde{B}) = R(M) \equiv 0$.

(this already show $\nexists\, W_{\text{non pert}}$ since $R(W)$ must be $= 2$).

Consider Λ^{2N_c}, $\det M$, $B\tilde{B}$

all of dim $= 2N_c$ and invariant under $SU(N_f = N_c) \times \widetilde{SU}(N_f) \times U(1)_B$.

Clearly I am allowed to add Λ^{2N_c}:

$$\det M - B\tilde{B} - \Lambda^{2N_c} = 0$$

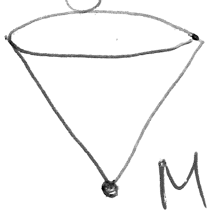
No other term is allowed by comparing with the classical limit: Terms w/ $(\Lambda^{2N_c})^{\text{NEGATIVE POWER}}$ are excluded by $\Lambda \to 0$.

terms like $(B\tilde{B})^2/\det M$ are excluded because they would survive $\Lambda \to 0$.

Terms like $\Lambda^{2N_c} \cdot \dfrac{B\tilde{B}}{\det M}$

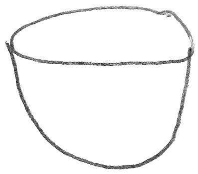
or anything else w/ a field at the denominator will give rise to extra solution valid for $\det M, B\tilde{B} \gg \Lambda^{2N_c}$. But these are excluded, again, by the classical limit.

classically: $\mathcal{M}_{cl} = \{ \det M - B\tilde{B} = 0 \}$



$M, B, \tilde{B} = 0$.

quantum: $\mathcal{M}_q = \{ \underbrace{\det M - B\tilde{B} - \Lambda^{2N_c}} = 0 \}$



smooth! $Y(M, B, \tilde{B})$

$Y = 0$ and $\nabla Y = 0$ has no solutions.

Some points of interest:

a): $M = \Lambda^2 \mathbb{1}$. $B = \tilde{B} = 0$.

Unbroken flavor: $SU(N_f = N_c)_{DIAG} \times U(1)_B$.

Massless modes (sol's to $\nabla Y = 0$) $B, \tilde{B}$ and the goldstone modes of χ-SB.

b) : $M = 0 \quad B = \tilde{B} = i\Lambda^{N_c}$.

Unbroken flavor $SU(N_f) \times \widetilde{SU}(N_f)$.

$U(1)_B$ broken and $B\tilde{B}$ = goldstone mode.

All M's massless.

So we conclude that for $N_f = N_c$ the theory admits as IR d.o.f. the fields M^i_j, B, $\tilde{B}$ with one constraint $\det M - B\tilde{B} = \Lambda^{2N_c}$. there are thus $N_c^2 + 1 + 1 - 1 = N_c^2 + 1$ light d.o.f. Consisten w/ the fact that the gauge symmetry is completely broken generically and thus out the the $2N_c^2$ UV d.o.f. Q, $\tilde{Q}$ there are $\dim(SU(N_c)) = N_c^2 - 1$ eaten goldst. bosons leaving $N_c^2 + 1$ fields.

The 't Hooft anomaly matching condition is satisfied. ✓

The (rather amazing at first sight) thing is that the anomaly matching is ALSO SATISFIED by $N_f = N_c + 1 \equiv N$.

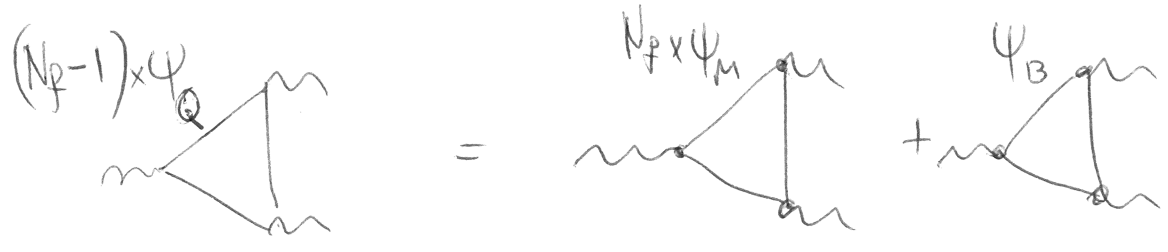
(But not more!).

