An Introduction to Supergravity, Duality and Black Holes

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1 Introduction to Supergravity

Why supersymmetry? Supersymmetry is a symmetry which relates fermionic and bosonic particles. There are general phenomenological arguments in favor of the idea that supersymmetry is actually an underlying symmetry of Nature. The presence of this symmetry makes field theories better behaved in the ultraviolet (UV) by virtue of the cancellation of fermionic and bosonic contributions (which have different signs) to divergent loop integrals. As a consequence of this UV divergences are less severe. This solves an important problem with the Standard Model of fundamental interactions (SM), namely the *hierarchy* problem (see for instance [1]): There is a huge hierarchy between the scale of the weak interaction (about 100 GeV) and that of gravity (the reduced Planck scale $M_P = \sqrt{\hbar c/(8\pi G_N)} \approx 2.4 \times 10^{18}\, GeV/c^2$). If we assume there is no new physics occurring between the two, we would then expect the SM to hold at energies up to M_P. Quantum corrections to the Higgs mass, however, contribute quadratic terms in the energy scale (energy cut-off). These terms of order M_P^2, unless an unnatural fine-tuning of the SM parameters is made, would push the value of the Higgs mass up to M_P, in contrast with the experimental value of $m_H \approx 125\, GeV/c^2$. Assuming supersymmetry, on the other hand, would require the existence in the theory, for each particle, of a *super-partner* obeying the opposite statistics. The contributions of these new particles have the effect of cancelling the quadratic divergences in the quantum corrections to the Higgs mass, leaving just the logarithmic ones.

Besides stabilizing the Higgs mass, and thus its ratio to M_P, against quantum corrections, the presence of an underlying supersymmetry also has the beneficial effect of unifying the coupling constants at some higher energy scale: The coupling constants of the weak, electromagnetic and strong interactions, if extrapolated to high energies through their renormalization-group evolution, meet at an energy scale of about $2\times 10^{16}\, GeV$, thus hinting towards a Grand Unified Theory (GUT) of the fundamental interactions.

Supersymmetry also has a more theoretical appeal, since it unifies space-time with internal symmetries (for a general introduction to supersymmetry see for instance [2]). The group SG containing supersymmetry transformations indeed generalizes the Poincaré group G_P in that its generators comprise, aside from those of the Lorentz transformations J_{ab} and space-time translations P_a, also fermionic generators Q (the supersymmetry generators) and generators B_i of an internal (compact) symmetry group G_i. The corresponding algebra of infinitesimal generators is therefore called *super-Poincaré* algebra. The supersymmetry generators Q belong to the spin-1/2 representation of the Lorentz group (the $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ of $\mathrm{SL}(2,\mathbb{C})$) as well as to a representation of the internal compact group. Transforming non-trivially under both G_P and G_i, the *fermionic* generators Q allow for a non-trivial interplay between space-time and internal symmetries. Moreover, having a spin quantum number, the action of Q on a states varies by 1/2 its spin, so that:

$$Q|\text{boson}\rangle = |\text{fermion}\rangle \; ; \quad Q|\text{fermion}\rangle = |\text{boson}\rangle \, . \tag{1.1}$$

If we require SG to be a symmetry of a local quantum field theory, consistency implies that the super-Poincaré algebra must be defined in terms of commutators $[\cdot,\, \cdot]$ between bosonic generators B (i.e. J_{ab}, P_a and B_i) and bosonic generators, or bosonic and fermionic generators F (in our case the Qs), and *anti-commutators* $\{\cdot,\, \cdot\}$ between two fermionic generators. Symbolically:

$$[B,\, B] = B \; ; \;\; [B,\, F] = F \;\; , \;\; \{F,\, F\} = B \, . \tag{1.2}$$

A Lie algebra containing fermionic generators obeying anti-commutation relations is called *graded* Lie algebra. In the super-Poincaré algebra the anti-commutator of two Qs yields the momentum operator P plus internal symmetry operators Z which are central charges of the superalgebra:

$$\{Q,\, Q\} \propto P + Z \, . \tag{1.3}$$

This implies that, modulo internal transformations Z, *the combination of two subsequent supersymmetry transformations amounts to a space-time translation.* Moreover Q commutes with P, and thus with the mass operator $m^2 = P^2/c^2$. As a consequence of this, irreducible representations of SG (supermultiplets), comprise one-particle states with the *same mass but different spins*. This is certainly a desirable feature if we ultimately aim at unifying all fundamental forces of Nature together and with matter. Indeed the gravitational force is mediated by the spin-2 graviton while the other interactions by spin-1 vector bosons and matter is made of spin-1/2 particles.

Supergravity. Here we come to gravity. The symmetry principle underlying Einstein's theory of gravity is its invariance under general coordinate transformations, which can be thought of as *local* space-time transformations generated by P_a. In a supersymmetric theory of gravity, called *supergravity*, such an invariance, by virtue of eq. (1.3), would be a consequence of a more fundamental symmetry principle: invariance of the theory under space-time dependent supersymmetry transformations (*local supersymmetry*). In *supergravity*[1] the gravitational field, described by Einstein's general theory of relativity, is coupled to its super-partners and possibly to other supermultiplets containing states with at most spin-1 (matter multiplets).

All theories describing, in a consistent way, the fundamental interactions and their coupling to matter are based on the *gauge principle*: the invariance under local (i.e. space-time dependent) transformations of some symmetry group (*gauge group*). This local symmetry is achieved only if matter is coupled to bosonic *gauge fields* (i.e. 1-forms) associated with (*gauging*) each infinitesimal generator (*gauge generator*) of the gauge group and transforming under the local group transformations in a suitable way (i.e. as *gauge connections*). These bosonic particles are the mediators of an interaction. Quantum-electrodynamics (QED), describing the coupling of matter to the electromagnetic field, is a gauge theory with gauge group U(1). Weak and electromagnetic interactions are unified in the SM and described by a local SU(2) × U(1) gauge group. The four gauge generators are in correspondence with the four mediating vector bosons: the $W^{\pm}$, Z^0 bosons and the photon γ^0. Strong interactions are described by an SU(3)-gauge-theory (QCD), the eight gluons being in correspondence with the SU(3) infinitesimal generators. Similarly Einstein's gravity can be viewed as the "gauge theory" of the Poincaré group and the graviton as the gauge boson associated with the local translation generators P_a. The quotes above indicate an important difference between the SM and general relativity: In the former the gauge group in an *internal* symmetry, namely acts on internal degrees of freedom, while in the latter the "gauge group" describes *external*, i.e. space-time, symmetries. While there is no dynamics along the internal directions, there is dynamics along the external ones: The dependence of the fields on the space-time coordinates is not the result of some (unphysical) gauge transformation, but is dictated by the field equations.

Finally supergravity can be viewed as the gauge theory of the super-Poincaré group, where the fermionic generators Q are *gauged* by the superpartner of the graviton, which is

[1] Good references for supergravity are [3, 5, 6].

a spin-3/2 particle Ψ called the *gravitino*. This field has a vector index (being a 1-form), corresponding to a spin-1 representation of the Lorentz group (the $(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$), and a spinor index, corresponding to the spinor components of Q and its spin is defined by the irreducible 3/2-representation in the product $1 \times 1/2$ of the corresponding two spin-representations.

Supersymmetric theories differ in the amount of supersymmetry, namely in the number $\mathcal{N}$ of the supersymmetry generators Q, and in the field content which should correspond to multiplets of SG. $\mathcal{N}$ supersymmetry generators define an $\mathcal{N}$- *extended supersymmetry*. The larger $\mathcal{N}$, the stronger the constraints on the interactions, the larger the maximum spin s_{max} of the fields in the supermultiplets. In general the least value of the maximum spin in the supermultiplets is related to $\mathcal{N}$: in four space-time dimensions we have $s_{max} \geq \mathcal{N}/4$. Theories which are only invariant under *global* super-Poincaré transformations (*rigid supersymmetry*), do not contain gravity and are thus defined on flat space-time. Renormalizability requires their fields not to have spin greater than 1, and thus $\mathcal{N} \leq 4$. The $\mathcal{N} = 4$ case is unique and describes a supersymmetric extension of the Yang-Mills theory (*super-YM theory*). Its high amount of supersymmetry makes it *perturbatively finite*.

As a theory of gravity, also supergravity is non-renormalizable. This follows from a simple power-counting argument: the coupling constant of gravity is Newton's constant G_N which has dimension of $length^2$ (in units $\hbar = c = 1$). The limit on the amount $\mathcal{N}$ of supersymmetry in supergravity comes from the possibility of a consistent coupling to gravity, which restricts the maximum spin of the fields to be 2, thus implying $\mathcal{N} \leq 8$. Supersymmetry however improves the UV properties of the theory, making it finite up to two loops[2](pure Einstein's gravity is only one-loop finite [7, 8] and this property is spoiled by the presence of matter). The maximal $\mathcal{N} = 8$ supergravity, just as the rigid $\mathcal{N} = 4$ super-YM theory, is unique (supersymmetry fixes its field content to be that of the supermultiplet containing the graviton as the maximum spin state). Though its perturbative finiteness has been tested, so far, up to four loops [9], some believe the maximal theory to be perturbatively finite just as its rigid $\mathcal{N} = 4$ counterpart.

Supergravity as an effective theory and dualities. Supergravity theories are defined also in $D > 4$ space-time dimensions. Of particular relevance are the theories in $D = 10$ and $D = 11$, since they describe the low-energy dynamics of superstring theory and M-theory, on flat space-time, respectively. Lower dimensional supergravities describe superstring/M-theories compactified on some internal compact manifold. Since the fundamental objects of superstring theories are not point-like particles, but oscillating strings, there is a natural cut-off given by the tension of these objects, which regularizes the loop integrals, thus making these theories perturbatively finite.

Even if infinite, at the perturbative level, supergravity models would make sense as effective "macroscopic" realizations of the more fundamental "microscopic" superstring/M-theories.

This microscopic description is however not unique. There are five kinds of superstring

[2]Here we refer to *ungauged* four-dimensional supergravities, namely theories in which the vector fields are not minimally coupled to other fields.

theories (*Type IIA*, *Type IIB*, two kinds of *Heterotic* string theories and *Type I*) in $D = 10$ and there is M-theory in $D = 11$, whose fundamental degrees of freedom are, as yet, not known. The discovery in the 90's of *dualities* has simplified the picture considerably: There exists an equivalence between the different superstring theories or M-theory realized on various backgrounds. This allows to think of these theories as different descriptions of the same microscopic degrees of freedom. Such correspondences, or dualities, between different realizations of superstring/M-theories, which can be either perturbative or non-perturbative, manifest themselves, at the level of the low-energy supergravity description, as *global symmetries* of the equations of motion (on-shell symmetries). For this reason the study of global symmetries of supergravity models and of their action on the corresponding solutions, plays an important role in understanding the non-perturbative aspects of superstring theories.

Finally the fruitful AdS/CFT conjecture, that is the conjectured equivalence between superstring theory realized on an anti-de Sitter space-time and the conformal field theory on its boundary at infinity, made supergravity (on anti-de Sitter space) a valuable tool for investigating non-perturbative properties of gauge theories. We shall not touch upon this issue here.

Black holes in supergravity. As a theory of gravity, supergravity has black hole solutions[3]. In general relativity it is known that in order for a static, asymptotically flat charged black hole solution (described by the Reissner-Nordström solution) not to be singular, that is for its spatial singularity to be hidden inside an event horizon, its mass M, electric and magnetic charges q, p should satisfy a *regularity bound* (here and in the following we set $8\pi G_N = c = \hbar = 1$):

$$M^2 \geq \frac{p^2 + q^2}{2}\,, \tag{1.4}$$

In general relativity there is a *cosmic censor conjecture* [13] according to which the above condition is satisfied by all black hole solutions in Nature, that is our Universe is clear of naked singularities which would make it unpredictable. There is so far no definite proof of this conjecture.

Things change in the presence of supersymmetry [14]. As solutions to a supersymmetric theory, supergravity black holes must belong to massive representations of the super-Poincaré algebra. If computed on a black hole background, the central charges Z of the super-Poincaré algebra, on the right hand side of eq. (1.3), have a non-vanishing value which depends on the electric and magnetic charges. In fact they are topological quantities associated with the solution. In an $\mathcal{N}$-extended theory the central charges are entries of an $\mathcal{N}\times\mathcal{N}$ antisymmetric matrix $Z_{ij} = -Z_{ji}$, $i, j = 1, \ldots, \mathcal{N}$. It can be shown that, under the general assumption that the norm in the Hilbert space of states be positive definite, supersymmetry implies that the mass M of the solution must be greater that the modulus of all the skew-eigenvalues z_ℓ of Z_{ij} [11, 12]:

$$M \geq |z_\ell| \;\;,\;\; \ell = 1, \ldots, \left[\frac{\mathcal{N}}{2}\right]\,. \tag{1.5}$$

[3]For a review of black hole solutions in supergravity see [10].

These can be thought of as the supergravity analogue of the Bogomol'nyi - Prasad - Sommerfield bound for solitonic solutions to gauge theories. On the Reissner-Nordström solution the above condition implies the regularity bound (1.4). In other words, at least for static solutions, supersymmetry acts as a cosmic censor in that it provides a general principle which rules out the existence of naked singularities.

If the inequalities (1.5) are not saturated for any ℓ, the solution is *non-extremal* and has a non-vanishing Hawking temperature T. By quantum mechanical effects it radiates (Hawking-evaporation effect) until its mass equals the largest $|z|_{max}$ of the $|z_\ell|$ and the temperature drops to zero. The resulting solution is called extremal and preserves a fraction of the $\mathcal{N}$ supersymmetries (at least $1/\mathcal{N}$). Supersymmetric black holes are called BPS (i.e. saturating the Bogomol'nyi - Prasad - Sommerfield bound) and are solutions to a set of first-order differential equations (the *Killing spinor equations*) which imply the second-order field equations. BPS solutions have played an important role in the study of superstring non-perturbative dualtities since $|z_\ell|$ are duality-invariant quantities and are protected, to a certain extent, from quantum corrections by supersymmetry.

Supergravity has more general solutions than the simple Reissner-Nordström one, which feature a non-trivial interplay between the scalar fields of the theory and the vector fields. They belong to different *topological sectors* of the theory and, after evaporating, they reach a lowest mass, zero-temperature extremal state in which M equals a new characteristic quantity $W > |z_\ell|$. The remarkable feature of these extremal solutions is that, although they do not preserve any supersymmetry and thus are non-BPS, they are still described by a set of first-order differential equations which imply the second order field equations.

Even a collection of the essential topics related to supergravity and its connection to superstring/M-theories would be far too vast to be covered in a limited series of lectures. This minicourse will therefore deal with a restricted selection of issues and is organized as follows.

- Part I: We give a brief introduction to supersymmetry and supergravity;
- Part II: We consider extended supergravity models in four space-time dimensions (i.e. models with $\mathcal{N} \geq 2$) and describe their on-shell global symmetry group G. We then examine the (asymptotically flat) black hole solutions to these models and their properties with respect to G.

2 Part I: Supersymmetry and Supergravity

2.1 The Super-Poincaré Algebra

Historically it was Haag, Lopuszanski and Sohnius [19] who proved that the largest possible symmetry group of the S-matrix of a four dimensional relativistic field theory was a supergroup. It was the *superconformal group* in four dimensions, whose contraction yields the super-Poincaré group SG to be discussed below. This result overcame a "no-go theorem" by Coleman and Mandula [20] which stated that such largest symmetry group ought to be the direct product of the Poincaré group times an internal symmetry one. Such theorem only considered ordinary Lie algebras whose structure is defined in terms of commutators only. The key ingredient of the Haag-Lopuszanski-Sohnius generalization was considering *graded Lie algebras* with fermionic generators obeying anti-cummutation rules. It is important to point out that supersymmetry appeared in the literature earlier in the works by Neveu, Schwarz and Ramond (1971), Gol'fand and Likhtman [21], Wess and Zumino [22], Volkov and Akulov [23]. For the notations we refer to Appendix A. The super-Poincaré algebra is spanned by the Poincaré generators $\hat{\mathcal{P}}_\mu$, $\hat{\mathcal{L}}_{\mu\nu}$, by (compact) *internal symmetry* generators B_r, and by a set of $\mathcal{N}$ fermionic generators, represented by spinor operators Q_i, $i = 1, \ldots, \mathcal{N}$, satisfying the Majorana condition:

$$Q_i = \begin{pmatrix} Q_{\alpha\, i} \\ \bar{Q}^{\dot{\alpha}\, i} \end{pmatrix} = C\bar{Q}_i^T \,, \tag{2.1}$$

where $\bar{Q}^{\dot{\alpha}\, i} = \epsilon^{\dot{\beta}\dot{\alpha}}\, (Q_{\alpha\, i})^\dagger$. If G_i is the compact group of internal transformations generated by B_i, $Q_{\alpha\, i}$ transform in the $\mathcal{N}$ -representation of G_i and $\bar{Q}^{\dot{\alpha}\, i}$ in the $\overline{\mathcal{N}}$. The group G_i will in general be contained inside $\mathrm{SU}(\mathcal{N}) \times \mathrm{U}(1)^k$, for some k. The graded-Lie algebra structure of the super-Poincaré algebra is defined by the following commutation/anti-commutation relations:

$$[\hat{\mathcal{L}}_{\mu\nu},\, \hat{\mathcal{L}}_{\rho\sigma}] = \eta_{\nu\rho}\, \hat{\mathcal{L}}_{\mu\sigma} + \eta_{\mu\sigma}\, \hat{\mathcal{L}}_{\nu\rho} - \eta_{\nu\sigma}\, \hat{\mathcal{L}}_{\mu\rho} - \eta_{\mu\rho}\, \hat{\mathcal{L}}_{\nu\sigma} \,, \tag{2.2}$$
$$[\hat{\mathcal{L}}_{\mu\nu},\, \hat{\mathcal{P}}_\rho] = \hat{\mathcal{P}}_\mu\, \eta_{\nu\rho} - \hat{\mathcal{P}}_\nu\, \eta_{\mu\rho} \,, \tag{2.3}$$
$$[Q_{\alpha\, i},\, \hat{\mathcal{L}}_{\mu\nu}] = -\frac{i}{2}\, (\sigma_{\mu\nu})_\alpha{}^\beta\, Q_{\beta\, i} \,, \tag{2.4}$$
$$[\bar{Q}^{\dot{\alpha}\, i},\, \hat{\mathcal{L}}_{\mu\nu}] = -\frac{i}{2}\, (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{Q}^{\dot{\beta}\, i} \,, \tag{2.5}$$
$$[\bar{Q}^{\dot{\alpha}\, i},\, \hat{\mathcal{P}}_\mu] = [Q_{\alpha\, i},\, \hat{\mathcal{P}}_\mu] = 0 \,, \tag{2.6}$$
$$[Q_{\alpha\, i},\, B_r] = (B_r)_i{}^j\, Q_{\alpha\, j} \,, \tag{2.7}$$
$$[\bar{Q}^{\dot{\alpha}\, i},\, B_r] = -(B_r)_j{}^i\, \bar{Q}^{\dot{\alpha}\, j} \,, \tag{2.8}$$
$$\{Q_{\alpha\, i},\, \bar{Q}^{\dot{\beta}\, j}\} = 2\, i\, \delta_i^j\, (\sigma^\mu)_\alpha{}^{\dot{\beta}}\, \hat{\mathcal{P}}_\mu \,, \tag{2.9}$$
$$\{Q_{\alpha\, i},\, Q_{\beta\, j}\} = 2\, \epsilon_{\alpha\beta}\, Z_{ij} \,, \tag{2.10}$$
$$\{\bar{Q}^{\dot{\alpha}\, i},\, \bar{Q}^{\dot{\beta}\, j}\} = -2\, \epsilon_{\dot{\alpha}\dot{\beta}}\, Z^{ij} \,, \tag{2.11}$$
$$[B_r,\, B_s] = f_{rs}{}^t\, (B_t) \,,$$
$$[B_r,\, \hat{\mathcal{L}}_{\mu\nu}] = [B_r,\, \hat{\mathcal{P}}_\nu] = 0 \,, \tag{2.12}$$

where $Z^{ij} = (Z_{ij})^*$. The above relations were shown by Haag, Lopuszanski and Sohnius to define the most general supersymmetry algebra of a relativistic field theory defined on Minkowski space.