		$SU(N_f)$	$\widetilde{SU}(N_f)$	$U(1)_B$	$U(1)_R$
UV	$\underline{(N_c \equiv N_f - 1)} \times \psi_Q$	N_f	1	1	$\frac{1}{N_f} - 1$
	$\underline{(N_f - 1)} \times \psi_{\tilde{Q}}$	1	$\overline{N_f}$	-1	$\frac{1}{N_f} - 1$
	$\underline{N_c^2 - 1 \equiv N_f^2 - 2N_f} \times \lambda$	1	1	0	1
IR	ψ_M	N_f	$\overline{N_f}$	0	$\frac{2}{N_f} - 1$
	ψ_B	$\overline{N_f}$	1	$N_c \equiv N_f - 1$	$-\frac{1}{N_f}$
	$\psi_{\tilde{B}}$	1	N_f	$-N_f + 1$	$-\frac{1}{N_f}$

The non zero matching anomalies are:

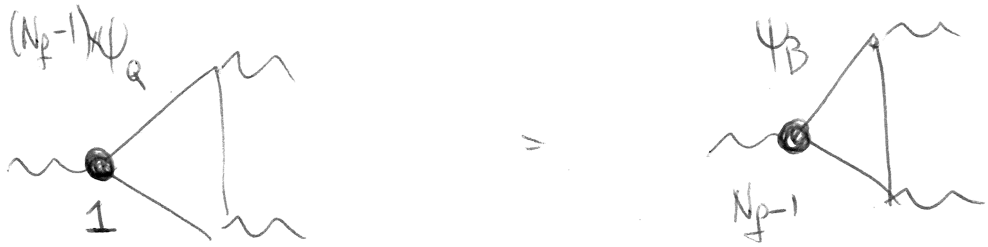
$SU(N_f)^3$: $N_f - 1$

UV IR

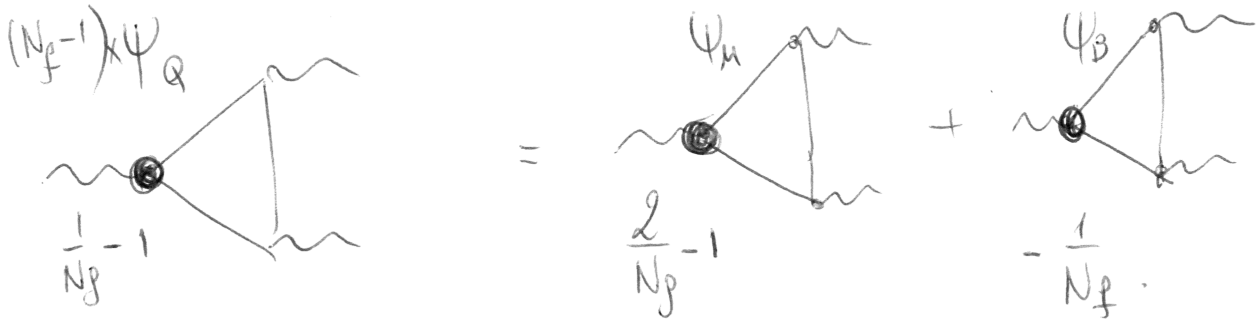


$(N_f-1)\cdot \underbrace{A(\square)}_{1}$ $\qquad$ $N_f \times A(\square) + 1 \cdot \underbrace{A(\bar{\square})}_{-1}$.