Let us generically denote by T_A the graded Lie algebra generators and by $[T_A,\, T_B\}$ the anti-commutator if T_A, T_B are both of fermionic type or the commutator otherwise. The commutation/anti-commutation relations satisfy consistency conditions given by the *graded-*

Jacobi identities:

$$[T_A,\,[T_B,\,T_C\}\} + \text{graded cyclic} = 0\,, \tag{2.13}$$

where "graded cyclic" are cyclic permutations with a minus sign if two fermionic generators are interchanged.

On the Hilbert space of states $(\hat{\mathcal{P}}_\mu)^\dagger = -\hat{\mathcal{P}}_\mu$, $(\hat{\mathcal{L}}_{\mu\nu})^\dagger = -\hat{\mathcal{L}}_{\mu\nu}$, $B_r^\dagger = -B_r$. From the graded-Jacobi identities it follows that entries of the anti-symmetric matrix $Z_{ij} = a_{ij}^r\,B_r = -Z_{ji} = R_{ij} + i\,I_{ij}$ are central charges of the algebra and thus generate the $\mathrm{U}(1)^k$ part of G_i.

Exercise 1.:Check from the $[\bar{Q},[Q,Q\}\}$ graded-Jacobi identity that Z_{ij} has to commute with $\bar{Q}$.

We can write the above commutation/anti-commutation relations in terms of the four-component spinor Q_i. In particular we find:

$$\begin{aligned}
&\{Q_i,\,Q_j^T\} = -2\,i\,\delta_i^j\,(\gamma^\mu C)\,\hat{\mathcal{P}}_\mu + 2\,(R_{ij}\,\gamma^5 - i\,I_{ij})C\,,\\
&\{Q_i,\,\bar{Q}_j\} = \{Q_i,\,Q_j^T\}C = 2\,i\,\delta_i^j\,\gamma^\mu\,\hat{\mathcal{P}}_\mu - 2\,(R_{ij}\,\gamma^5 - i\,I_{ij}) = 2\,i\,\left(\delta_i^j\,\gamma^\mu\,\hat{\mathcal{P}}_\mu + \mathbb{Z}_{ij}\right)\,,\\
&\{\bar{Q}_i,\,\bar{Q}_j\} = -C\{Q_i,\,\bar{Q}_j\} = -2\,i\,\delta_i^j\,(C\gamma^\mu)\,\hat{\mathcal{P}}_\mu + 2\,C(R_{ij}\,\gamma^5 - i\,I_{ij}) = -2\,i\,\delta_i^j\,(C\gamma^\mu)\,\hat{\mathcal{P}}_\mu - 2i\,C\mathbb{Z}_{ij}\,,\\
&[Q_i,\,\hat{\mathcal{L}}_{\mu\nu}] = D(\mathcal{L}_{\mu\nu})Q_i = \frac{\gamma_{\mu\nu}}{2}\,Q_i\;;\quad [\bar{Q}_i,\,\hat{\mathcal{L}}_{\mu\nu}] = -\bar{Q}_i\,\frac{\gamma_{\mu\nu}}{2}\,,
\end{aligned} \tag{2.14}$$

having defined $\mathbb{Z}_{ij} = i\,R_{ij}\,\gamma^5 + I_{ij}$.

Exercise 2.: Derive the above relations from Eq.s (2.2)-(2.11).

Consider two finite supersymmetry transformations:

$$g_1 = e^{-i\,\bar{\epsilon}_1\,Q}\;;\quad g_2 = e^{-i\,\bar{\epsilon}_2\,Q}\,, \tag{2.15}$$

where, for the sake of simplicity, we have suppressed the internal index i and $\bar{\epsilon}_1$, $\bar{\epsilon}_2$ are two parameters of the Q-generators. The combination $g_1^{-1}\cdot g_2^{-1}\cdot g_1\cdot g_2$ must belong to the group. If we expand this operator in powers of the parameters, we find:

$$g_1^{-1}\cdot g_2^{-1}\cdot g_1\cdot g_2 = \mathbf{1} + \mathbf{T} + \text{higher order terms}\,. \tag{2.16}$$

The lowest order term is $\mathbf{T} \propto [\bar{\epsilon}_1\,Q,\,\bar{\epsilon}_2\,Q]$, which should belong to the super-Poincaré algebra:

$$[\bar{\epsilon}_1\,Q,\,\bar{\epsilon}_2\,Q] \in \text{ super-Poincaré algebra}\,. \tag{2.17}$$

However Eq. (2.10) tells us that the anti-commutator of two Q's should belong to the algebra. The only way to reconcile these two properties is to assume the parameters of the fermionic generators in the graded-algebra to be Grassmannian numbers, namely 4-spinors. Moreover we also require ϵ_1, ϵ_2 to be Majorana, so that:

$$[\bar{\epsilon}_1\,Q,\,\bar{\epsilon}_2\,Q] = [\bar{\epsilon}_1\,Q,\,\bar{Q}\,\epsilon_2] = \bar{\epsilon}_1\{Q,\,\bar{Q}\}\epsilon_2 \in \text{ super-Poincaré algebra}\,. \tag{2.18}$$

The conditions that both Q and its parameter ϵ be described by Majorana spinors ($\epsilon = C\bar{\epsilon}^T$) derives from the requirement that the infinitesimal supersymmetry transformation $-i\,\bar{\epsilon}Q$ be an anti-hermitian operator on the space of states.

$$(-i\,\bar{\epsilon}Q)^\dagger = i\,(\bar{\epsilon}Q)^\dagger = i\,\bar{\epsilon}Q\,, \tag{2.19}$$

where we have used the property (see Appendix A) that, if χ, λ are both Majorana spinors, $\bar{\chi}\lambda$ is real. Being Q an operator, this property implies that $\bar{\epsilon}Q$ is hermitian. The reason we require an infinitesimal transformation $T = \epsilon^A T_A$ to be anti-hermitian is in order for the corresponding variation of an observable operator (hermitian) be still hermitian:

$$\delta\hat{O} = [\hat{O}, T] \;\Leftrightarrow\; \delta\hat{O}^\dagger = -[\hat{O}^\dagger, T^\dagger] = [\hat{O}, T] = \delta\hat{O}\,, \tag{2.20}$$

having assumed $T^\dagger = -T$ and being $\hat{O}^\dagger = \hat{O}$.

We can define the local correspondence between the super-Poincaré algebra $\mathfrak{sg}$ and the corresponding group SG through the following *exponential map*

$$\frac{\lambda^{\mu\nu}}{2}\,\hat{\mathcal{L}}_{\mu\nu} + x_0^\mu\,\hat{\mathcal{P}}_\mu - i\,\bar{\theta}\,Q \,\in\, \mathfrak{sg} \;\to\; U(\lambda,\, x_0,\, \theta) = e^{\frac{\lambda^{\mu\nu}}{2}\,\hat{\mathcal{L}}_{\mu\nu}}\, e^{x_0^\mu\,\hat{\mathcal{P}}_\mu - i\,\bar{\theta}\,Q} \,\in\, SG\,. \tag{2.21}$$

Consider now the effect of an infinitesimal supersymmetry transformation of a field operator $\hat{\Phi}(x)$

$$\begin{aligned}\hat{\Phi}^m(x) \longrightarrow\; \hat{\Phi}'^m(x') &= e^{i\,\bar{\theta}\,Q}\,\hat{\Phi}^m(x')\,e^{-i\,\bar{\theta}\,Q} = \left[\mathcal{O}_{(\Lambda,\,x_0)}\cdot\hat{\Phi}\right]^m(x') \approx \\ &\approx \hat{\Phi}^m(x') - i\,\left[\bar{\theta}\,\mathcal{O}(Q)\cdot\hat{\Phi}\right]^m(x') = \hat{\Phi}^m(x') + \delta\hat{\Phi}^m(x')\,,\end{aligned} \tag{2.22}$$

where

$$\delta\hat{\Phi}^m(x) = -i\,[\hat{\Phi}^m(x),\,\bar{\theta}\,Q] = -i\,\left[\bar{\theta}\,\mathcal{O}(Q)\cdot\hat{\Phi}\right]^m(x)\,, \tag{2.23}$$

$\mathcal{O}(Q)$ being the *realization* of the supersymmetry generator Q on $\hat{\Phi}(x)$ (we shall suppress its Lorentz index for the sake of notational simplicity). We can compute the effect of two consecutive infinitesimal supersymmetry transformations on $\hat{\Phi}(x)$ parametrized by ϵ_1, ϵ_2. In particular we can evaluate the following commutator (we are using the *passive* description of transformations, see Appendix A):

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})\hat{\Phi} = [\hat{\Phi},\,[\bar{\epsilon}_1\,Q, \bar{\epsilon}_2\,Q]] = 2i\,\bar{\epsilon}_1\gamma^\mu\,\bar{\epsilon}_2\,\partial_\mu\,\hat{\Phi} = \delta_{x_0}\hat{\Phi}\,. \tag{2.24}$$

Exercise 3.: Prove the above equation using the super-Jacobi identities and Eq. (2.18).

Equation (2.24) implies that *the commutator of two consecutive infinitesimal supersymmetry transformations amounts to an infinitesimal translation by a quantity*:

$$x_0^\mu = 2i\,\bar{\epsilon}_1\gamma^\mu\,\bar{\epsilon}_2\,. \tag{2.25}$$

As it is apparent from the relations (2.2)-(2.12), the super-Poincaré algebra has an *automorphism group* G_R acting non trivially on the internal indices $i, j, \ldots$ only:

$$Q_{\alpha i} \to\; U_i{}^j\,Q_{\alpha i} \;;\quad \bar{Q}^i_{\dot{\alpha}} \to\; U^{-1}{}_j{}^i\,\bar{Q}^j_{\dot{\alpha}} \;;\quad Z_{ij} \to\; U_i{}^k\,U_j{}^l\,Z_{kl} \;;\quad Z^{ij} \to\; U^{-1}{}_k{}^i\,U^{-1}{}_l{}^j\,Z^{kl}\,, \tag{2.26}$$

where $U \in G_R$. G_R is named *R-symmetry group*. An *automorphism group* is a transformation on the basis of generators which leaves the structure constants unchanged. As we shall see, massless states arrange themselves in irreducible representations of G_R. From (2.9) it follows that $G_R \subset \mathrm{U}(\mathcal{N})$.

Since $Q_{\alpha\, i}$ and $\bar{Q}^i_{\dot{\alpha}}$ transform differently under a $\mathrm{U}(\mathcal{N})$ transformation, the full R-symmetry group is not manifest in the 4-component Majorana representation of the spinor generators Q_i. The action of G_R on a Majorana spinor can be described as follows. Let us write a generic $\mathrm{U}(\mathcal{N})$ transformation U in the form:

$$U = \exp(A + i\, S) \ \ , \ \ A = -A^T \ \ , \ \ S = S^T\,, \tag{2.27}$$

A, S being real matrices. From (2.26) it follows that (we suppress the i, j indices)

$$Q_\alpha \ \to \ U\, Q_\alpha = e^{A+i\,S}\, Q_\alpha \ ; \quad \bar{Q}_{\dot{\alpha}} \ \to \ U^{-1\,T}\, \bar{Q}_{\dot{\alpha}} = e^{A-i\,S}\, \bar{Q}_{\dot{\alpha}}\,. \tag{2.28}$$

The 4-component Majorana spinor Q_i transforms under the action of a $(2\mathcal{N}) \times (2\mathcal{N})$ matrix $\mathbf{U}$ defined as follows:

$$Q \to \ \mathbf{U}\, Q \ \ , \quad \mathbf{U} = \exp(A \otimes \mathbf{1}_4 - i\, S \otimes \gamma^5) = \begin{pmatrix} e^{A+i\,S} & \mathbf{0} \\ \mathbf{0} & e^{A-i\,S} \end{pmatrix}\,. \tag{2.29}$$

The matrix $\mathbf{U}$ defines the action of the full R-symmetry group G_R on Majorana spinors. We see that only the subgroup of G_R generated by A, that is the part contained in $\mathrm{SO}(\mathcal{N})$, is manifest.

Alternatively we can use the 4-component Weyl representation of spinors and define:

$$\mathbf{Q}_i \equiv \begin{pmatrix} Q_{\alpha\, i} \\ \mathbf{0} \end{pmatrix} = \frac{1-\gamma^5}{2}\, Q_i \ \ ; \quad \mathbf{Q}^i \equiv \begin{pmatrix} \mathbf{0} \\ \bar{Q}^i_{\dot{\alpha}} \end{pmatrix} = \frac{1+\gamma^5}{2}\, Q_i\,. \tag{2.30}$$

The chiral spinors $\mathbf{Q}_i$, $\mathbf{Q}^i$ transform under a $\mathrm{U}(\mathcal{N})$-transformation as $Q_{\alpha\, i}$ and $\bar{Q}^i_{\dot{\alpha}}$, respectively. Moreover one can easily verify that:

$$Q_i = \mathbf{Q}_i + \mathbf{Q}^i \ \ , \quad \mathbf{Q}_i = (\mathbf{Q}^i)_c = C\, \overline{(\mathbf{Q}^i)}^T\,. \tag{2.31}$$

2.2 Poincaré Superspace

If $\hat{\Phi}(x)$ is a scalar field-operator, from Eq.s (A.5) and (A.10) we see that, representing an element of the Poincaré group, on the Hilbert space of states, as

$$U(\Lambda,\, x_0) = U(\Lambda)\, U(x_0) = e^{\frac{\lambda^{\mu\nu}}{2}\, \hat{\mathcal{L}}_{\mu\nu}}\, e^{x_0^\mu \hat{\mathcal{P}}_\mu}\,, \tag{2.32}$$

we have:

$$\hat{\Phi}(x) \overset{(\Lambda,\, x_0)}{\longrightarrow} \ \hat{\Phi}'(x') = U(\Lambda,\, x_0)^\dagger\, \hat{\Phi}(x')\, U(\Lambda,\, x_0) = \hat{\Phi}(\Lambda^{-1}\, x + x_0) = \left[\mathcal{O}_{(\Lambda,\, x_0)} \cdot \hat{\Phi}\right](x')\,. \tag{2.33}$$

In particular:

$$\hat{\Phi}(x_0) = U(x_0)^\dagger U(\Lambda)^\dagger\, \hat{\Phi}(0)\, U(\Lambda)\, U(x_0)\,, \tag{2.34}$$

that is the field operator in x_0 is obtained from its value at the origin $x^\mu = 0$ by means of the Poincaré transformation $U(\Lambda)\, U(x_0)$. Notice that the same correspondence (2.34) is defined by any other element $U(\Lambda')\, U(x_0)$ differing from $U(\Lambda)\, U(x_0)$ by the Lorentz factor. The reason for this is that the Lorentz group leaves the origin of space-time invariant (it is *the stabilizer or little group* of the origin)[4]. We can define an equivalence between Poincaré transformations:

$$U \sim U' \quad \Leftrightarrow \quad U'\, U^{-1} \in \mathrm{O}(1,3)\,, \tag{2.35}$$

and correspondingly group Poincaré transformations in equivalence classes:

$$[U(x_0)] = \{U \in \text{Poincaré group}\ :\ \ U = U(\Lambda)\, U(x_0)\ ,\ \text{for some}\ \ U(\Lambda) \in \mathrm{O}(1,3)\}\,. \tag{2.36}$$

Each space-time point x^μ is therefore in one-to-one correspondence with the equivalence class $[U(x)]$:

$$\hat{\Phi}(0) \stackrel{[U(x)]}{\longrightarrow} \hat{\Phi}(x)\,. \tag{2.37}$$

The equivalence classes $[U(x)]$ are called *left-cosets* and their collection is dubbed *left-coset space*:

$$\{[U(x)]\}_{x\in\mathcal{M}_4} \;=\; \mathrm{SO}(1,3)\backslash\mathrm{ISO}(1,3)\,, \tag{2.38}$$

where $\mathcal{M}_4$ is Minkowski space-time and $\mathrm{ISO}(1,3)$ the Poincaré group. Due to the one-to-one correspondence between $x \in \mathcal{M}_4$ and $[U(x)]$ we can then represent Minkowski space-time by the left-coset space:

$$\mathcal{M}_4 \;=\; \mathrm{SO}(1,3)\backslash\mathrm{ISO}(1,3)\,. \tag{2.39}$$

It is customary to describe Poincaré transformations in terms of an operator L defined as:

$$L(\Lambda,\, x_0) \equiv U(\Lambda,\, x_0)^\dagger = U(x_0)^\dagger\, U(\Lambda)^\dagger = L(x_0)\, L(\Lambda)\,,$$

so that

$$\hat{\Phi}(x_0) = L(x_0) L(\Lambda)\, \hat{\Phi}(0)\, L(\Lambda)^\dagger\, L(x_0)^\dagger\,. \tag{2.40}$$

Just as with the operators $U(x)$, we can define a one-to-one correspondence between points $x \in \mathcal{M}_4$ and *right-cosets* $[L(x)]$:

$$\begin{aligned}\{[L(x)]\}_{x\in\mathcal{M}_4} \;&\equiv\; \{L \in \text{Poincaré group}\ :\ \ L = L(x)\, L(\Lambda)\ ,\ \text{for some}\ \ L(\Lambda) \in \mathrm{O}(1,3)\} \;= \\ &=\; \mathrm{ISO}(1,3)/\mathrm{SO}(1,3)\,. \end{aligned} \tag{2.41}$$

We shall adopt the above description of $\mathcal{M}_4$ in terms of right-cosets.

By the same token, we can define a *superspace* $\mathcal{M}^{(4|\mathcal{N})}$ as a manifold parametrized by the 4 space-time coordinates x^μ and $4\mathcal{N}$ Grassmannian coordinates θ_i as:

$$\mathcal{M}^{(4|\mathcal{N})} \;=\; SG/\mathrm{SO}(1,3)\,, \tag{2.42}$$

[4]In fact the stabilizer of any space-time point x is a group $\mathrm{O}(1,3)_x$ which is isomorphic to the Lorentz group $\mathrm{O}(1,3)$ which stabilizes the origin.

where SG is the super-Poincaré group, whose elements are described as in (2.21):

$$L(\Lambda,\, x,\theta) = L(x,\theta)\, L(\Lambda) = e^{-x^\mu\,\hat{\mathcal{P}}_\mu + i\,\bar{\theta}\,Q}\, e^{-\frac{\lambda^{\mu\nu}}{2}\,\hat{\mathcal{L}}_{\mu\nu}} = U(\Lambda,\, x,\theta)^\dagger\,. \tag{2.43}$$

and the coset space $SG/\mathrm{SO}(1,3)$ is the set of right-cosets $[L(x,\theta)]$, in one-to-one correspondence with the points in $\mathcal{M}^{(4|\mathcal{N})}$. The scalar field in a generic point $(x,\,\theta)$ is then defined as follows

$$\hat{\Phi}(x,\,\theta) \equiv U(x,\,\theta)^\dagger\,\hat{\Phi}(0)\,U(x,\,\theta) = L(x,\,\theta)\,\hat{\Phi}(0)\,L(x,\,\theta)^\dagger\,. \tag{2.44}$$

A field defined over $\mathcal{M}^{(4|\mathcal{N})}$ is called *superfield*. Since Grassmannian numbers ξ are nilpotent, $\xi^2 = 0$, if we Taylor-expand a superfield in θ_i, the expansion terminates at order $4\mathcal{N}$, beyond which any monomial in θ_i would contain some of its Grassmann-components squared, which gives zero. Each coefficient in the θ-expansion in a local field.

Let us now consider the $\mathcal{N} = 1$ case for the sake of simplicity and compute the realization on superfields of the infinitesimal generators $\mathcal{P}_\mu$ and Q_i. Consider the effect of an infinitesimal transformation $L(\xi,\,\epsilon)$ on a superfield $\hat{\Phi}(x,\theta)$:

$$\begin{aligned}\hat{\Phi}(x',\theta') &= L(\xi,\,\epsilon)\hat{\Phi}(x,\theta)L(\xi,\,\epsilon)^\dagger = L(x',\theta')\,\hat{\Phi}(0,0)\,L(x',\theta')^\dagger = \hat{\Phi}(x,\theta) + \delta\hat{\Phi}(x,\theta)\,,\\ L(x',\theta') &= L(\xi,\,\epsilon)L(x,\theta)\;\;,\;\;\delta\hat{\Phi}(x,\theta) = (\xi^\mu\mathcal{O}(\mathcal{P}_\mu) - i\,\bar{\epsilon}\,\mathcal{O}(Q))\cdot\hat{\Phi}(x,\theta)\,.\end{aligned} \tag{2.45}$$

To evaluate this effect we should compute x', θ' in terms of x, θ and expand $\hat{\Phi}(x',\theta')$ up to first order terms in the infinitesimal parameters. To this end let us use the Baker-Campbell-Hausdorff (BCH) formula:

$$e^A\, e^B = e^C\;\;;\;\; C = A + B + \frac{1}{2}\,[A,B] + \dots \tag{2.46}$$

to compute $U(x',\theta')$

$$\begin{aligned}L(x',\theta') &= L(\xi,\,\epsilon)L(x,\theta) = e^{-\xi\cdot\hat{\mathcal{P}} + i\,\bar{\epsilon}\,Q}\, e^{-x\cdot\hat{\mathcal{P}} + i\,\bar{\theta}\,Q} = \\ &= \exp\left(-(x+\xi)\cdot\hat{\mathcal{P}} - \frac{1}{2}\,\bar{\epsilon}\{Q,\,\bar{Q}\}\,\theta + i\,(\bar{\theta}+\bar{\epsilon})\,Q\right) = e^{-x'\cdot\hat{\mathcal{P}} + i\,\bar{\theta}'\,Q}\,,\end{aligned} \tag{2.47}$$

where the ellipses in (2.46) represent higher order terms involving commutators of commutators, which vanish when A and B are combinations of $\mathcal{P}_\mu$ and Q, since their commutator only involves $\mathcal{P}_\mu$ which commutes with both A and B. From the second of Eq.s (2.14) we find:

$$x'^\mu = x^\mu + \xi^\mu - i\,\bar{\theta}\gamma^\mu\,\epsilon = x^\mu + \delta x^\mu\;,\quad \theta' = \theta + \epsilon\,. \tag{2.48}$$

In the 2-component notation:

$$\delta x^\mu = \xi^\mu - i\,\left(\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\,\bar{\epsilon}^{\dot{\beta}} + \bar{\theta}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\,\epsilon_\beta\right)\,. \tag{2.49}$$

Expanding $\hat{\Phi}(x',\theta')$ to first order in the infinitesimal parameters ξ^μ, ϵ, as in the last of Eq.s (2.45), we derive the expression for $\mathcal{O}(\mathcal{P}_\mu)$, $\mathcal{O}(Q)$:

$$\begin{aligned}\hat{\Phi}(x',\theta') &= \hat{\Phi}(x,\theta)+\delta x^\mu\partial_\mu\hat{\Phi}(x,\theta)+\bar{\epsilon}\,\partial\hat{\Phi}(x,\theta)=\\ &=\hat{\Phi}(x,\theta)+(\xi^\mu-i\,\bar{\theta}\gamma^\mu\,\epsilon)\,\partial_\mu\hat{\Phi}(x,\theta)+\bar{\epsilon}\,\frac{\partial}{\partial\bar{\theta}}\hat{\Phi}(x,\theta)=\\ &=\hat{\Phi}(x,\theta)+(\xi^\mu\mathcal{O}(\mathcal{P}_\mu)-i\,\bar{\epsilon}\,\mathcal{O}(Q))\cdot\hat{\Phi}(x,\theta)\,,\end{aligned}\tag{2.50}$$

where we have defined the *spinorial derivative* ∂ as follows:

$$\frac{\partial}{\partial\bar{\theta}}\equiv\begin{pmatrix}\frac{\partial}{\partial\theta^\alpha}\\ \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\end{pmatrix}\quad,\quad\frac{\partial}{\partial\theta}\equiv\left(\frac{\partial}{\partial\theta_\alpha},\,\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right)\,.\tag{2.51}$$

Using the Majorana condition on θ ($\bar{\theta}=\theta^T\,C$), the reader can verify that:

$$\left(\frac{\partial}{\partial\theta}\right)^T=C\,\frac{\partial}{\partial\bar{\theta}}\,.\tag{2.52}$$

From (2.50) we find:

$$\mathcal{O}(Q)=i\,\frac{\partial}{\partial\bar{\theta}}-\gamma^\mu\theta\,\partial_\mu\ \ ;\quad\mathcal{O}(\mathcal{P}_\mu)=\partial_\mu\,.\tag{2.53}$$

From the Majorana condition on Q and (2.52) we derive:

$$\mathcal{O}(\bar{Q})=\mathcal{O}(\bar{Q})^T\,C=i\left(\frac{\partial}{\partial\bar{\theta}}\right)^T C-\theta^T(\gamma^\mu)^T\,C\,\partial_\mu=-i\,\frac{\partial}{\partial\theta}+\bar{\theta}\,\gamma^\mu\,\partial_\mu\,.\tag{2.54}$$

Exercise: Verify that the generator $\mathcal{O}(Q)$, in the two-component notation, reads:

$$\begin{aligned}\mathcal{O}(Q_\alpha)&=i\,\frac{\partial}{\partial\theta^\alpha}-(\sigma^\mu\,\bar{\theta})_\alpha\,\partial_\mu\,,\\ \mathcal{O}(\bar{Q}_{\dot{\alpha}})&=-i\,\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}+(\theta\sigma^\mu)_{\dot{\alpha}}\,\partial_\mu\,,\end{aligned}\tag{2.55}$$

Exercise: Verify that $\mathcal{O}(Q)$ and $\mathcal{O}(\bar{Q})$, in (2.53) and (2.54), satisfy the second of Eq.s (2.14).

Notice that $\mathcal{O}(\mathcal{P}_\mu)$, $\mathcal{O}(Q)$ describe the effect of a *left-multiplication* on $L(x,\theta)$ by means of an infinitesimal transformation $L(\xi,\epsilon)$, see (2.47). The reader can indeed easily verify that:

$$\begin{aligned}L(x',\theta')&=L(\xi,\,\epsilon)L(x,\theta)=e^{-\xi\cdot\hat{\mathcal{P}}+i\,\bar{\epsilon}\,Q}e^{-x\cdot\hat{\mathcal{P}}+i\,\bar{\theta}\,Q}=L(x+\delta x,\theta+\epsilon)=\\ &=(\mathbf{1}+\xi^\mu\mathcal{O}(\mathcal{P}_\mu)-i\,\bar{\epsilon}\,\mathcal{O}(Q))\cdot L(x,\theta)\,.\end{aligned}\tag{2.56}$$

Under the action of a supersymmetry transformation on a superfield, the local fields over Minkowski space-time defining the coefficients of its θ-expansion, transform into one another: fermionic fields into bosonic ones and viceversa. These field-components define a representation of the super-Poincaré algebra. In general such representation is not irreducible. In order for the components of a superfield to define irreducible representations, certain differential constraints on it have to be imposed. These are defined in terms of spinorial differential operators in superspace (just like $\mathcal{O}(Q)$), called *supercovariant derivatives* $D = (D_\alpha,\, \bar{D}^{\dot\alpha})$. In order for these constraints on superfields to be supersymmetry-invariant, the supercovariant derivatives must commute with the supersymmetry transformations, implemented on superfields by the differential operators $\mathcal{O}(Q)$:

$$[\bar\epsilon_1\, D,\, \bar\epsilon_2\, \mathcal{O}(Q)] = 0\,. \tag{2.57}$$

The above condition is satisfied if we define D as the generator of a *right-multiplication* on $L(x, \theta)$ by a supersymmetry transformation. Left and right-multiplications, by group elements L_1 and L_2 respectively, on a third element L commute by virtue of the associative property of the group product:

$$(L_1 \cdot L) \cdot L_2 = L_1 \cdot (L \cdot L_2)\,. \tag{2.58}$$

We then define D_μ and D as follows:

$$L(x,\theta)L(\xi,\,\epsilon) = e^{-x\cdot\hat{\mathcal{P}}+i\,\bar\theta\, Q}\, e^{-\xi\cdot\hat{\mathcal{P}}+i\,\bar\epsilon\, Q} = (\mathbf{1} + \xi^\mu\, D_\mu + \bar\epsilon\, D)\cdot L(x,\theta)\,. \tag{2.59}$$

Using the BCH formula (2.46) as above, we find:

$$D = \frac{\partial}{\partial\bar\theta} - i\,\gamma^\mu\theta\,\partial_\mu \;\Rightarrow\; \begin{cases} D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\bar\theta)_\alpha\,\partial_\mu \\ \bar{D}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\,(\bar\sigma^\mu\theta)^{\dot\alpha}\,\partial_\mu \;\Rightarrow\; \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\,(\theta\sigma^\mu)_{\dot\alpha}\,\partial_\mu \end{cases}. \tag{2.60}$$

Exercise: Verify Eq. 2.57. The reader can verify the following anticommutation relations:

$$\{D_\alpha,\,\bar{D}_{\dot\alpha}\} \;=\; 2i\,(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu\,,\;\; \{D_\alpha,\,D_\beta\} = \{\bar{D}_{\dot\beta},\,\bar{D}_{\dot\alpha}\} = 0 \tag{2.61}$$

2.3 Representations of the Super-Poincaré Algebra

We construct the this section the single-particle (unitary) irreducible representations of SG. We start noticing that, since Q commutes with $\hat{\mathcal{P}}_\mu$, all SG commutes with the mass-squared operator $\propto \hat{\mathcal{P}}_\mu\hat{\mathcal{P}}^\mu$. Therefore, just as for the Poincaré representations, all states in an irreducible representation of SG have the same mass. This is not the case for the spin since, as the reader can easily verify, the fermionic operators Q do not commute with the Pauli-Lubanski operator $\hat{W}_\mu$ (see Appendix A for the definition of $\hat{W}_\mu$). A representation of SG will therefore contain states with different spin. In general, if V is the carrier of such a representation, we can split it in the direct sum of two spaces V_F, V_B consisting of of fermionic and bosonic states, respectively:

$$V = V_B \oplus V_F\,. \tag{2.62}$$

The action of Q on a state changes its spin by 1/2, and thus its statistics. We have:

$$V_B \;\xrightarrow{Q}\; V_F \;\xrightarrow{Q}\; V_B\,, \tag{2.63}$$

that is $Q \cdot V_B \subset V_F$ and $Q \cdot Q \cdot V_B \subset Q \cdot V_F \subset V_B$. The latter space of states resulting from the consecutive action of two infinitesimal supersymmetry transformations contains those states originating from the action of the commutator of two supersymmetry transformations which in turn coincides with infinitesimal translations:

$$\hat{\mathcal{P}} \cdot V_B = \{Q,\, Q\} \cdot V_B \subset V_B\,. \tag{2.64}$$

In a unitary (infinite-dimensional) representation of SG, $\hat{\mathcal{P}}$ is a *semisimple* operator and the action of translations is *free*, namely there exists no state which is invariant under all space-time translations (the only state with such property is the vacuum which defines a trivial representation of SG). As a consequence of this $\hat{\mathcal{P}} \cdot V_B = V_B$ and therefore (2.63) is onto and thus invertible. This implies

$$\dim(V_B) = \dim(V_F)\,, \tag{2.65}$$

that is *the numbers of fermionic and bosonic states in an irreducible representation of SG coincide.*

Supersymmetry on Minkowski space (described by the super-Poincaré group) has an important implication on the energy of a state. Consider the following expectation value on a state $|a\rangle$:

$$\langle a|\{Q_{\alpha\, i},\, (Q_{\alpha\, i})^\dagger\}|a\rangle = \langle a|\{Q_{\alpha\, i},\, \bar{Q}_{\dot\alpha}{}^i\}|a\rangle = 2\,(\sigma^\mu)_{\alpha\dot\alpha}\, p_\mu\,, \tag{2.66}$$

where $p_\mu = \langle a|\hat{P}_\mu|a\rangle$. If we trace over α, $\dot\alpha$, the only matrix σ^μ surviving is $\sigma^0 = \mathbf{1}$, so that we have

$$\sum_{\alpha=1}^{2}\langle a|\{Q_{\alpha\, i},\, (Q_{\alpha\, i})^\dagger\}|a\rangle = 4\, E\,, \tag{2.67}$$

E being the expectation energy of $|a\rangle$. If we assume the inner product $\langle\cdot|\cdot\rangle$ in the Hilbert space of states to be positive definite, the left hand side is a non-negative number, being

$$\sum_{\alpha=1}^{2}\langle a|\{Q_{\alpha\, i},\, (Q_{\alpha\, i})^\dagger\}|a\rangle = \sum_{\alpha=1}^{2}\left(||(Q_{\alpha\, i})^\dagger|a\rangle||^2 + ||Q_{\alpha\, i}|a\rangle||^2\right)\,. \tag{2.68}$$

As a consequence of (2.67) then $E \geq 0$. Since for a particle state $E \neq 0$, we conclude that *super-Poincaré algebra implies positivity of energy.* An other implication of the above derivation is that single particle states are never annihilated by supersymmetry generators ($(Q_i)^\dagger|a\rangle$ or $Q_i|a\rangle \neq 0$, for any i). Indeed if, for some i, $Q_i|p, s\rangle = (Q_i)^\dagger|a\rangle = 0$, computing the norm of the state and summing over the spinor indices of Q_i, we would find

$$0 = \sum_{\alpha=1}^{2}\left(||(Q_{\alpha\, i})^\dagger|a\rangle||^2 + ||Q_{\alpha\, i}|a\rangle||^2\right) = 4E\,, \tag{2.69}$$

that is $E = 0$, which cannot be for a particle state. This last implication, as we shall see, only holds for those states on which the central charge generators have a vanishing expectation value Z_{ij}. In the presence of a non-trivial central charge Z_{ij}, suitable combinations of Q

and $Q^\dagger$ may annihilate the state, and thus a fraction of the original supersymmetries may be preserved by it. A non-vanishing Z_{ij} is related to the electric-magnetic charges of the solution.

The only state for which E can be 0 is the *vacuum* $|0\rangle$. Since Eq. (2.69) holds for any i, a vacuum state with zero energy must preserve all supersymmetries

$$Q_i|0\rangle = (Q_i)^\dagger|0\rangle = 0 \qquad i = 1, \dots, \mathcal{N}\,. \tag{2.70}$$

The above argument seems to imply that there cannot be a *spontaneous partial supersymmetry breaking*: Either all supersymmetries are broken by the vacuum (and this occurs if and only if the vacuum energy is non-vanishing) or they are all preserved by the vacuum ($E = 0$ case). This no-go theorem was proven in the eighties not to be correct [15]. Indeed it can be proven that in the presence of a partial supersymmetry breaking the anticommutator of supercharges is not well defined. The appropriate way of describing symmetries in a local field theory is in terms of symmetry currents (Noether currents) and their algebra. In particular we can write the (local) anticommutator between a supersymmetry generator and a supercurrent which has the form:

$$\{Q_{\alpha i},\, \bar{J}^{\dot\beta j}_\mu(x)\} = 2\,i\,\delta_i^j\,(\sigma^\nu)_\alpha{}^{\dot\beta}\,T_{\nu\mu}(x)\,. \tag{2.71}$$

The reader can verify that integrating the $\mu = 0$ component of the above equation one finds the relation (2.9). Eq. (2.71) can however be generalized as follows:

$$\{Q_{\alpha i},\, \bar{J}^{\dot\beta j}_\mu(x)\} = 2\,i\,\delta_i^j\,(\sigma^\nu)_\alpha{}^{\dot\beta}\,T_{\nu\mu}(x) + (\sigma_\mu)_\alpha{}^{\dot\beta}\,C_i{}^j\,. \tag{2.72}$$

where the matrix $C_i{}^j$ consists of constant c-numbers whose effect on any field of the theory is trivial. It describes a central extension of the current algebra (2.71). It implies a constant shift in the energy density $\mathcal{H} \to \mathcal{H}+c_i$, which depends on the direction in the supersymmetry parameter space.[5] This feature is crucial in order to evade the aforementioned no-go theorem. The first $\mathcal{N} = 2$ globally supersymmetric theory featuring a non-vanishing matrix $C_i{}^j$ and thus a partial spontaneous supersymmetry breaking was constructed in [16].