$SU(N_f)^2\, U(1)_B$: $N_f - 1$



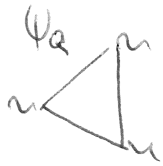
$SU(N_f)^2\, U(1)_R$: $\frac{1}{N_f} - 1$.

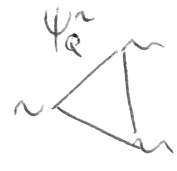


One can similarly check

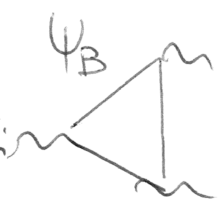
B^3, B^2R, BR^2, R^3, B, R ✓

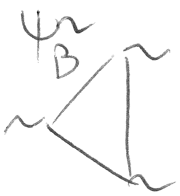
example $U(1)_B^2\, U(1)_R$:

<u>UV</u> : 

$+$ 

$$= (N_f-1)\cdot 1^2 \cdot \left(\frac{1}{N_f}-1\right) + (N_f-1)\cdot(-1)^2\cdot\left(\frac{1}{N_f}-1\right) = -2\,\frac{(N_f-1)^2}{N_f}$$

<u>IR</u> : 

$+$ 

$$= (N_f-1)^2\cdot\left(-\frac{1}{N_f}\right) + (-N_f+1)^2\cdot\left(-\frac{1}{N_f}\right) = -2\,\frac{(N_f-1)^2}{N_f}$$

Remedial notes on large N.

Consider pure YM for simplicity.

$$\langle 0 | T\left(A_{\mu}{}^{i}{}_{j}(x)\, A_{\nu}{}^{k}{}_{\ell}(y)\right) | 0 \rangle = \underline{\underline{\delta^k_j \delta^i_\ell}} \times D_{\mu\nu}(x-y)$$

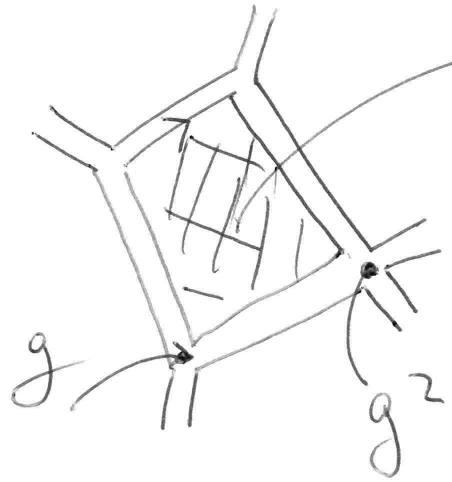
(For U(N) otherwise for SU(N), subtract $\propto \frac{1}{N}$ term).

Write the prop. in double line notation: $i \longrightarrow \ell$, $j \longleftarrow k$

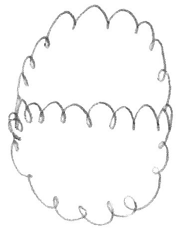
Every $i \rightarrow \ell \equiv \delta^i_\ell$

(Ghosts are the same)

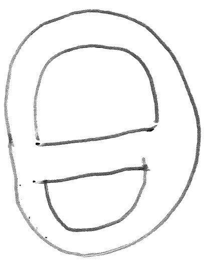
every loop $\propto N$.



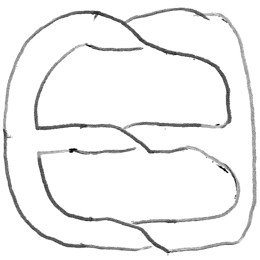
A vacuum diagram like



can be written as.