In supergravity the no-go theorem does not apply in the first place and spontaneous partial supersymmetry breaking can occur [17]. As a last remark, we notice that in supersymmetric models featuring spontaneous partial supersymmetry breaking, and thus a non-vanishing matrix $C_i{}^j$, integrating both sides of (2.72) over an infinite volume in order to retrieve the anticommutation relation between supercharges, the central extension gives an infinite contribution. This implies that in the presence of spontaneous partial supersymmetry breaking the anticommutation relations between supersymmetry generators are ill defined.

To construct single-particle irreducible representations of SG, we use the method of *induced representations*: We consider a basis of eigenstates $|p, s\rangle$ of the 4-momentum operator $\hat{P}_\mu$ and of its little group (helicity group SO(2) for massless particles or spin group SU(2) for massive ones). We construct the states $|\bar{p}, s\rangle$ in a given frame of reference S_0 where the 4-momentum is simplest $\bar{p} = (\bar{p}_\mu)$, and then Lorentz-boost them to a frame S where the momentum is generic p_μ.

[5] Constant shifts in the energy density are irrelevant in the absence of gravity, that is in globally supersymmetric theories.

Massless states. We start from a frame S_0 in which

$$p_\mu = \bar{p}_\mu = (E, 0, 0, -E) = (E, -\bar{\mathbf{p}})\,,$$

The P.-L. operator reads:

$$\langle \hat{W}_\mu \rangle = \bar{p}_\mu \, \langle \hat{\Gamma} \rangle \;;\; \Rightarrow \quad \langle \hat{W}_0 \rangle = E \, \langle \hat{\Gamma} \rangle\,, \tag{2.73}$$

where $\hat{\Gamma}$ is the helicity operator:

$$\langle \hat{\Gamma} \rangle = \frac{\bar{\mathbf{p}} \cdot \langle \hat{\mathbf{J}} \rangle}{|\bar{\mathbf{p}}|} = \langle \hat{J}_3 \rangle\,. \tag{2.74}$$

We can describe the states in S_0 in terms of eigenmatrices $|E,\, \pm s\rangle$ of $\hat{\Gamma}$:

$$\hat{\Gamma}|E,\, \pm s\rangle \,=\, \pm s\, |E,\, \pm s\rangle \;;\;\; \hat{W}_0\, |E,\, \pm s\rangle \,=\, \pm s\, E\, |E,\, \pm s\rangle\,. \tag{2.75}$$

Next we compute the commutator of the angular momentum operator with the supersymmetry generator $Q_i = (Q_{\alpha\, i})$:

$$[\hat{J}_I,\, Q_i] = -(S_I\, Q_i) = -\frac{\sigma_I}{2}\, Q_i\,, \tag{2.76}$$

so that

$$[\hat{\Gamma},\, Q_i] = -\frac{\sigma_3}{2}\, Q_i\,. \tag{2.77}$$

If we act on $|E,\, \lambda\rangle$ by means of Q_i, we change the helicity λ of the state as follows:

$$\hat{\Gamma} Q_i |E,\, \lambda\rangle = ([\hat{\Gamma},\, Q_i] + Q_i \hat{\Gamma})|E,\, \lambda\rangle = (\lambda\, \mathbf{1} - \frac{\sigma_3}{2}) Q_i |E,\, \lambda\rangle\,, \tag{2.78}$$

so that, modulo a normalization factor,

$$Q_{1,\, i}|E,\, \lambda\rangle = |E,\, \lambda - \frac{1}{2}\rangle \;;\;\; Q_{2,\, i}|E,\, \lambda\rangle = |E,\, \lambda + \frac{1}{2}\rangle\,. \tag{2.79}$$

The action of $Q_{1,\, i}$ "lowers" the helicity by 1/2 while that of $Q_{2,\, i}$ "raises" the helicity by the same amount. The opposite holds for $\bar{Q}^i_{\dot{\alpha}}$: $\bar{Q}^i_{\dot{1}}$ "raises" while $\bar{Q}^i_{\dot{2}}$ "lowers" the helicity by 1/2.

Exercise 4.: Check this.

We choose $Z_{ij} = 0$, the motivation for this will become clear when we deal with massive representations. From the supersymmetry algebra we have:

$$\begin{aligned} \{Q_{\alpha\, i},\, \bar{Q}_{\dot{\alpha}}{}^j\} &= 2\, \delta_i^j\, \sigma^\mu \bar{p}_\mu = 2\, \delta_i^j\, E\, (\mathbf{1} - \sigma_3)\,, \\ \{Q,\, Q\} &= \{\bar{Q},\, \bar{Q}\} = 0\,, \end{aligned} \tag{2.80}$$

so that

$$\{Q_{1\, i},\, \bar{Q}_{\dot{1}}{}^j\} = 0 \;;\;\; \{Q_{2\, i},\, \bar{Q}_{\dot{2}}{}^j\} = 4\, E\, \delta_i^j\,. \tag{2.81}$$

The first of the above relations implies that $Q_{1\,i}$, $(Q_{1\,i})^\dagger$ vanish on the states:

$$0 = \langle a|\{Q_{1\,i},\, \bar{Q}_{\dot{1}}{}^{i}\}|a\rangle = ||\bar{Q}_{\dot{1}}{}^{i}|a\rangle||^2 + ||Q_{1\,i}|a\rangle||^2 \;\Rightarrow\; \bar{Q}_{\dot{1}}{}^{i}|a\rangle = Q_{1\,i}|a\rangle = 0\,, \tag{2.82}$$

for any $|a\rangle$. Let us define the generators

$$q_i \equiv \frac{1}{2\sqrt{E}}\, Q_{2\,i}\,. \tag{2.83}$$

These operators satisfy the relations:

$$\{q_i,\, \bar{q}^j\} = \delta_i^j\,, \tag{2.84}$$

and thus generate the *Clifford algebra* of $\mathcal{N}$ fermionic degrees of freedom. Any irreducible representation of such an algebra is constructed out of a Clifford *ground state* $|E,\, \lambda_0\rangle$ ($\lambda_0 > 0$), defined by the condition:

$$q_i\,|E,\, \lambda_0\rangle = 0 \qquad \forall i = 1, \dots, \mathcal{N}\,, \tag{2.85}$$

by applying to its the "raising" operators $\bar{q}^i$. The manifest automorphism group of this algebra is $G_R = \mathrm{U}(\mathcal{N})$. The states arrange in irreducible representations of G_R:

$$\begin{aligned} &|E,\, \lambda_0\rangle \\ |E,\, \lambda_0 - \tfrac{1}{2},\, [i]\rangle &\propto \bar{q}^i\,|E,\, \lambda_0\rangle \\ |E,\, \lambda_0 - 1,\, [ij]\rangle &\propto \bar{q}^j\,\bar{q}^i\,|E,\, \lambda_0\rangle \\ &\vdots \\ |E,\, \lambda_0 - \frac{k}{2},\, [i_1 i_2 \dots i_k]\rangle &\propto \bar{q}^{i_1}\,\dots\,\bar{q}^{i_k}\,|E,\, \lambda_0\rangle\,. \end{aligned} \tag{2.86}$$

The states with a given helicity $\lambda = \lambda_0 - \frac{k}{2}$, define the k-fold antisymmetric representation of the R-symmetry group $\mathrm{U}(\mathcal{N})$. Therefore for each λ, there are $\binom{\mathcal{N}}{k}$ such states. The lowest helicity state corresponds to $k = \mathcal{N}$ and has degeneracy 1.

Invariance under CPT of any Lorentz-invariant field theory, requires any representation of the symmetry group SG to contain both helicity states for each spin, since

$$CPT|E,\, \lambda\rangle = |E,\, -\lambda\rangle\,. \tag{2.87}$$

The action of CPT on a spinor is $\chi \to \eta\,\chi^*$, η being a phase. CPT therefore maps q_i into $\bar{q}^i$. If an irreducible representation contains the states:

$$\bar{q}^{i_1}\,\dots\,\bar{q}^{i_\mathcal{N}}\,|E,\, \lambda_0\rangle\;;\;\dots\;;\;\bar{q}^i|E,\, \lambda_0\rangle\;;\;|E,\, \lambda_0\rangle\,, \tag{2.88}$$

it should also contain their CPT conjugates

$$|E,\, -\lambda_0\rangle\;;\;q_i|E,\, -\lambda_0\rangle\;;\;\dots\;;\;q_{i_1}\,\dots\,q_{i_\mathcal{N}}\,|E,\, \lambda_0\rangle\,, \tag{2.89}$$

obtained by acting on $|E, -\lambda_0\rangle$ by means of q_i. To construct the particle states in S_0 with highest spin λ_0 we start from λ_0 and construct the states with helicities $\lambda_0-1/2, \ldots, \lambda_0-\mathcal{N}/2$ by applying $\bar{q}^i$. At the same time we start from the unique state with $-\lambda_0$ and, by consecutive applications of q_i, we construct the states with helicities $-\lambda_0+1/2, \ldots, -\lambda_0+\mathcal{N}/2$:

$$\begin{bmatrix} & & \overset{\bar{q}^i}{\longleftarrow} & \lambda_0-\frac{1}{2} & \lambda_0 \\ -\lambda_0 & -\lambda_0+\frac{1}{2} & \underset{}{\overset{q_i}{\longrightarrow}} & & \end{bmatrix}, \tag{2.90}$$

where each helicity $\pm(\lambda_0-k/2)$ comes with a multiplicity $\begin{pmatrix}\mathcal{N}\\ k\end{pmatrix}$. For instance one can construct the $\mathcal{N}=1$ supermultiplet with maximum spin $\lambda_0=1/2$, called the Wess-Zumino massless multiplet:

$$\begin{bmatrix} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \end{bmatrix} = 1\times(\frac{1}{2})+2\times(0)\,, \tag{2.91}$$

where we have used the notation (s) to denote the collection of the two spin-s helicity states $\pm s$. The supermultiplet contains one massless spin-1/2 field and two real (massless) scalars. In contains 2 fermionic and two bosonic on-shell degrees of freedom.

An other example is the $\mathcal{N}=2$ vector multiplet, with $\lambda_0=1$:

$$\begin{bmatrix} & & 0 & \frac{1}{2} & 1 \\ -1 & -\frac{1}{2} & 0 & & \end{bmatrix} = 1\times(1)+2\times(\frac{1}{2})+2\times(0)\,. \tag{2.92}$$

It contains one spin-1 field A_μ, two spin-1/2 fields λ^i and two real scalar fields which arrange in a complex one. In contains 4 fermionic and 4 bosonic on-shell degrees of freedom.

Each row in (2.90) contains $2^{\mathcal{N}}$ states since

$$2^{\mathcal{N}} = \sum_{k=0}^{\mathcal{N}} \begin{pmatrix}\mathcal{N}\\ k\end{pmatrix} = n_F + n_B\,, \tag{2.93}$$

where n_F and n_B are the number of fermionic and bosonic states corresponding to odd (even) and even (odd) k, respectively, when λ_0 is integer (half-integer). From elementary number theory it follows that $n_F=n_B$. If the supermultiplet contains both rows of (2.90) then the number of states is $2^{\mathcal{N}+1}$. There are two important exceptions to this, i.e. cases in which $\lambda_0-\mathcal{N}/2=-\lambda_0$. They occur for $\mathcal{N}=8$ and $\lambda_0=2$, $\mathcal{N}=4$, $\lambda_0=1$ and $\mathcal{N}=2$, $\lambda_0=1/2$. In the first two cases the supermultiplet consists of only one of the two rows in (2.90), since each of them are separately *CPT-self-conjugate*:

$\underline{\mathcal{N}=8}$

$$\begin{aligned}&\left[-2\,,\ -\frac{3}{2}\,,\ -1\,,\ -\frac{1}{2}\,,\ 0\,,\ \frac{1}{2}\,,\ 1\,,\ \frac{3}{2}\,,\ 2\right] = 1\times(2)+8\times(\frac{3}{2})+28\times(1)+\\ &+56\times(\frac{1}{2})+70\times(0)\,,\end{aligned} \tag{2.94}$$

$\underline{\mathcal{N}=4}$

$$\left[-1\,,\ -\frac{1}{2}\,,\ 0\,,\ \frac{1}{2}\,,\ 1\right] = 1\times(1)+4\times(\frac{1}{2})+6\times(0)\,. \tag{2.95}$$

The former supermultiplet contains a spin-2 state which can be identified with the graviton, eight spin 3/2 states to be identified with the *gravitino* fields Ψ^i_μ in the **8** of SU(8), 28 vector fields A^{ij}_μ in the 2-fold antisymmetric representation of SU(8), 56 spin-1/2 states described by the so-called *dilatino* fields λ^{ijk} in the 3-fold antisymmetric representation of SU(8) and, finally 70 fields ϕ^{ijkl} in the 4-fold antisymmetric representation of SU(8). The matching of the 128-fermionic and 128-bosonic on-shell degrees of freedom requires ϕ^{ijkl} to satisfy the following reality condition:

$$\phi^{ijkl} = \frac{1}{4!}\,\epsilon^{ijklpqrs}\,(\phi^{pqsr})^*\,. \tag{2.96}$$

The above condition breaks the automorphism group G_R from U(8) to SU(8), which is the R-symmetry group of the maximal theory.

Similarly the second (2.95) describes the gauge-supermultiplet of the $\mathcal{N} = 4$ theory. It consists of a single gauge field A_μ, 4 spin-1/2 states in the fundamental representation of SU(4) and 6 real fields ϕ^{ij} in the 2-fold antisymmetric representation of SU(4). Also in this case ϕ^{ij} are subject to a reality condition:

$$\phi^{ij} = \frac{1}{2}\,\epsilon^{ijkl}\,(\phi^{kl})^*\,, \tag{2.97}$$

which reduces the R-symmetry group to SU(4). The two multiplets (2.94) and (2.95) are called *self-dual* and contain $2^{\mathcal{N}}$ degrees of freedom.

The $\mathcal{N} = 2$, $\lambda_0 = 1/2$ multiplet is different from the previous ones since it contains a complex SU(2)-doublet of scalars ϕ^i on which a reality condition cannot be imposed[6]. Each row in (2.90) is therefore not CPT-self-conjugate and we need to consider both both of them, one containing ϕ^i and the other its CPT-conjugate $\phi_i = (\phi^i)^*$:

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = 2 \times (\frac{1}{2}) + 4 \times (0)\,. \tag{2.98}$$

This is the *massless* $\mathcal{N} = 2$ *hypermultiplet*. It represents the matter sector of an $\mathcal{N} = 2$ theory. As we shall see, there is also a *massive* $\mathcal{N} = 2$ *hypermultiplet* with a non-vanishing central charge.

As a other example let us give the massless $\mathcal{N} = 3$, $\lambda_0 = 2$ and $\lambda_0 = 1$ multiplets:

$$\begin{bmatrix} & & & & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ -2 & -\frac{3}{2} & -1 & -\frac{1}{2} & & & & \end{bmatrix} = 1 \times (1) + 3 \times (\frac{3}{2}) + 3 \times (1) + 1 \times (\frac{1}{2})\,,$$

$$\begin{bmatrix} & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \\ -1 & -\frac{1}{2} & 0 & \frac{1}{2} & \end{bmatrix} = 1 \times (1) + (3+1) \times (\frac{1}{2}) + (3+3) \times (0)\,,$$

Exercise 5.: Compute the massless $\lambda_0 = 2$ supermultiplet (supergravity multiplet) in $\mathcal{N} = 4$.

Massive case with $Z_{ij} = 0$. Let us now consider massive states on which the central charges Z_{ij} vanish. The corresponding irreducible representations are called *long multiplets*.

[6]A condition $\phi^i = \epsilon^{ij}\,(\phi^j)^*$, in analogy with Eq.s (2.96), (2.97), is not consistent, being non-involutive.

We start from the rest frame S_0 in which $\bar{p}^\mu = (m, 0, 0, 0)$. The states define representations of the spin group SU(2): $|m, s, s_3\rangle$. The action of $Q_{\alpha\, i}$ changes the spin by 1/2:

$$Q_{\alpha\, i}|m, s, s_3\rangle = \sum_{s_3'=-(s-\frac{1}{2})}^{s-\frac{1}{2}} a^{(-)}_{s,s_3'}|m, s-\frac{1}{2}, s_3'\rangle + \sum_{s_3'=-(s+\frac{1}{2})}^{s+\frac{1}{2}} a^{(+)}_{s,s_3'}|m, s+\frac{1}{2}, s_3'\rangle \,., \tag{2.99}$$

If we define:

$$q_{\alpha\, i} \equiv \frac{1}{\sqrt{2m}}\, Q_{\alpha\, i} \;;\;\; \bar{q}^i_{\dot\alpha} \equiv (q_{\alpha\, i})^\dagger = \frac{1}{\sqrt{2m}}\, \bar{Q}^i_{\dot\alpha}\,, \tag{2.100}$$

then, from Eq. (2.9) we have:

$$\{q_{\alpha\, i},\, \bar{q}^j_{\dot\alpha}\} = \delta^j_i\, \delta_{\alpha\dot\alpha}\,. \tag{2.101}$$

The operators $q_{\alpha\, i}$, $\bar{q}^i_{\dot\alpha}$ generate a Clifford algebra of a system with $2\mathcal{N}$ fermionic degrees of freedom. As in the previous case, the states of an irreducible representation of the algebra are obtained by acting by means of $\bar{q}^i_{\dot\alpha}$ on a Clifford-ground-state $|\Omega\rangle \equiv \{|m, s_0, s_3\rangle\}_{s_3=-s_0,\dots,s_0}$, described in this case by a spin-s_0 irreducible representation of SU(2), and defined by the condition:

$$q_{\alpha\, i}|\Omega\rangle = 0 \quad , \quad \forall \alpha,\, i\,. \tag{2.102}$$

The states have the form:

$$\bar{q}^{i_1}_{\dot\alpha_1} \dots \bar{q}^{i_k}_{\dot\alpha_k}|m, s_0, s_3\rangle\,, \tag{2.103}$$

and, for each s_3, are $2^{2\mathcal{N}}$, so that:

$$\text{total number of states} \;=\; (2s_0+1)\times 2^{2\mathcal{N}}\,. \tag{2.104}$$

Recall that:

$$\hat{J}_3\, q_i|m, s_0, s_3\rangle = ([\hat{J}_3,\, q_i] + q_i\hat{J}_3)|m, s_0, s_3\rangle = (s_3\,\mathbf{1} - \frac{\sigma_3}{2})q_i|m, s_0, s_3\rangle\,, \tag{2.105}$$

so that $q_{1\, i}$ lowers s_3 by 1/2 and $q_{2\, i}$ raises s_3 by 1/2, while $\bar{q}^i_1$ raises s_3 by 1/2 and $\bar{q}^i_2$ lowers s_3 by 1/2 . The highest spin is $s_0 + \mathcal{N}/2$, while the lowest is $s_0 - \mathcal{N}/2$ if $s_0 > \mathcal{N}/2$, zero otherwise.