or



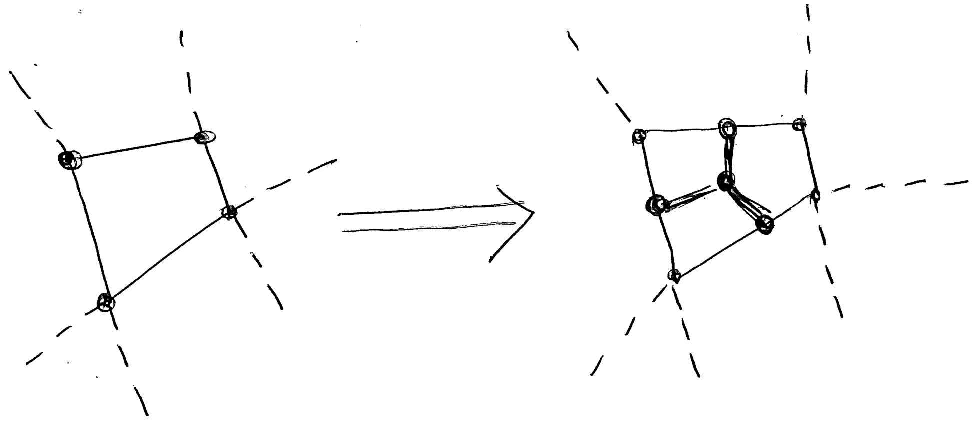
3 "color loops" $\propto g^2 \cdot N^3$

1 "color loop" $\propto g^2 \cdot N$

't Hooft observed that if I set $\lambda = g^2 N$ and let $N \to \infty$ keeping λ finite the diagrams scale like the EULER CHARACTERISTIC of the surface on which they can be drawn. For a TRIANGULATED surface

$$X = F - E + V$$

It is easy to show that χ does not depend on the specifics of the tiling:



$$V \longrightarrow V+4$$

$$E \longrightarrow E-4+10 = E+6$$

$$F \longrightarrow F-1+3 = F+2$$

$$\chi = F-E+V \longrightarrow \chi \quad \checkmark$$

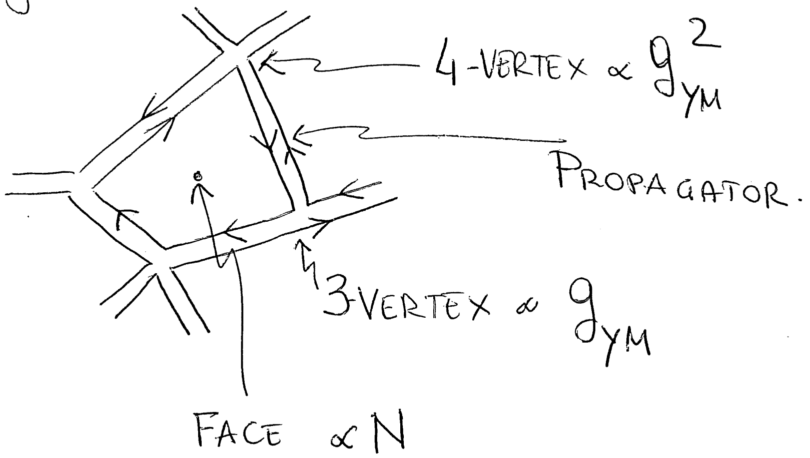
So I can pick any tiling I want

e.g. S^2: 

$$\chi = 4-6+4 = 2.$$

(If it stresses you that $\langle \mathrm{tr}F^2(x)\,\mathrm{tr}F^2(y)\rangle_{S^2} \sim N^2$, just rescale $\mathrm{tr}F^2 \to \frac{1}{N}\mathrm{tr}\,F^2$.)

Proof: A generic diagram (written in double line notation a' la 't Hooft) will define an oriented, closed 2d surface.