Since $Q_{\alpha\, i}$ and $\bar{Q}^{\dot\alpha\, i}$ transform under the spin-SU(2) by the same matrices, it is useful to define the following $2\mathcal{N}$-component vector $Q_{\alpha\, a}$ for each spinor component α:

$$Q_{\alpha\, a} \equiv (Q_{\alpha\, i},\, \bar{Q}^{\dot\alpha\, i})\;,\;\; a = 1, \dots, 2\mathcal{N}\,. \tag{2.106}$$

Similarly we define

$$\bar{Q}^a_{\dot\alpha} \equiv (Q_{\alpha\, a})^\dagger = (\bar{Q}^i_{\dot\alpha},\, Q^\alpha_i)\;,\;\; a = 1, \dots, 2\mathcal{N}\,. \tag{2.107}$$

In the presence of Z_{ij}, Eq.s (2.10), (2.9) and (2.11) in the rest frame can be recast in a more compact form:

$$\{Q_{\alpha\, a},\, Q_{\beta\, b}\} = 2\,\epsilon_{\alpha\beta}\,\Lambda_{ab}\,, \tag{2.108}$$

where

$$\Lambda_{ab} \equiv \begin{pmatrix} Z_{ij} & -m\,\delta_i{}^k \\ m\delta^l{}_j & Z^{lk} \end{pmatrix}\,. \tag{2.109}$$

Exercise 6.: Derive Eq. (2.108).

Similarly

$$\{\bar{Q}^a_{\dot\alpha},\, \bar{Q}^b_{\dot\beta}\} = -2\,\epsilon_{\dot\alpha\dot\beta}\,\Lambda^{ab} = 2\,\epsilon_{\alpha\beta}\,\Lambda^{ab}\,, \tag{2.110}$$

where $\Lambda^{ab} \equiv (\Lambda_{ab})^*$.

One can verify that a vector $V_{\alpha\, a}$, like $Q_{\alpha\, a}$, satisfies the following *reality condition*

$$V_{\alpha\, a} = -\epsilon_{\alpha\beta}\,\mathbb{C}_{ab}\,(V_{\beta\, b})^* \tag{2.111}$$

where

$$\mathbb{C} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}\,, \tag{2.112}$$

is the symplectic-invariant matrix. The most general unitary transformation of the form

$$V_{\alpha\, a} \to V'_{\alpha\, a} = S_a{}^b\, V_{\alpha\, b}\,, \tag{2.113}$$

preserving this reality condition is a matrix S in $\mathrm{USp}(2\mathcal{N})$, namely a unitary matrix leaving $\mathbb{C}_{ab}$ invariant:

$$\mathrm{USp}(2\mathcal{N}) = \mathrm{U}(2\mathcal{N}) \cap \mathrm{Sp}(2\mathcal{N}, c) \quad : \quad S \in \mathrm{USp}(2\mathcal{N}) \quad \Leftrightarrow \quad S^\dagger S = \mathbf{1}\;\;, \quad S^T \mathbb{C}\, S = \mathbb{C}\,. \tag{2.114}$$

The reality condition (2.111) is also preserved by a generic SU(2) transformation on the index α. The transformation group preserving condition (2.111) is therefore $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$:

$$V_{\alpha\, a} \to V'_{\alpha\, a} = T_\alpha{}^\beta\, S_a{}^b\, V_{\beta\, b}\;\;, \quad T \in \mathrm{SU}(2)\,, \;\; S \in \mathrm{USp}(2\mathcal{N})\,, \tag{2.115}$$

Exercise 7.: Prove the above statement.

If $Z_{ij} = 0$, Eq. (2.108) reduces to:

$$\{Q_{\alpha\, a},\, Q_{\beta\, b}\} = -2m\,\epsilon_{\alpha\beta}\,\mathbb{C}_{ab}\,. \tag{2.116}$$

It is useful to define a composite index $A = (\alpha,\, a)$ and to rewrite Eq. (2.116) in the following form:

$$\{Q_A,\, Q_B\} = 2m\,\eta_{AB}\,, \tag{2.117}$$

where the $(4\mathcal{N}) \times (4\mathcal{N})$ symmetric matrix η_{AB} has $2\mathcal{N}$ eigenvalues $+1$ and $2\mathcal{N}$ eigenvalues -1. The anti-commutation relations (2.117) have a manifest automorphism group which is $\mathrm{O}(2\mathcal{N}, 2\mathcal{N}, c) \equiv \mathrm{O}(4\mathcal{N}, c)$ consisting of all complex matrices $S_A{}^B$ which leave η_{AB} invariant:

$$Q_A \to Q'_A = S_A{}^B\, Q_B \;\;;\;\; S\,\eta\, S^T = \eta\,. \tag{2.118}$$

The compact part of this group is $\mathrm{O}(4\mathcal{N})$ of which, however, only the subgroup $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$ is manifest, where the two factors act on the two indices α, i separately: SU(2) is the spin-group acting only on $\alpha, \beta, \dots$, $\mathrm{USp}(2\mathcal{N})$ is the compact group acting on the $a, b, \dots$ indices only and defined in (2.114). The states in a supermultiplet will therefore group in representations of $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$. The $2^{2\mathcal{N}}$ states in (2.103) for each value of s_3 define the spinorial representation of $\mathrm{O}(4\mathcal{N})$, half of which are fermions and half bosons. If the ground state has spin-0, the group $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$ has a non trivial action only on the products of the creation operators $\bar{q}^i_{\dot{\alpha}}$, which complete, as mentioned above, the spinorial representation $\mathbf{2^{2\mathcal{N}}}$ of $\mathrm{O}(4\mathcal{N})$. In this case the $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$ representation content of the supermultiplet, which is called *fundamental multiplet*, is obtained by branching this spinorial representation as follows:

$$\mathbf{2^{2\mathcal{N}}} \overset{\mathrm{SU}(2)\times\mathrm{USp}(2\mathcal{N})}{\longrightarrow} \left(\frac{\mathcal{N}}{\mathbf{2}}, \mathbf{1}\right) + \left(\frac{\mathcal{N}-\mathbf{1}}{\mathbf{2}}, \mathbf{2\mathcal{N}}\right) + \left(\frac{\mathcal{N}-\mathbf{2}}{\mathbf{2}}, [\mathbf{2\mathcal{N}}]_{\mathbf{2}}\right) + \dots$$
$$\dots + \left(\frac{\mathcal{N}-\mathbf{k}}{\mathbf{2}}, [\mathbf{2\mathcal{N}}]_{\mathbf{k}}\right) + \dots + (\mathbf{0}, [\mathbf{2\mathcal{N}}]_{\mathcal{N}}) , \tag{2.119}$$

where $[2\mathcal{N}]_k$ denotes the k-fold antisymmetric, traceless product of the fundamental representation of $\mathrm{USp}(2\mathcal{N})$:

$$[2\mathcal{N}]_k = (2\mathcal{N}) \wedge (2\mathcal{N}) \wedge \dots \wedge (2\mathcal{N}) . \tag{2.120}$$

The first entry in each couple on the right hand side of (2.119) is the spin. The first representation on the right hand side of Eq. (2.119) has spin $\mathcal{N}/2$, while the last has spin-0 and corresponds to the state

$$\bar{q}^{i_1}_{\dot{1}} \bar{q}^{j_1}_{\dot{2}} \dots \bar{q}^{i_\mathcal{N}}_{\dot{1}} \bar{q}^{j_\mathcal{N}}_{\dot{2}} |m, 0, 0\rangle . \tag{2.121}$$

In the ground state $|\Omega\rangle$ has spin s_0, it is in the representation $(\mathbf{s_0},\, \mathbf{1})$ of $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$, so that the $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$-representation content of the corresponding supermultiplet is obtained by multiplying the spinorial representation (2.119) by $(\mathbf{s_0},\, \mathbf{1})$:

$$(\mathbf{s_0},\, \mathbf{1}) \times \mathbf{2^{2\mathcal{N}}} . \tag{2.122}$$

As an example, let us consider the fundamental (i.e. having $s_0 = 0$) multiplet in $\mathcal{N} = 1$. Being the ground state a singlet with respect to $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N}) = \mathrm{SU}(2) \times \mathrm{USp}(2) \equiv \mathrm{SU}(2) \times \mathrm{SU}(2)$, the states of the supermultiplet arrange themselves in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ representations according to the branching (2.119):

$$\mathbf{4} \to (\frac{\mathbf{1}}{\mathbf{2}},\, \mathbf{1}) + (\mathbf{0},\, \mathbf{2}) = 1 \times (\frac{1}{2}) + 2 \times (0) . \tag{2.123}$$

The multiplet contains one fermion and two real scalars. The other multiplets are obtained from ground states $|\Omega\rangle$ with different spins s_0. For instance, if $s_0 = 1/2$, the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ representation content of the multiplet is computed by multiplying the representations of the fundamental multiplet by $(\mathbf{1/2}, \mathbf{1})$ and we find:

$$(\mathbf{1/2}, \mathbf{1}) \times \left[(\frac{\mathbf{1}}{\mathbf{2}},\, \mathbf{1}) + (\mathbf{0},\, \mathbf{2})\right] = (\mathbf{1},\, \mathbf{1}) + (\mathbf{0},\, \mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\, \mathbf{2}) = 1 \times (1) + 2 \times (\frac{1}{2}) + 1 \times (0) . \tag{2.124}$$

The supermultiplet contains a massive vector field (3 on-shell degrees of freedom), 2 massive spinors (4 on-shell degrees of freedom) and one massive scalar field (1 on-shell degrees of freedom).

Exercise 8.: Compute the states of the $\mathcal{N}=1$ representations with $s_0 = 1$ and $s_0 = 3/2$.

Let us compute now some $\mathcal{N}=2$ long multiplets. The manifest automorphism group is $\mathrm{SU}(2)\times\mathrm{USp}(2\mathcal{N}) = \mathrm{SU}(2)\times\mathrm{USp}(4)$ with respect to which the fundamental multiplet has the following representation content:

$$\mathbf{16} \to (\mathbf{1},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\,\mathbf{4}) + (\mathbf{0},\,\mathbf{5}) = 1\times(1) + 4\times(\frac{1}{2}) + 5\times(0)\,. \tag{2.125}$$

Notice that the dimension of the 2-fold antisymmetric product of the fundamental **4** of USp(4) is 5 since it is computed as the dimension of a 4×4 antisymmetric tensor T^{ab}, which is 6, minus its symplectic trace $T^{ab}\mathbb{C}_{ab}$, which contributes one parameter.

The supermultiplet with ground state of spin-1/2 is obtained as usual:

$$\begin{aligned}(\mathbf{1/2},\mathbf{1})\times\left[(\mathbf{1},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\,\mathbf{4}) + (\mathbf{0},\,\mathbf{5})\right] &= (\frac{\mathbf{3}}{\mathbf{2}},\,\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\,\mathbf{1}+\mathbf{5}) + (\mathbf{1},\,\mathbf{4}) + (\mathbf{0},\,\mathbf{4}) = \\ &= 1\times(\frac{3}{2}) + 4\times(1) + (5+1)\times(\frac{1}{2}) + 4\times(0)\,.\end{aligned} \tag{2.126}$$

This is the massive spin-3/2 $\mathcal{N}=2$ long-multiplet.

Exercise 9.: Compute the states of the $\mathcal{N}=2$ long-multiplet with $s_0 = 1$. What is its maximum spin state?

Massive representations with central charge. We consider now representations on which the central charge matrix Z_{ij} is non-vanishing. It is known that [18], by means of a transformation U in G_R, see (2.26), this matrix can be reduced to a skew-diagonal form (also called *normal form*) :

$$Z_{ij} \to Z'_{ij} = U_i{}^k U_j{}^l\, Z_{kl} = \begin{cases} \begin{pmatrix} z_1\,\epsilon & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & z_1\,\epsilon & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & z_{\frac{\mathcal{N}}{2}}\,\epsilon \end{pmatrix} & \mathcal{N}\text{ even} \\ \begin{pmatrix} z_1\,\epsilon & \mathbf{0} & \dots & \mathbf{0} & \begin{matrix}0\\0\end{matrix} \\ \mathbf{0} & z_1\,\epsilon & \dots & \mathbf{0} & \begin{matrix}0\\0\end{matrix} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & z_{[\frac{\mathcal{N}}{2}]}\,\epsilon & \begin{matrix}0\\0\end{matrix} \\ 0\,0 & 0\,0 & \dots & 0\,0 & 0 \end{pmatrix} & \mathcal{N}\text{ odd}\,, \end{cases} \tag{2.127}$$

where the blocks in boldface are 2×2, $\epsilon = (\epsilon_{xy})$, $x,y = 1,2$, while z_k are complex numbers (skew-eigenvalues). If $G_R = \mathrm{U}(\mathcal{N})$, z_k can be made real, while if $G_R = \mathrm{SU}(\mathcal{N})$, as it is the case for $\mathcal{N}=4,\,8$, z_k can be made real *modulo* an overall phase. We shall consider in what follows z_k complex. If $\mathcal{N}$ is even, we can write the index i as a couple of indices: $i=(x,u)$, where $x=1,2$ and $u=1,\dots,\mathcal{N}/2$, so that the entries of Z'_{ij} in the normal form can be written as:

$$Z'_{ij} = Z'_{(x,u)(y,v)} = z_u\,\epsilon_{xy}\,\delta_{uv}\,. \tag{2.128}$$

If $\mathcal{N}$ is odd, we only write the first $\mathcal{N}-1$ values of the index i as a couple (x,u), so that $i=\{(x,u),N\}$ and

$$Z'_{(x,u)(y,v)} = z_u\,\epsilon_{xy}\,\delta_{uv}\ ,\ \ Z'_{(x,u)N} = -Z'_{N(x,u)} = 0\,. \tag{2.129}$$

If we consider the supersymmetry generators in the basis $Q_{\alpha\,a}$, the above unitary transformation is implemented by a USp$(2\mathcal{N})$-transformation $\mathbb{U}=(\mathbb{U}_a{}^b)$:

$$Q_{\alpha\,a}\ \rightarrow\ Q'_{\alpha\,a} = \mathbb{U}_a{}^b\,Q_{\alpha\,b}\,, \tag{2.130}$$

where

$$\mathbb{U} \equiv \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & U^* \end{pmatrix}\,. \tag{2.131}$$

Exercise 10.: Verify that the matrix $\mathbb{U}$ *defined above belongs to* USp$(2\mathcal{N})$.

In this new basis the matrix Λ_{ab} in (2.110) has the following form:

$$\Lambda_{ab}\ \rightarrow\ \Lambda'_{ab} = \mathbb{U}_a{}^c\mathbb{U}_b{}^d\,\Lambda_{cd} = \begin{pmatrix} Z'_{ij} & \mathbf{0} \\ \mathbf{0} & Z'^{kl} \end{pmatrix} - m\,\mathbb{C}_{ab}\,. \tag{2.132}$$

Next we perform one further change of basis of the supersymmetry generators through an other USp$(2\mathcal{N})$ matrix S. The transformation for $\mathcal{N}$ odd, must be thought of as acting only on the first $\mathcal{N}-1$ values of the index i, of the type (x,u), leaving the last component $i=\mathcal{N}$ inert. Consider for the sake of simplicity the case $\mathcal{N}=2\,\ell$ even and perform the following transformation:

$$Q'_{\alpha\,a}\ \rightarrow\ \tilde{Q}_{\alpha\,a} = S_a{}^b\,Q'_{\alpha\,b} = (S\mathbb{U})_a{}^b\,Q_{\alpha\,b}\,, \tag{2.133}$$

where

$$\begin{aligned} S &= \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^* & \mathbf{A}^* \end{pmatrix} \in \mathrm{USp}(2\mathcal{N})\,,\\ \mathbf{A} &= \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_\ell \end{pmatrix}\ ;\ \mathbf{B} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_\ell \end{pmatrix}\,,\\ A_k &\equiv \begin{pmatrix} 1 & 1 \\ \sqrt{\frac{\bar{z}_k}{z_k}} & \sqrt{\frac{\bar{z}_k}{z_k}} \end{pmatrix}\ ;\ B_k \equiv \begin{pmatrix} -\sqrt{\frac{z_k}{\bar{z}_k}} & \sqrt{\frac{z_k}{\bar{z}_k}} \\ 1 & -1 \end{pmatrix}\,. \end{aligned} \tag{2.134}$$

The explicit relation between the new supersymmetry generators $\tilde{Q}$ and Q' reads:

$$\tilde{Q}_{\alpha\,i} = \tilde{Q}_{\alpha\,(x,u)} = \frac{1}{2}\,\left((A_u)_x{}^y\,Q'_{\alpha\,(y,u)} + (B_u)_{xy}\,\bar{Q}'^{\dot\alpha\,(y,u)}\right)\,, \tag{2.135}$$

$$\overline{\tilde{Q}}^{\dot\alpha\,i} = \overline{\tilde{Q}}^{\dot\alpha\,(x,u)} = \frac{1}{2}\,\left((A_u^*)^x{}_y\,\overline{Q}'^{\,\dot\alpha\,(y,u)} - (B_u^*)^{xy}\,Q'_{\alpha\,(y,u)}\right)\,, \tag{2.136}$$

where $u = 1, \ldots, \ell$.

The reader can verify that:

$$\Lambda'_{ab} \;\rightarrow\; \tilde{\Lambda}_{ab} = S_a{}^c\, S_b{}^d\, \Lambda'_{cd} = \begin{pmatrix} \mathbf{0} & -\mathbf{D} \\ \mathbf{D} & \mathbf{0} \end{pmatrix} \;\; ; \;\; \mathbf{D} = \begin{pmatrix} m\,\mathbf{1}_2 + |z_1|\,\sigma_3 & & \\ & \ddots & \\ & & m\,\mathbf{1}_2 + |z_\ell|\,\sigma_3 \end{pmatrix} . \tag{2.137}$$

In the new basis, the only non-vanishing anticommutator is the following:

$$\begin{aligned} \{\tilde{Q}_{\alpha\,(x,u)},\, \overline{\tilde{Q}}_{\dot{\beta}}{}^{(y,v)}\} &= \{\tilde{Q}_{\alpha\,(x,u)},\, (\tilde{Q}_{\beta\,(y,v)})^\dagger\} = 2\delta_{\alpha\beta}\,(m\,\mathbf{1}_2 + |z_u|\,\sigma_3)_{xy}\,\delta_{uv} = \\ &= 2\delta_{\alpha\beta}\,(m - (-)^x |z_u|)\,\delta_{xy}\,\delta_{uv}\,. \end{aligned} \tag{2.138}$$

Since, as earlier emphasized, $\{\tilde{Q}_{\alpha\,(x,u)},\, (\tilde{Q}_{\beta\,(y,v)})^\dagger\}$ is a positive definite operator in the space of states, the above equation implies:

$$m \geq |z_u| \;\; ; \;\; u = 1, \ldots, \ell\,, \tag{2.139}$$

namely the mass should be larger than the moduli of all the skew-eigenvalues of the central charge. As a consequence of this massless representations must have vanishing central charge. In the case of odd $\mathcal{N}$ ($\mathcal{N} = 2\ell + 1$), as mentioned above, if we write $i = \{(x, u), \mathcal{N}\}$, Eq.s (2.135), (2.136) still hold for $i = 1, \ldots, 2\ell$, while

$$\tilde{Q}_{\alpha\mathcal{N}} = Q'_{\alpha\mathcal{N}}\,, \tag{2.140}$$

so that:

$$\{\tilde{Q}_{\alpha\mathcal{N}},\, (\tilde{Q}_{\beta\mathcal{N}})^\dagger\} = 2m\,\delta_{\alpha\beta}\,. \tag{2.141}$$

Let us consider first the case $m > |z_u|$, $u = 1, \ldots, \ell$. The numbers $m - (-)^x |z_u|$ are positive and we can define the generators:

$$\begin{aligned} \tilde{q}_{\alpha\,(x,u)} &= \frac{1}{\sqrt{2(m - (-)^x |z_u|)}}\,\tilde{Q}_{\alpha\,(x,u)}\,, \\ \tilde{q}_{\alpha\mathcal{N}} &= \frac{1}{\sqrt{2m}}\,\tilde{Q}_{\alpha\mathcal{N}} \quad (\mathcal{N} \text{ odd.}) \end{aligned} \tag{2.142}$$

From Eq.s (2.138) (and (2.141) for $\mathcal{N}$ odd) we find:

$$\{\tilde{q}_{\alpha\,i},\, \bar{\tilde{q}}_{\dot{\beta}}{}^j\} = \delta_{\alpha\beta}\,\delta_i^j\,, \tag{2.143}$$

that is $\tilde{q}_{\alpha\,i}$, $\bar{\tilde{q}}_{\dot{\beta}}{}^j$ generate the Clifford algebra of a system of $2\mathcal{N}$ fermionic degrees of freedom. The states of an irreducible representation are constructed, as usual, by applying $\bar{\tilde{q}}_{\dot{\beta}}{}^j$ on a ground state $|\Omega\rangle$ annihilated by all the $\tilde{q}_{\alpha\,i}$. The procedure for constructing the states parallels the one illustrated in the case $Z_{ij} = 0$, with the states groped in representations of the manifest automorphism group $\mathrm{SU}(2) \times \mathrm{USp}(2\mathcal{N})$. In this case, however, invariance of the theory under CPT requires a doubling of the states. Indeed under CPT

$$Z_{ij} \stackrel{CPT}{\longrightarrow} \eta^2 Z^{ij}, \tag{2.144}$$

η being a phase. Being Z_{ij} complex, it is not inert under CPT. This means that a same representation should contain states in which the expectation value of the central charge is Z_{ij} together with states in which it is $\eta^2 Z^{ij}$. This requires a doubling of the states, starting from the ground state:

$$|\Omega\rangle \stackrel{CPT}{\longrightarrow} |\Omega'\rangle . \tag{2.145}$$

Therefore, for a given spin s_0 of the ground state, we have:

$$\begin{pmatrix}\text{Number of states of a massive multiplet}\\ \text{with } Z_{ij} \neq 0\end{pmatrix} = 2 \times \begin{pmatrix}\text{Number of states of a massive}\\ \text{multiplet with } Z_{ij} = 0\end{pmatrix} . \tag{2.146}$$

Suppose now a number q of the skew-eigenvalues z_k of the central charge matrix coincides with m:

$$m = |z_1| = \cdots = |z_q| > |z_{q+1}|, \ldots, |z_\ell| . \tag{2.147}$$

From Eq. (2.138) we find:

$$\{\tilde{Q}_{\alpha\,(2,u)},\, (\tilde{Q}_{\beta\,(2,u)})^\dagger\} = 0 \quad (u = 1, \ldots, q) , \tag{2.148}$$

which implies $\tilde{Q}_{\alpha\,(2,u)} = 0$, $u = 1, \ldots, q$. The states of the multiplet are annihilated by the q supercharges $\tilde{Q}_{\alpha\,(2,u)}$ and therefore preserve a fraction $q/\mathcal{N}$ of the original $\mathcal{N}$-supersymmetries. They are therefore called $(q/\mathcal{N})$-BPS. As a consequence of this property the generators of the Clifford algebra are effectively reduced from $2\mathcal{N}$ to $2(\mathcal{N}-q)$ and the manifest symmetry in the rest frame is $\mathrm{SU}(2) \times \mathrm{USp}(2(\mathcal{N}-q))$. The number of states are twice that of a $Z_{ij} = 0$ massive representation of $(\mathcal{N}-q)$- extended supersymmetry. These supermultiplets are called *short.*

As an example, let us consider the 1/2-BPS fundamental (i.e. $s_0 = 0$) representation of $\mathcal{N} = 2$ supersymmetry. The number of states is twice that of a long $\mathcal{N} = 1$ multiplet:

$$2 \times [1 \times \left(\frac{1}{2}\right) + 2 \times (0)] = 2 \times \left(\frac{1}{2}\right) + 4 \times (0) . \tag{2.149}$$

This representation, consisting of 2 fermions and four scalar fields is the *massive hypermultiplet.* Its field content is the same as that of the massless hypermultiplet (2.98).

By the same token, the 1/2-BPS representation of $\mathcal{N} = 2$ supersymmetry with $s_0 = 1/2$ is:

$$2 \times [1 \times (1) + 2 \times \left(\frac{1}{2}\right) + 1 \times (0)] , \tag{2.150}$$

where we have used the structure of the long $s_0 = 1/2$ $\mathcal{N} = 1$ multiplet in (2.124).

Exercise 9.: Compute the 1/2*-BPS fundamental representation of* $\mathcal{N} = 2$ *supersymmetry.*

See Appendix (B) for a list of the long and short massive representations.

BPS-states. Let us re-derive the above results in the 4-component notation for the spinor fields. We consider massive states and define the longitudinal unit-vector $\zeta^\mu = p^\mu/m$ in a generic frame of reference. Let us consider the anticommutation relation between the supersymmetry generators, as written in the second of Eq.s (2.14):

$$\{Q_i,\, \bar{Q}_j\} = 2\,i\,\left(\delta_i^j\,(\gamma^\mu C)\,\hat{\mathcal{P}}_\mu + \mathbb{Z}_{ij}\right)\,, \tag{2.151}$$

where $\mathbb{Z}_{ij} = i\,R_{ij}\,\gamma^5 + I_{ij}$. Consider, for the sake of simplicity, the even-$\mathcal{N}$ case. Using the unitary transformation U in Eq. (2.127), we can bring the central charge matrix to its normal (skew-diagonal) form (we suppress the prime):

$$\mathbb{Z}_{ij} = \mathbb{Z}_{(x,u)(y,v)} = \mathbb{Z}_u \epsilon_{xy}\,\delta_{uv}\,, \tag{2.152}$$

where $\mathbb{Z}_u \equiv \mathrm{Im}(z_u) + i\,\mathrm{Re}(z_u)\,\gamma^5$. Define the following projectors:

$$\mathcal{S}^{(\pm)}_{(x,u),(y,v)} \equiv \frac{1}{2}\left(\delta_{xy}\delta_{uv} \pm i\,\zeta_\mu\,\gamma^\mu\,\frac{\mathbb{Z}_u}{|z_u|}\,\epsilon_{xy}\delta_{uv}\right)\,, \tag{2.153}$$

$$\hat{\mathcal{S}}^{(\pm)}_{(x,u),(y,v)} \equiv \frac{1}{2}\left(\delta_{xy}\delta_{uv} \pm i\,\zeta_\mu\,\gamma^\mu\,\frac{\bar{\mathbb{Z}}_u}{|z_u|}\,\epsilon_{xy}\delta_{uv}\right)\,. \tag{2.154}$$

One can verify that:

$$\mathcal{S}^{(\pm)}\cdot\mathcal{S}^{(\pm)} = \mathcal{S}^{(\pm)}\ ;\quad \mathcal{S}^{(\pm)}\cdot\mathcal{S}^{(\mp)} = 0\,, \tag{2.155}$$

$$\hat{\mathcal{S}}^{(\pm)}\cdot\hat{\mathcal{S}}^{(\pm)} = \hat{\mathcal{S}}^{(\pm)}\ ;\quad \hat{\mathcal{S}}^{(\pm)}\cdot\hat{\mathcal{S}}^{(\mp)} = 0\,, \tag{2.156}$$

$$\gamma^0\,(\mathcal{S}^{(\pm)})^\dagger\,\gamma^0 = \hat{\mathcal{S}}^{(\pm)}\ ;\quad C^{-1}\,\hat{\mathcal{S}}^{(\pm)}\,C = (\mathcal{S}^{(\pm)})^T\,, \tag{2.157}$$

and, moreover, that the action of $\mathcal{S}^{(\pm)}$, $\hat{\mathcal{S}}^{(\pm)}$ on a 4-spinor $\xi_i = \xi_{(x,u)}$ preserves the Majorana condition:

$$\xi\quad\text{Majorana spinor}\quad\Rightarrow\quad \bar{\xi}\cdot\mathcal{S}^{(\pm)}\ ,\ \hat{\mathcal{S}}^{(\pm)}\cdot\xi\quad\text{Majorana spinors}\,, \tag{2.158}$$

The matrices $\mathcal{S}^{(\pm)}$, $\hat{\mathcal{S}}^{(\pm)}$ have rank $\ell = \mathcal{N}/2$ each.

Exercise 10.: Prove Eq.s (2.155), (2.156), (2.157) and (2.158).

Define now the projected supersymmetry generators:

$$Q_i^{(\pm)} = Q^{(\pm)}_{(x,u)} \equiv \hat{\mathcal{S}}^{(\pm)}_{(x,u),(y,v)}\,Q_{(y,v)}\quad\Rightarrow\quad \overline{Q^{(\pm)}}_{(x,u)} = \bar{Q}_{(y,v)}\,\mathcal{S}^{(\pm)}_{(y,v),(x,u)}\,. \tag{2.159}$$

With some γ-matrix algebra one finds:

$$\begin{aligned}\{Q^{(\pm)},\,\overline{Q^{(\pm)}}\} &= \hat{\mathcal{S}}^{(\pm)}\cdot\{Q,\,\bar{Q}\}\cdot\mathcal{S}^{(\pm)}\ ;\ \{Q^{(\pm)},\,\overline{Q^{(\mp)}}\} = \hat{\mathcal{S}}^{(\pm)}\cdot\{Q,\,\bar{Q}\}\cdot\mathcal{S}^{(\mp)} = 0\,,\\ \{Q^{(\pm)}_{(x,u)},\,\overline{Q^{(\pm)}}_{(y,v)}\} &= 2\,\hat{\mathcal{S}}^{(\pm)}_{(x,u),\,(y,v)}\,\zeta_\mu\,\gamma^\mu\,(m \pm |z_u|)\ ;\quad \{Q^{(\pm)},\,\overline{Q^{(\mp)}}\} = 0\,.\end{aligned} \tag{2.160}$$

In the rest frame $\zeta^\mu = (1,0,0,0)$ and

$$\{Q^{(\pm)}_{(x,u)},\,(Q^{(\pm)}_{(yv)})^\dagger\} = 2\,\hat{\mathcal{S}}^{(\pm)}_{(x,u),\,(y,v)}\,(m \pm |z_u|)\,, \tag{2.161}$$

from which we find the general property derived above: $m \geq |z_u|$.

Consider now an infinitesimal supersymmetry transformation of the form

$$\bar{Q}\,\epsilon = \overline{Q^{(+)}}\,\epsilon^{(+)} + \overline{Q^{(-)}}\,\epsilon^{(-)}\,, \tag{2.162}$$

where

$$\epsilon^{(\pm)}_{(x,u)} = \mathcal{S}^{(\pm)}_{(x,u),(y,v)}\,\epsilon_{(y,v)}\,. \tag{2.163}$$

Suppose now, on a state $|BPS\rangle$, $m = |z_1| = \cdots = |z_q|$. By equation (2.161) we have that:

$$Q^{(-)}_{(x,u)}\,|BPS\rangle = 0 \ \ , \ \ \ u = 1,\dots,q\,. \tag{2.164}$$

Notice that, in spite of the index $x = 1, 2$, there are only ℓ independent $Q^{(-)}$ (the same for $Q^{(+)}$). For this reason, Eq. (2.164) implies that the state $|BPS\rangle$ preserves only a fraction $q/\mathcal{N}$ of the original supersymmetries. The preserved supersymmetry is parametrized by $\epsilon^{(-)}_{(x,u)}$, with $u = 1,\dots,q$, which are defined by the condition:

$$\mathcal{S}^{(+)}_{(x,u),(y,v)}\,\epsilon_{(y,v)} = \epsilon_{(x,u)} + i\,\zeta_\mu\,\gamma^\mu\,\frac{\mathbb{Z}_u}{|z_u|}\,\epsilon_{xy}\epsilon_{(y,u)} = 0\ ,\ \ u = 1,\dots,q\,, \tag{2.165}$$

$$\epsilon_{(x,u)} = 0\ ,\ \ u = q+1,\dots,\mathcal{N}/2\,, \tag{2.166}$$

and are named *Killing spinors.*

Let us now show that condition (2.164) amounts to a set of first order differential equations on the fields describing the state. Let $\hat{\Phi}(x)$ denote a generic field-operator of the theory. The state $|BPS\rangle$ is escribed by a set of (bosonic and fermionic) fields, generically denoted by $\Phi(x)$, defined as:

$$\Phi(x) = \langle 0|\hat{\Phi}(x)|BPS\rangle\,. \tag{2.167}$$

Condition (2.164) implies that:

$$\delta_{\epsilon^{(-)}}\Phi(x) = -i\,\langle 0|[\hat{\Phi}(x),\,\overline{Q^{(-)}}\,\epsilon^{(-)}]|BPS\rangle = f(\Phi(x),\,\epsilon^{(-)}) = 0\,, \tag{2.168}$$

where $\epsilon^{(-)}$ satisfy Eq.s (2.166). The fields $\Phi(x)$ describe the state as a solution of the theory and are therefore solutions to the field equations. The function $f(\Phi(x),\,\epsilon^{(-)})$ is the supersymmetry transformation rule, which expresses the infinitesimal transformation of a field in terms of the supersymmetry parameter and all the fields. It depends in general on the the fields and their first space-time derivatives. Schematically, in a supergravity theory, the supersymmetry transformation rules have the general form:

$$\delta_\epsilon\Phi_F \sim \partial_\mu\Phi_B\,\gamma^\mu\epsilon\ ;\ \ \delta_\epsilon\Phi_B \sim \overline{\Phi}_F\epsilon\,. \tag{2.169}$$

In a *bosonic solution* the fermion fields vanish $\Phi_F(x) = 0$ and therefore the only non-trivial condition comes from :

$$\delta_{\epsilon^{(-)}}\Phi_F(x) = 0\,. \tag{2.170}$$

The above conditions define a set of first order differential equations on the (bosonic) background fields $\Phi_B(x) = 0$ called *Killing spinor equations.*

2.4 Local Symmetries

In this section we introduce supergravity as the "gauge theory" of the super-Poincaré group, highlighting analogies and differences with ordinary gauge theories. We shall then illustrate in detail the construction of pure $\mathcal{N} = 1$ supergravity.

2.4.1 Gauge Theories

Three of the fundamental interactions (the strong, weak and electro-magnetic) are mediated by spin-1 particles and are well described by *gauge theories*, namely relativistic field theories which are invariant under *local* (i.e. space-time dependent) transformations of some suitable internal symmetry group (color-SU(3) for the strong interactions, $\mathrm{SU}(2) \times \mathrm{U}(1)$ for the weak and electro-magnetic ones). Gauge theories provide a renormalizable description of these fields and of their coupling to matter. The BEH mechanism of spontaneous symmetry breaking then allows the interaction fields to have an effective mass without spoiling the renormalizability property of the theory.

Let us briefly recall how the requirement of local invariance under some internal gauge group requires the coupling of matter to suitable massless spin-1 fields. Consider a theory describing a complex massive scalar field $\phi(x)$ (in flat Minkowski space-time) through a Lagrangian density

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \, |\phi|^2 \,. \tag{2.171}$$

The theory is clearly invariant under *global* U(1)-transformations $\phi \;\to\; e^{i\alpha}\,\phi$, where α is a constant parameter, but not under *local* ones, since if $\alpha = \alpha(x)$ the kinetic term transforms in a non-trivial way. We can make the theory invariant under local U(1)-transformations provided the field $\phi(x)$ is coupled to a vector potential $A_\mu(x)$. Such coupling is introduced by replacing the ordinary derivatives by *covariant* ones:

$$\partial_\mu \;\to\; D_\mu = \partial_\mu - i\, e\, A_\mu \,, \tag{2.172}$$

so that the new Lagrangian density for the scalar field reads:

$$\mathscr{L}' = (D_\mu \phi)^* D^\mu \phi - m^2 \, |\phi|^2 \,. \tag{2.173}$$

The theory is invariant under the following local U(1)-transformations:

$$\phi(x) \;\to\; \phi'(x) = e^{i\, e \alpha(x)}\, \phi(x) \quad , \quad A_\mu(x) \;\to\; A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \,, \tag{2.174}$$

by virtue of the following property of the covariant derivative:

$$D_\mu \phi \;\to\; D'_\mu \phi' = e^{i\, e \alpha(x)}\, D_\mu \phi \,. \tag{2.175}$$

Since A_μ is a dynamical field, the full Lagrangian density of the theory should also contain a kinetic term for it:

$$\mathscr{L} = -\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 \, |\phi|^2 \,, \tag{2.176}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength associated with A_μ.

This theory describes the coupling of a charged scalar field ϕ, with charge e, to the electromagnetic field $A_\mu(x)$ (scalar-QED), and this coupling is fixed by the gauge-invariance requirement, namely the requirement of invariance under local-U(1).

The above construction is generalized to a theory with gauge group $\mathcal{G}$. Let the compact Lie group $\mathcal{G}$ be locally generated by a Lie algebra $\mathfrak{g}$ with generators T_A, $A = 1, \dots, \dim(\mathcal{G})$, so that, in a neighborhood of the identity element a generic $\mathcal{G}$-transformation can eb written as:

$$g \in \mathcal{G} \ , \quad g = \exp(\alpha^A T_A) \, . \tag{2.177}$$

The structure of $\mathfrak{g}$ is described by the following commutation relations:

$$[T_A, T_B] = f_{AB}{}^C \, T_C \, , \tag{2.178}$$

$f_{AB}{}^C$ being the structure constants of $\mathfrak{g}$ satisfying the Jacobi identity $f_{[AB}{}^D \, f_{C]D}{}^E = 0$.

Let $\Phi(x)$ be some field transforming in a representation $\mathscr{R}$ of $\mathcal{G}$ (we suppress the internal index associated with this representation) and let the $\Phi \cdot \Phi$ denote a $\mathcal{G}$-invariant inner product in the representation $\mathscr{R}$:

$$\forall g \in \mathcal{G} \ \ : \quad (\mathscr{R}(g) \, \Phi) \cdot (\mathscr{R}(g) \, \Phi) = \Phi \cdot \Phi \, . \tag{2.179}$$

In the previous example $\phi \cdot \phi = \phi^* \, \phi = |\phi|^2$.