If: F = # Faces.
E = # Propagators = # Edges.
V_3 = # 3-vertices
V_4 = # 4-vertices

We can also define $V = V_3 + V_4$ = total # of vertices,

and notice that $E = \frac{1}{2}(4V_4 + 3V_3) \equiv 2V_4 + \frac{3}{2}V_3$

(each 4 vertex contributes HALF of 4 propagators:

 and so on)

Then, a generic diagram will scale like: $g_{YM}^{V_3+2V_4} \times N^F \equiv$

$$\equiv (g_{YM}^2 N)^{\frac{1}{2}V_3 + V_4} \times N^{F - \frac{1}{2}V_3 - V_4}$$

$$\equiv (g_{YM}^2 N)^{E-V} \; N^{F-E+V}$$

$$\equiv (g_{YM}^2 N)^{\ell - 1} \cdot N^{\chi}$$

•) $\ell = E - V + 1$ is the usual QFT "loop" counting: it is the number of unconstrained 4-momenta to be integrated over: each propagator contributes to one, each vertex has a δ-function and removes one and there is one overall δ-function that does not help because it constrains only the external momenta. For $g_{YM}^2 N \ll 1$ I can still do perturbation theory.

••) $\chi = F - E + V$ is the EULER CHARACTERISTIC!

What happens for $N_c+2 \le N_f < 3N_c$?
The d.o.f. $M, B, \tilde{B}$ are no longer a good description of the physics AT THE ORIGIN of the moduli space (the 't Hooft anomalies don't match)

Consider the 2 loops β-function:

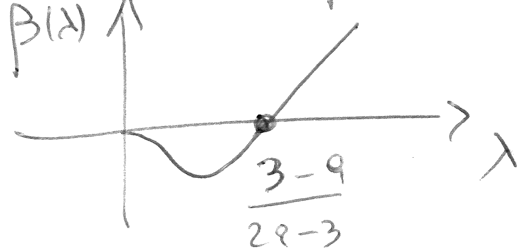
$$\beta(g) = -\frac{g^3}{16\pi^2}(3N_c - N_f) + \frac{g^5}{128\pi^4}\left(2N_cN_f - 3N_c^2 - \frac{N_f}{N_c}\right)$$

We would like to see if there is a reliable ($\approx$ pert.) fixed point near the origin.
To do that let us consider the region $N_c \sim N_f \to \infty$ $\left(\frac{N_f}{N_c} = \text{a const.}\right)$
Then 't Hooft showed us that the true coupling is $\sqrt{N_c}\cdot g$, or better $\lambda = \frac{N_c g^2}{16\pi^2}$.

$$\mu \frac{d}{d\mu} \lambda = \beta(\lambda) = \mu \frac{d}{d\mu} \frac{N_c g^2}{16\pi^2} =$$

$$= \frac{N_c \cdot 2g}{16\pi^2} \beta(g) \cong -2\lambda^2(3-a) + 4\lambda^3(-3+2a)$$

For $a \lesssim 3$ there is a near pert. large N_c fixed pt:



In fact the "conformal window" extends to $\frac{3}{2} N_c < N_f < 3N_c$ and is already open for $N_c = 2$ ($N_f = 4$ or 5)

There is also a "left over" interval $N_c + 2 \leq N_f < \frac{3}{2} N_c$ open from $N_c = 4$ (for which $N_f = 6$)

To argue why this should be the case we need to recall a couple of facts of $\mathcal{N}=1$ SCFT.

Remember that the conformal group can be obtained by adding to the Poincaré group (generated by P^μ, $J^{\mu\nu}$) the Dilatation op D. and conformal boosts K^μ

Alltogether $SO(4,2)$ (15 generators)

(The only not "obvious" relation being: $[P^\mu, K^\nu] = -i\left(\eta^{\mu\nu} D + J^{\mu\nu}\right)$)

For the SUPER-conf. algebra we need to add extra supercharges S^α, $\bar{S}^{\dot\alpha}$ AND an extra bosonic generator R (REQUIRED) R-charge.

$$\{Q^\alpha, S^\beta\} = -i\left(\epsilon^{\alpha\beta}\left(D - \frac{3}{2}R\right) + \sigma_{\mu\nu}^{\alpha\beta} J^{\mu\nu}\right)$$

IMPORTANT RESULT: For a chiral (PRIMARY) superfield $D = \frac{3}{2} R$.

So, if I know the R-charge, I know the scaling dimensions!