The Lagrangian density:

$$\mathscr{L} = \frac{1}{2} \, \partial_\mu \Phi \cdot \partial^\mu \Phi \, , \tag{2.180}$$

is invariant under global $\mathcal{G}$ transformations:

$$\Phi(x) \ \rightarrow \ \mathscr{R}(g) \, \Phi(x) \ \ , \quad g = e^{\alpha^A T_A} \in \mathcal{G} \, ,$$

α^A being constant parameters. Just as in the previous case, the above Lagrangian is not invariant under local transformations, parametrized by $\alpha^A(x)$.

To construct a theory which is invariant under local- $\mathcal{G}$ transformations we associate with each generator T_A a vector field (gauge field) $A^A_\mu(x)$ and define a *gauge connection*:

$$\Omega_g = A^A_\mu(x) \, dx^\mu \, T_A \ \in \ T^* \mathcal{M}_4 \times \mathfrak{g} \, , \tag{2.181}$$

which is a 1-form on Minkowski space-time (i.e. an element of $T^* \mathcal{M}_4$) with values in the Lie algebra $\mathfrak{g}$. We then define a covariant derivative on Φ:

$$D_\mu \Phi \ \equiv \ (\partial_\mu + \mathscr{R}(\Omega_g)) \ \Phi = \big(\partial_\mu + A^A_\mu(x) \, dx^\mu \, \mathscr{R}(T_A)\big) \ \Phi \, , \tag{2.182}$$

where $\mathscr{R}(T_A)$ are the matrices representing the action of T_A on Φ. If $\mathscr{R}$ is the adjoint representation $\mathscr{R}(T_C)_A{}^B = -\mathscr{R}(T_C)^B{}_A = f_{AC}{}^B$. On a p-form field μ in the representation $\mathscr{R}$, we define an exterior covariant derivative which yields the following $(p+1)-$form:

$$D\mu \ \equiv \ (d + \mathscr{R}(\Omega_g)\wedge) \ \mu = \big(d + A^A \wedge \mathscr{R}(T_A)\big) \ \mu \, . \tag{2.183}$$

The reader can verify that, under a generic *local* $\mathcal{G}$-transformation $g(x) \in \mathcal{G}$:

$$D'_\mu(\mathscr{R}(g)\,\Phi) = \left(\partial_\mu + \mathscr{R}(\Omega'_g)\,\Phi\right)\,(\mathscr{R}(g)\,\Phi) = \mathscr{R}(g)\,D_\mu\Phi\,, \tag{2.184}$$

provided Ω_g transforms as follows:

$$\Omega_g \stackrel{g}{\longrightarrow} \Omega'_g = g\,\Omega_g\,g^{-1} + g\,dg^{-1}\,. \tag{2.185}$$

Exercise: Prove this.

This implies a corresponding transformation property of $A^A_\mu(x)$. In particular, under an infinitesimal transformation:

$$U = \mathbf{1} + \epsilon^A\,T_A \;\;,\;\; \epsilon^a \ll 1\,, \tag{2.186}$$

we find:

$$\delta\Omega_g = \delta A^A\,T_A = \Omega'_g - \Omega_g = [\epsilon^C\,T_C,\,\Omega_g] - d\epsilon^A\,T_A = -A^B\,\epsilon^C\,f_{BC}{}^A\,T_A - d\epsilon^A\,T_A = -D\epsilon^A\,T_A \;\Rightarrow$$

$$\Rightarrow\;\; \delta A^A = -D\epsilon^A\,. \tag{2.187}$$

Next we define a curvature 2-form:

$$\mathcal{F} \equiv d\Omega_g + \Omega_g\wedge\Omega_g = F^A\,T_A = \frac{1}{2}\,F^A_{\mu\nu}\,dx^\mu\wedge dx^\nu\,T_A\,. \tag{2.188}$$

The two-forms F^A are the *field strengths* of the gauge fields A^A_μ and read:

$$F^A = dA^A + \frac{1}{2}\,f_{BC}{}^A\,A^B\wedge A^C \;\;\Rightarrow\;\; F^A_{\mu\nu} = \partial_\mu\,A^A_\nu - \partial_\nu\,A^A_\mu + f_{BC}{}^A\,A^B_\mu\,A^C_\nu\,. \tag{2.189}$$

Under a gauge transformation (2.185), $\mathcal{F}$ transforms covariantly in the co-adjoint representation of $\mathcal{G}$:

$$\mathcal{F} \to\;\; g\,\mathcal{F}\,g^{-1} \;\;\Rightarrow\;\; F^A \;\to\;\; F'^A = F^B\,(g^{-1})_B{}^A\,, \tag{2.190}$$

where $g_A{}^B$ is the adjoint representation of g, and we have used the property $g\,T_A\,g^{-1} = (g^{-1})_A{}^B\,T_B$.

On the field strengths F^A the covariant derivative reads:

$$DF^A = \frac{1}{2}\,D_\mu F^A_{\nu\rho}\,dx^\mu\wedge dx^\nu\wedge dx^\rho = \,dF^A + A^B\wedge\mathscr{R}(T_B)_C{}^A\,F^C = dF^A + A^B\wedge f_{BC}{}^A\,F^C\,, \tag{2.191}$$

and the reader can verify, using the Jacobi identities, the following Bianchi identities:

$$DF^A = 0\,. \tag{2.192}$$

On the generic field Φ, we have:

$$D^2\Phi = D_\mu D_\nu\Phi\,dx^\mu\wedge dx^\nu = \mathscr{R}(\mathcal{F})\,\Phi = F^A\,\mathscr{R}(T_A)\,\Phi\,. \tag{2.193}$$

Exercise: Prove this.

Using the above properties, it is straightforward to verify the $\mathcal{G}$-invariance of the Lagrangian density:

$$k\,\mathrm{Tr}\,[\mathscr{R}(\mathcal{F})\wedge\mathscr{R}(\mathcal{F})] + \frac{1}{2}\,D_\mu\Phi\cdot D^\mu\Phi\,, \tag{2.194}$$

where k is a positive normalization constant depending on the representation $\mathscr{R}$. Also in this general case, the couplings among the gauge fields A^A and between these and the matter fields Φ, are completely fixed by the requirement of gauge invariance.

2.4.2 Gauge Transformations as Diffeomorphisms

To appreciate the difference between an ordinary gauge theory described above, and general relativity, seen as the "gauge theory" of the Poincaré group, it is useful to describe the former in a somewhat more formal framework. We extend the definition of the one-forms $A_\mu\, dx^\mu$ to a larger manifold P which can be locally described as a product of space-time $\mathcal{M}_4$ and the gauge group $\mathcal{G}$.[7] An element of this larger space can thus be locally described as $(x, g) = (x^\mu,\, g)$, where x^μ define a point on $\mathcal{M}_4$ and $g \in \mathcal{G}$. The tangent and co-tangent spaces to P at any point (x, g) are the direct sum of the tangent and co-tangent spaces to $\mathcal{M}_4$ and $\mathcal{G}$. It is known that the structure of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ can be described either in terms of the commutation relations among its generators $T_A \in T\mathcal{G}$, as in (2.178), or in terms of *dual forms* $\sigma^A = \sigma^A_B\, d\alpha^B \in T^*\mathcal{G}$, defined by the property:

$$\sigma^A(T_B) = \delta^A_B\,. \tag{2.195}$$

These forms can be defined through the right-invariant 1-form

$$\Omega^{(0)} = g\, dg^{-1} = \sigma^A\, T_A\,. \tag{2.196}$$

It is straightforward to verify that:

$$d\Omega^{(0)} = dg \wedge dg^{-1} = dg\, g^{-1}\, g \wedge dg^{-1} = -g\, dg^{-1} \wedge g\, dg^{-1} = -\Omega^{(0)} \wedge \Omega^{(0)}\,, \tag{2.197}$$

which, if expanded in the basis of generators T_A, yields the Maurer-Cartan equations for σ^A:

$$d\sigma^A + \frac{1}{2}\, f_{BC}{}^A\, \sigma^B \wedge \sigma^C = 0\,. \tag{2.198}$$

Equations (2.198) and (2.178) are equivalent.[8] Exterior differentiation of noth sides of the above equation yields $f_{AB}{}^D\, f_{CD}{}^E\, \sigma^A \wedge \sigma^B \wedge \sigma^C = 0$.

Next we define, on the larger manifold P, the following 1-form with value in $\mathfrak{g}$:[9]

$$\begin{aligned}\tilde{\Omega}_g(x, g) \equiv{}& g\, \Omega_g(x)\, g^{-1} + \Omega^{(0)} = A^A(x)\, g\, T_A\, g^{-1} + \Omega^{(0)} = \\ ={}& \tilde{A}^A(x, g)\, T_A\,,\end{aligned} \tag{2.199}$$

$$\tilde{A}^A(x, g) = A^B(x)\, g^{-1}{}_B{}^A + \sigma^A\,. \tag{2.200}$$

[7]The correct mathematical framework is that of *principal bundles.* We shall not enter however the mathematical details of the subject.

[8]The equivalence between (2.178) and (2.198) can be easily verified by starting from the latter and computing both its sides on the couple of vectors T_B, T_C. One needs to use the property that, for any 1-form ω and vectors X, Y, $d\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X,\, Y])$. We then find $d\sigma^A(T_B,\, T_C) = T_B[\sigma^A(T_C)] - T_C[\sigma^A(T_B)] - \sigma^A([T_B,\, T_C]) = T_B[\delta^A_C] - T_C[\delta^A_B] - \sigma^A([T_B,\, T_C]) = -\sigma^A([T_B,\, T_C])$. Being moreover $\frac{1}{2}\, f_{EF}{}^A\, \sigma^E \wedge \sigma^F(T_B, T_C) = f_{BC}{}^A$, we end up with the equation: $\sigma^A([T_B,\, T_C] - f_{BC}{}^D\, T_D) = 0$, which is equivalent to (2.178).

[9]From the theory of principal bundles (see for instance [24]), the form $\tilde{\Omega}_g(x, g)$ consistently defines a connection on P provided transition functions are defined through the right action of $\mathcal{G}$ on the fiber, namely on the $\mathcal{G}$-component of a point $(x, g) \in P$. Gauge transformations here are implemented through local $\mathcal{G}$ elements acting on the fiber to the left. We shall refrain from going into these mathematical subtleties.

The 1-forms $\tilde{A}^A$ extend the definition of $A^A(x)$ on the whole P. On the tangent space TP to the principal bundle at any point, we can move along *inner or vertical* directions tangent to $\mathcal{G}$ or *outer* directions tangent to the base Minkowski space. These two kind of directions define an orthogonal decomposition of TP

$$TP = T\mathcal{M}_4 \oplus T\mathcal{G} = \text{Span}(\partial_\mu, T_A) \ , \ \ T^*P = T^*\mathcal{M}_4 \oplus T^*\mathcal{G} = \text{Span}(dx^\mu, \sigma_A)\,,$$
$$dx^\mu(\partial_\nu) = \delta^\mu_\nu \ , \ \ \sigma^A(T_B) = \delta^A_B \ , \ \ dx^\mu(T_A) = \sigma^B(\partial_\mu) = 0\,. \tag{2.201}$$

From the above properties we can verify that:

$$\tilde{A}^A(T_B) = \sigma^A(T_B) = \delta^A_B\,. \tag{2.202}$$

This means that we can find a basis of T^*P such that $\tilde{A}^A$ are the dual forms to T_A.

This framework allows to describe gauge transformations as diffeomorphisms on the larger manifold P which map (x, g) into $(x, h(x)\, g)$ and are thus implemented by a spacetime-dependent group element $h(x)$. All the quantities that were previously defined on space-time can be extended on P. For instance we define the curvature 2-form on P:

$$\tilde{\mathcal{F}}(x, g) \equiv d\tilde{\Omega}_g + \tilde{\Omega}_g \wedge \tilde{\Omega}_g = g\,\mathcal{F}(x)\,g^{-1} = \tilde{F}^A(x, g)\,T_A\,,$$
$$\tilde{F}^A = d\tilde{A}^A + \frac{1}{2}\,f_{BC}{}^A\,\tilde{A}^B \wedge \tilde{A}^C\,. \tag{2.203}$$

Notice that $\tilde{F}^A(x, g) = \frac{1}{2}\tilde{F}^A_{\mu\nu} dx^\mu \wedge dx^\nu$ are 2-forms on $\mathcal{M}_4$, and thus they are orthogonal to the vertical (gauge) directions T_A, so that

$$\iota_{T_B}\tilde{F}^A = 0\,, \tag{2.204}$$

where ι_{T_B} denotes the *contraction* of the form along the direction T_B of the tangent space to P. We also extend the definition of the field $\Phi(x)$ transforming in a representation $\mathscr{R}$ of $\mathcal{G}$, to a field $\tilde{\Phi}(x, g)$ on P, by defining $\tilde{\Phi}(x, g) \equiv \mathscr{R}(g)\,\Phi(x)$. Just as we did in our earlier treatment, we define a covariant derivative on $\tilde{\Phi}(x, g)$:

$$\tilde{D}\tilde{\Phi} = d\tilde{\Phi} + \mathscr{R}(\tilde{\Omega}_g)\tilde{\Phi}\,. \tag{2.205}$$

The reader can then verify that:

$$\tilde{D}\tilde{\Phi}(x, g) = \mathscr{R}(g)\,D\Phi(x)\,, \tag{2.206}$$

where D was defined in (2.182).

The following Bianchi identities hold:

$$d\tilde{\mathcal{F}} + \tilde{\Omega}_g \wedge \tilde{\mathcal{F}} - \tilde{\mathcal{F}} \wedge \tilde{\Omega}_g = 0 \ \Leftrightarrow \ \tilde{D}\tilde{F}^A = d\tilde{F}^A + f_{BC}{}^A\tilde{A}^B \wedge \tilde{F}^C = 0\,. \tag{2.207}$$

Exercise: Check this.

The reader can easily verify that the effect of a gauge transformation $(x, g) \to (x, g') = (x, h(x)\, g)$ is:

$$\begin{aligned}
\tilde{\Omega}_g \to \;& \tilde{\Omega}'_g = h\, \tilde{\Omega}_g\, h^{-1} + h d h^{-1}\,, \\
\tilde{D}\tilde{\Phi} \to \;& \tilde{D}'\tilde{\Phi}' = \tilde{D}'(\mathscr{R}(h)\, \tilde{\Phi}) = \mathscr{R}(h)\, \tilde{D}\tilde{\Phi}\,, \\
\tilde{\mathcal{F}} \to \;& \tilde{\mathcal{F}}' = h\, \tilde{\mathcal{F}}\, h^{-1}\,.
\end{aligned} \tag{2.208}$$

Under an infinitesimal transformation $h(x) = \mathbf{1} + \epsilon^A(x)\, T_A$,

$$\delta \tilde{A}^A = \tilde{A}'^A - \tilde{A}^A = -D\epsilon^A\,. \tag{2.209}$$

We notice that the above transformation property can be described as a *diffeomorphism* on P and thus expressed in terms of the *Lie derivative* of $\tilde{A}^A$ along the (inner) vector $\epsilon = \epsilon^A\, T_A$:

$$\begin{aligned}
\delta \tilde{A}^A &= -\ell_\epsilon \tilde{A}^A = -d\left(\iota_\epsilon \tilde{A}^A\right) - \iota_\epsilon d\tilde{A}^A = \\
&= -d\epsilon^A - \iota_\epsilon \left(\tilde{F}^A - \frac{1}{2}\, f_{BC}{}^A\, \tilde{A}^B \wedge \tilde{A}^C\right) = \\
&= -d\epsilon^A - f_{BC}{}^A\, \tilde{A}^B\, \epsilon^C - \iota_\epsilon \tilde{F}^A = -\tilde{D}\epsilon^A\,,
\end{aligned} \tag{2.210}$$

where we have used the horizontality property (2.204) of $\tilde{F}^A$.

If, on the manifold P, we were given a set of connection 1-forms $\tilde{A}^A$, whose curvature 2-forms are defined by (2.203) and satisfy the horizontality property (2.204), then the vectors $T_A \in TP$ dual to $\tilde{A}^A$ would generate the group $\mathcal{G}$, namely satisfy the commutation relations (2.178). This can be easily verified by evaluating (2.203) in components along the basis dx^μ, σ^A of T^*P. Horizontality implies that $\tilde{F}^A_{BC} = \tilde{F}^A_{B\mu} = 0$, and let σ^A be the restriction of $\tilde{A}^A$ to $T^*\mathcal{G}$. Thus restricting the equation along $T^*\mathcal{G}$ we find:

$$0 = \frac{1}{2}\, \tilde{F}^A_{BC}\, \sigma^B \wedge \sigma^C = d\sigma^A + \frac{1}{2}\, f_{BC}{}^A\, \sigma^B \wedge \sigma^C\,, \tag{2.211}$$

which are the Maurer-Cartan equations for the algebra of the group $\mathcal{G}$, which imply that the dual vectors T_A to σ^A (i.e. to $\tilde{A}^A$ in T^*P) satisfy (2.178). Moreover from the definition (2.203) and of the covariant derivatives, and from the Jacobi identities satisfied by $f_{BC}{}^A$, the reader can easily verify that the Bianchi identities (2.207) are satisfied, together with the property

$$\tilde{D}^2\tilde{\Phi} = \mathscr{R}(\tilde{\mathcal{F}})\tilde{\Phi} = \tilde{F}^A \mathscr{R}(T_A)\tilde{\Phi}\,. \tag{2.212}$$

To make contact with our previous discussion we need to project all quantities defined over the larger space P down to space-time. This is done using the notion of *section* of P, defined as a mapping from $\mathcal{M}_4$ to P:

$$s\,:\, x \in \mathcal{M}_4 \;\longrightarrow\; s(x) = (x, g(x))\,, \tag{2.213}$$

which (locally) associates with each point x on the base manifold an element $g(x)$ in the fiber $\mathcal{G}$. The pull-back of $\tilde{\Omega}_g(x, g)$ by s is the following one-form on $\mathcal{M}_4$

$$s_*\tilde{\Omega}_g = g(x)\, \Omega_g\, g(x)^{-1} + g(x)\, \partial_\mu g(x)^{-1}\, dx^\mu\,, \tag{2.214}$$

which is nothing but the transformed gauge connection $\Omega_g(x) = A^A(x)\, T_A$ under the gauge transformation $g(x)$. Similarly

$$s_* \tilde{\mathcal{F}} = g(x)\, \mathcal{F}\, g(x)^{-1} \,.$$

Therefore the choice of a section $s(x)$ of the bundle amounts to a gauge-choice. Choosing in particular the *canonical local trivialization* $s_0(x) = (x, e)$ we have $s_{0*}\tilde{A}^A = A^A$ and $s_{0*}\tilde{F}^A = F^A$.