(The R-charge can be tricky to get if there are extra $U(1)$ symm. that could mix. For this case one uses a-maximization).

Another corollary that follows from $D = \frac{3}{2} R$ is that the SCALING dims of CHIRAL OPS are ADDITIVE

$\Rightarrow$ No extra ren. needed (~~sing~~).

chiral Ring

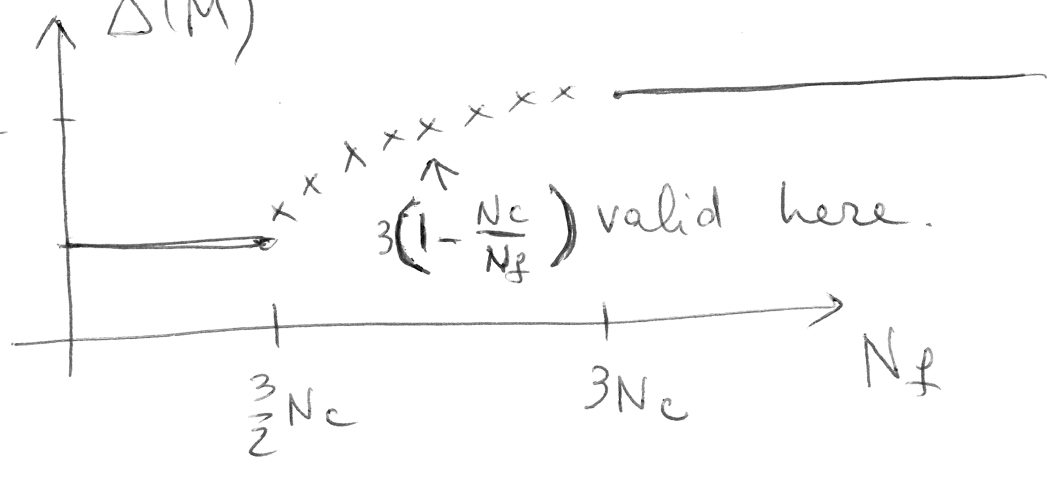
Applying these results to

$M^{f'}_{f} = \tilde{Q}^{c}_{f} Q^{f'}_{c}$ yields:

$$\Delta(M) = \frac{3}{2} R(M) = 3\left(1 - \frac{N_c}{N_f}\right)$$

Where can we trust this formula?

* We know that for $N_f \geq 3N_c$ the theory is IR free $\Rightarrow \Delta(M) = 2\Delta(Q) = 2$.
  (Q is free).
* We know that $\Delta(M) \geq 1$ where $\Delta(M) = 1$ means M (Not Q!) is a free field. (Note $\Delta = 1 \Rightarrow N_f = \frac{3}{2} N_c$)



Seiberg thought to keep this field as a free field in the IR for $N_c + 2 \leq N_f < \frac{3}{2} N_c$.

But we also need something to make the baryons. It must obey the $SU(N_f) \times \widetilde{SU(N_f)} \times U(1)_B \times U(1)_R$ symmetry $\Rightarrow$ N_f flavors of $q, \tilde{q}$ in the "$R, \bar{R}$ of some G".

It turns out that chosing $R = \square$ and $G = SU(N_f - N_c)$ works!

More precisely:

UV

	$SU(N_c)$	$SU(N_f)$	$\widetilde{SU(N_f)}$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	/	1	$1 - N_c/N_f$
$\tilde{Q}$	$\bar{\square}$	/	$\bar{\square}$	-1	$1 - N_c/N_f$

IR

	$SU(N_f - N_c)$	$SU(N_f)$	$\widetilde{SU(N_f)}$	$U(1)_B$	$U(1)_R$
q	$\square$	$\bar{\square}$	/	$N_c/N_f - N_c$	N_c/N_f
$\tilde{q}$	$\bar{\square}$	/	$\square$	$-N_c/N_f - N_c$	N_c/N_f
M	/	$\square$	$\bar{\square}$	0	$2(N_f - N_c)/N_f$.

It is obvious (in hindsight!) that if I want to make the same baryons out of Q and q I need $SU(N_f - N_c)$:

$$\epsilon^{c_1 \cdots c_{N_c}} Q^{f_1}_{c_1} \cdots Q^{f_{N_c}}_{c_{N_c}} \in \binom{N_f}{N_c} \text{ of } SU(N_f)$$

$$\epsilon_{c_1 \cdots c_{N_f - N_c}} q^{c_1}_{f'_1} \cdots q^{c_{N_f-N_c}}_{f'_{N_f-N_c}} \in \overline{\binom{N_f}{N_f - N_c}} \text{ of } SU(N_f)$$

But they are the same!
(Use $\epsilon_{f_1 \cdots f_{N_f}}$).

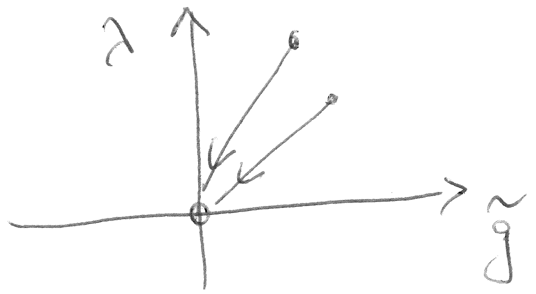
However for the "dual" theory we need to REMOVE the extra meson $m^{f}_{f'} = \tilde{q}^{f}_{c} q^{c}_{f'}$

This we can do by adding

$$W_{tree} = \lambda M^{f'}_{f} \tilde{q}^{f}_{c} q^{c}_{f'}$$

Now, for $N_f < \frac{3}{2} N_c$ the dual theory has $\beta_{1loop} = -\frac{\tilde{g}^3}{16\pi^2}\left(3(N_f - N_c) - N_f\right) > 0$

Hence $\tilde{g}$ (and λ) are $\xrightarrow{IR} 0$:

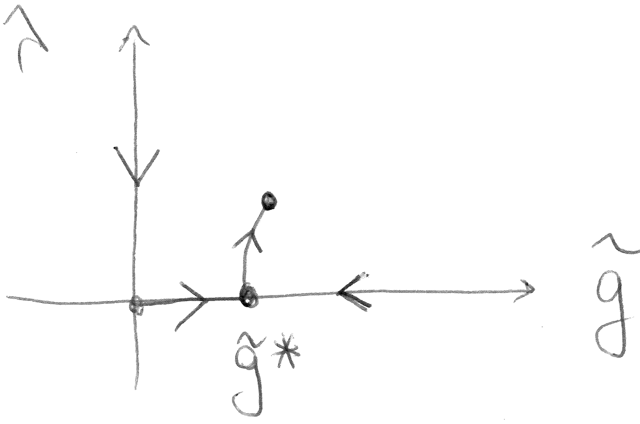


Even more interestingly, for

$\frac{3}{2} N_c < N_f < 3N_c$

The dual theory is ALSO A.F. and has a Fixed pt near the origin for N_f near $\frac{3}{2} N_c$

ALL 't Hooft anomalies MATCH!



at $\tilde g^*$ $\quad M\tilde q q$ is RELEVANT!

$$\Delta = 1 + \frac{3}{2}\frac{N_c}{N_f} + \frac{3}{2}\frac{N_c}{N_f} \lesssim 3.$$

NOT $\Delta(\tilde Q Q)$! Here M is a free field ($\lambda = 0$).

Lastly: the dual of the dual is the original theory.

$$Q, \tilde Q \xrightarrow{\text{dual}} q, \tilde q, M \xrightarrow{\text{dual}} Q' \tilde Q' M, N$$

$$W = 0 \qquad W = M\tilde q q \qquad W = MN + N\tilde Q' Q'$$

same