In the case of gravity or supergravity, the horizontality property (2.204) for some of the local symmetry generators (generators of general coordinate transformations or of local supersymmetry transformations) no longer holds: the generators of local coordinate transformations have a component tangent to space-time. As a consequence of this the gauge potentials (in particular the vielbein and the gravitino field) are no longer dual to the generators of the gauge group, but to vectors whose commutators close an algebra with *structure functions* depending on the space-time point. An other consequence of this is that the transformation law for the gauge potentials, which can still be expressed as a Lie derivative, is no longer a gauge transformation of the form (2.210).

2.5 Curved Space-Time

In general relativity gravity is related to the curvature of space-time. Let us briefly recall, in order to fix the notations, the main facts about the description of the geometry of a curved manifold.

Let $\mathcal{M}_4$ be a curved space-time whose metric is described by a $(0, 2)$ tensor $g_{\mu\nu}(x)$, in terms of which the squared invariant distance between two nearby points reads:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu \,. \tag{2.215}$$

The local geometry and curvature of $\mathcal{M}_4$ can be described in terms of an *affine connection* ∇, that is a prescription of how to *parallel transport* tensor along a curve. More specifically ∇ is defined as a mapping:

$$\begin{aligned} \nabla \,:\, T\mathcal{M}_4 \times T\mathcal{M}_4 \;&\longrightarrow\; T\mathcal{M}_4 \,, \\ \forall X, Y \in T\mathcal{M}_4 \;\; \nabla(X, Y) &= \nabla_X(Y) \,. \end{aligned} \tag{2.216}$$

The quantity $\nabla_X(Y)$ defines the infinitesimal variation of the vector Y when transported along a curve with tangent vector X. The vector Y is parallel transported along X if $\nabla_X(Y)$. We recall the main properties of ∇:

$$\begin{aligned} \nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z) \;;\;\; \nabla_{(X+Y)}(Z) = \nabla_X(Z) + \nabla_Y(Z) \,, \\ \nabla_{(f\,X)}(Y) = f\, \nabla_X(Y) \;;\;\; \nabla_X(f\, Y) = X(f)\, Y + f\, \nabla_X(Y) \,, \end{aligned} \tag{2.217}$$

for any $X, Y, Z \in T\mathcal{M}_4$ and $f(x)$ function over $\mathcal{M}_4$. Let $(\partial_\mu) \equiv (\frac{\partial}{\partial x^\mu})$ be a basis of $T\mathcal{M}_4$ and let (dx^μ) the dual basis of $T^*\mathcal{M}_4$: $dx^\mu(\partial_\nu) = \delta^\mu_\nu$. The connection is defined by the quantity $\Gamma^\rho_{\mu\nu}$:

$$\nabla_\mu(\partial_\nu) \equiv \nabla_{\partial_\mu}(\partial_\nu) = \Gamma^\rho_{\mu\nu}\, \partial_\rho \,. \tag{2.218}$$

Denoting by $\nabla_\mu X^\nu$ the components of the vector $\nabla_\mu X$ along ∂_ν, we then have:

$$\nabla_\mu X^\nu \equiv (\nabla_\mu X)^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\rho}\, X^\rho\,. \tag{2.219}$$

If $x^\mu(t)$ is a curve on $\mathcal{M}_4$, $V^\mu = \frac{dx^\mu}{dt}$ its tangent vector, a vector $X = X^\mu\partial_\mu$ is parallel transported along the curve if $\nabla_V(X) = V^\mu\nabla_\mu(X) = 0$. If the tangent vector V^μ is parallel transported along its own curve, $\nabla_V(V) = 0$, the curve is a *geodesic*.

The action of covariant derivative is extended to 1-forms by defining

$$\nabla_\mu(dx^\nu) \equiv -\Gamma_{\mu\rho}{}^{\nu}\, dx^\rho\,, \tag{2.220}$$

so that, if $\omega = \omega_\mu\, dx^\mu$ is a 1-form, the components $\nabla_\mu\omega_\nu \equiv (\nabla_\mu\omega)_\nu$ of $\nabla_\mu\omega$ along dx^μ read:

$$\nabla_\mu\omega_\nu \equiv (\nabla_\mu\omega)_\nu = \partial_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\,\omega_\rho\,. \tag{2.221}$$

The action of ∇_μ is then extended to tensor products of dx^μ and ∂_ν using Lifshitz rule, so that it is defined on a generic tensor.

Under a coordinate transformation $x^\mu \to y^{\mu'}(x)$, the symbol Γ transforms as:

$$\Gamma^\rho_{\mu\nu} \;\to\; \Gamma^{\rho'}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial y^{\rho'}}{\partial x^\rho}\Gamma^\rho_{\mu\nu} + \frac{\partial^2 x^\rho}{\partial y^{\mu'}\partial y^{\nu'}}\frac{\partial y^{\rho'}}{\partial x^\rho}\,. \tag{2.222}$$

Because of last term on the right hand side, $\Gamma^\rho_{\mu\nu}$ does not transform as a tensor.

We can constrain the connection to be of *metric type*, namely the parallel transport to preserve the inner product of two vectors, defined by the metric. This requires the metric to be *covariantly constant*:

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\, g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\, g_{\nu\sigma} = 0\,. \tag{2.223}$$

Writing the above condition for the triplets of indices $(\mu\nu\rho)$, $(\nu\rho\mu)$, $(\rho\mu\nu)$:

$$\partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\, g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\, g_{\nu\sigma} = 0\,, \tag{2.224}$$

$$\partial_\nu g_{\rho\mu} - \Gamma^\sigma_{\nu\rho}\, g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu}\, g_{\rho\sigma} = 0\,, \tag{2.225}$$

$$\partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu}\, g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu}\, g_{\mu\sigma} = 0\,. \tag{2.226}$$

To solve the above equations in $\Gamma^\sigma_{\mu\nu}$ we sum and subtract them, namely consider the equation (2.224)+(2.225)-(2.226). We find:

$$\Gamma^\sigma_{(\mu\nu)} = \frac{1}{2}\, g^{\sigma\gamma}\left(\partial_\mu g_{\gamma\nu} + \partial_\nu g_{\gamma\mu} - \partial_\gamma g_{\mu\nu}\right) + \frac{1}{2}\left(T_\mu{}^\sigma{}_\nu + T_\nu{}^\sigma{}_\mu\right)\,, \tag{2.227}$$

where we have defined the *torsion*:

$$T^\mu{}_{\nu\rho} \equiv \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu} = 2\,\Gamma^\mu_{[\nu\rho]}\,. \tag{2.228}$$

The symbol $\Gamma^\sigma_{\mu\nu}$ can then be computed as follows:

$$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{[\mu\nu]} + \Gamma^\sigma_{(\mu\nu)} = \frac{1}{2}\, g^{\sigma\gamma}\left(\partial_\mu g_{\gamma\nu} + \partial_\nu g_{\gamma\mu} - \partial_\gamma g_{\mu\nu}\right) + K^\sigma{}_{\mu\rho}\,, \tag{2.229}$$

where:

$$K^{\sigma}{}_{\mu\rho} \equiv \frac{1}{2}\left(T_{\mu}{}^{\sigma}{}_{\nu} + T_{\nu}{}^{\sigma}{}_{\mu} + T^{\sigma}{}_{\mu\nu}\right) , \tag{2.230}$$

is called the *contorsion*. If we consider the effect of a general coordinate transformation on $T^{\sigma}{}_{\mu\nu}$, from the definition of this tensor in terms of the connection, it follows that the non-homogeneous term in (2.222) drops out, being symmetric in $\mu\nu$, and thus that the *torsion transforms as a tensor*. The same holds for the contorsion.

If the torsion vanishes, $T^{\sigma}{}_{\mu\nu} = 0$, $\Gamma_{\mu\nu}{}^{\sigma}$ becomes symmetric in its lower indices and ∇ called the Levi-Civita connection.

In supergravity we shall see that the coupling of the gravitational field to its spin-3/2 superpartner, the gravitino, produces a torsion in the connection.

We recall the expression of the Riemann curvature tensor:

$$R_{\mu\nu}{}^{\sigma}{}_{\rho} \equiv \partial_\mu \Gamma^{\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \Gamma^{\gamma}_{\mu\rho}\Gamma^{\sigma}_{\nu\gamma} + \Gamma^{\gamma}_{\nu\rho}\Gamma^{\sigma}_{\mu\gamma} = -R_{\nu\mu}{}^{\sigma}{}_{\rho} = -R_{\mu\nu\,\rho}{}^{\sigma} , \tag{2.231}$$

where last equality follows from the metric compatibility of the connection.

If the connection is Levi-Civita, the following properties hold:

$$R_{\mu\nu}{}^{\sigma}{}_{\rho} = R^{\sigma}{}_{\rho\mu\nu} \quad , \quad R_{[\mu\nu\rho]}{}^{\sigma} = 0 \, . \tag{2.232}$$

The Ricci tensor and scalar are defined as:

$$\mathcal{R}_{\mu\nu} \equiv R_{\mu\rho\nu}{}^{\rho} \quad , \quad \mathcal{R} = R_{\mu\nu}{}^{\mu\nu} \, . \tag{2.233}$$

For a Levi-Civita connection $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\nu\mu}$.

2.6 Fermions on Curved Space-Time

Fermions are defined as fields transforming in the spinorial representation of the Lorentz group. In a curved spacetime the metric is no longer invariant under the Lorentz group, which is manifest only in inertial frames. Although we cannot define global inertial frames, we can define at any point a *local inertial one*. This is the free-falling (or moving) frame which is, in good approximation, inertial and thus in which the action of the Lorentz group is manifest and fermion fields can be defined. This frame can be defined about any space-time point p and is such that *at that point* the metric in this frame is the flat Lorentz one $\eta_{\mu\nu}$. About that point this is no longer true if the space-time curvature is non-vanishing in p and tidal forces manifest themselves. To define this frame we notice that the metric tensor at any point can be written in the form:

$$g_{\mu\nu}(x) = V_{\mu}{}^{a}(x)\, V_{\nu}{}^{b}(x)\, \eta_{ab} \ , \quad a, b = 0, 1, 2, 3 \, . \tag{2.234}$$

If the curvature is non-vanishing in p the matrices $V_{\mu}{}^{a}(x)$ cannot coincide with the Jacobian of some local coordinate transformation $x^\mu \to y^a(x)$ about p but it can coincide with a Jacobian $\partial_\mu y^a(x)$ in that point. The coordinates $y^a(x)$ define the local inertial frame at p. In such frame, in an infinitesimal neighborhood of p, the metric is $g_{ab} \approx \eta_{ab}$ and, as anticipated above, the Lorentz group is manifest and acts though matrices $\Lambda_a{}^b$ leaving η_{ab} invariant. The matrices $V_\mu{}^a(x)$ are called vierbein and their index a, labeling the basis of the local inertial frame and acted on by the Lorentz group, is called *rigid index*, as opposed to the curved ones $\mu, \nu, \dots$. They define a basis $V^a(x) \equiv V_\mu{}^a\, dx^\mu$ of the dual space $T_x^*\mathcal{M}_4$ in x. Similarly their inverse $V_a{}^\mu(x)$ ($V_a{}^\mu(x)\, V_\mu{}^b(x) = \delta_a^b$) define a basis $V_a = V_a{}^\mu \partial_\mu$ of $T_x\mathcal{M}_4$. At any point x there is a Lorentz group $\mathrm{O}(1,3)$ acting on $T_x\mathcal{M}_4$ and $T_x^*\mathcal{M}_4$:

$$V^a(x) \;\rightarrow\; V^b(x)\,\Lambda_b{}^a\,. \tag{2.235}$$

The vierbein matrix $V_\mu{}^a$ captures the degrees of freedom on the metric. Indeed their independent entries, modulo action of the local Lorentz group, are $4\times 4 - 6 = 10$ which are the independent entries of $g_{\mu\nu}(x)$.

Let us define the action of the connection on the vierbein basis:

$$\nabla_\mu V^a = -\omega_\mu{}^a{}_b\, V^b \;\;,\;\; \nabla_\mu V_b = \omega_\mu{}^a{}_b\, V_a\,. \tag{2.236}$$

From the first of the above equations we find:

$$0 = \nabla_\mu V_\nu{}^a + \omega_\mu{}^a{}_b\, V^b = \partial_\mu V_\nu{}^a + \Gamma^\rho_{\mu\nu}\, V_\rho{}^a + \omega_\mu{}^a{}_b\, V_\nu{}^b\,. \tag{2.237}$$

Antisymmetrizing in $\mu\nu$ we find:

$$T^a{}_{\mu\nu} \equiv V_\rho{}^a\, T^\rho{}_{\mu\nu} = 2\Gamma^\rho_{[\mu\nu]}\, V_\rho{}^a = \partial_{[\mu} V_{\nu]}{}^a + \omega_{[\mu}{}^a{}_b\, V_{\nu]}{}^b\,. \tag{2.238}$$

We the define then the *torsion 2-form*:

$$T^a \equiv \frac{1}{2}\, T^a{}_{\mu\nu}\, dx^\mu \wedge dx^\nu = dV^a + \omega^a{}_b \wedge V^b\,. \tag{2.239}$$

Similarly one can compute the Riemann curvature tensor in the new basis and find:

$$R_{\mu\nu}{}^a{}_b = 2(\partial_{[\mu}\omega_{\nu]}{}^a{}_b + \omega_{[\mu}{}^a{}_c\, \omega_{\nu]}{}^c{}_b)\,, \tag{2.240}$$

so that, defining the *curvature 2-form* as follows:

$$R^a{}_b \equiv \frac{1}{2}\, R_{\mu\nu}{}^a{}_b\, dx^\mu \wedge dx^\nu\,, \tag{2.241}$$

we have:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\,. \tag{2.242}$$

The quantities $\omega_\mu{}^a{}_b$ define the *spin connection 1-form*:

$$\omega^a{}_b = \omega_\mu{}^a{}_b\, dx^\mu\,, \tag{2.243}$$

It can be easily verified that, from the condition of metric compatibility of the connection $\nabla_\mu g_{\nu\rho} = 0$ and the definition of the vierbein matrices, the following property holds:

$$\omega^a{}_b = -\omega_b{}^a\,, \tag{2.244}$$

where the rigid indices $a, b, c, \ldots$ are raised and lowered with the Lorenzian $\eta_{ab} = \eta^{ab}$. This property implies that $R^a{}_b = R_b{}^a$, namely $R^a{}_b$ is a 2-form with values in the algebra of Lorentz generators.

The vanishing of the torsion tensor $T^a = 0$ defines the Levi-Civita connection and allows to determine ω^{ab} in terms of V^a. To this end we write $T^a = 0$ in components and define $\omega_{a,bc} \equiv V_a{}^\mu\, \omega_{\mu,bc}$:

$$\partial_{[\mu} V_{\nu]}{}^a + \omega_{[\mu}{}^a{}_b\, V_{\nu]}{}^b = 0 \;\Leftrightarrow\; \frac{1}{2}\,(\omega_{a,bc} - \omega_{b,ac}) = V_a{}^\mu V_b{}^\nu\, \partial_{[\mu} V_{\nu]c} = \Omega_{ab,c}\,, \tag{2.245}$$

where we have defined $\Omega_{ab,c} \equiv V_a{}^\mu V_b{}^\nu\, \partial_{[\mu} V_{\nu]c}$ and is antisymmetric in the first two indices. Write now three versions of (2.245) obtained from one another by cyclic permutation of the three indices:

$$\omega_{a,bc} - \omega_{b,ac} = 2\Omega_{ab,c}\,, \tag{2.246}$$
$$\omega_{b,ca} - \omega_{c,ba} = 2\Omega_{bc,a}\,, \tag{2.247}$$
$$\omega_{c,ab} - \omega_{a,cb} = 2\Omega_{ca,b}\,. \tag{2.248}$$
$$\tag{2.249}$$

Evaluating now (2.246)-(2.247)+(2.248), and using the antisymmetry of ω in its last two indices, one finds

$$\omega_{a,bc} = \Omega_{ab,c} - \Omega_{bc,a} + \Omega_{ca,b} = V_a{}^\mu V_b{}^\nu\, \partial_{[\mu} V_{\nu]c} + V_c{}^\mu V_a{}^\nu\, \partial_{[\mu} V_{\nu]b} - V_b{}^\mu V_c{}^\nu\, \partial_{[\mu} V_{\nu]a}\,, \tag{2.250}$$

from which we derive $\omega_\mu{}^a{}_b$.

Eq.s (2.239), (2.242) are the *Cartan's structure equations*. Notice that, if we interpret V^a and ω^{ab} as the "gauge potentials" associated with the Poincaré generators $\mathcal{P}_a$ and $\mathcal{L}_{ab}$, we see that Eq.s (2.239), (2.242) are nothing but the definition of the corresponding curvatures, see Eq.s (2.203). To appreciate this we compute the structure constants of the Poincaré algebra (A.3), (A.4):

$$\begin{aligned} \{T_A\} &= \{\mathcal{L}_{ab},\, \mathcal{P}_a\} \;\;,\;\; [T_A,\, T_B] = C_{AB}{}^C\, T_C\,, \\ [\mathcal{L}_{ab},\, \mathcal{L}_{cd}] &= \frac{1}{2}\, C_{ab,cd}{}^{ef}\, \mathcal{L}_{ef} \;\;,\;\; [\mathcal{L}_{ab},\, \mathcal{P}_c] = C_{ab,c}{}^d\, \mathcal{P}_d\,, \\ C_{ab,cd}{}^{ef} &= 2\left(\delta^{ef}_{bc}\, \eta_{ad} + \delta^{ef}_{ad}\, \eta_{bc} - \delta^{ef}_{ac}\, \eta_{bd} - \delta^{ef}_{bd}\, \eta_{ac}\right)\,, \\ C_{ab,c}{}^d &= \delta^d_a\, \eta_{bc} - \delta^d_b\, \eta_{ac}\,. \end{aligned} \tag{2.251}$$

The structure of the Poincaré algebra can alternatively be described by 1-forms σ^A, dual to T_A, and satisfying the Maurer-Cartan equations:

$$d\sigma^A + \frac{1}{2}\, C_{BC}{}^A\, \sigma^B \wedge \sigma^C = 0\,. \tag{2.252}$$

Next, just as we did for a generic gauge group $\mathcal{G}$, we define the 1-form $\tilde{\Omega}_g$ valued in the Lie algebra of the Poincaré group

$$\tilde{\Omega}_g = \tilde{A}^A\, T_A = \tilde{A}^a\, \mathcal{P}_a + \frac{1}{2}\, \omega^{ab}\, \mathcal{L}_{ab} = -V^a\, \mathcal{P}_a + \frac{1}{2}\, \omega^{ab}\, \mathcal{L}_{ab}\,, \tag{2.253}$$

and the corresponding Lie-algebra-valued curvature 2-form:

$$\tilde{\mathcal{F}} \equiv d\tilde{\Omega}_g + \tilde{\Omega}_g \wedge \tilde{\Omega}_g = R^A\, T_A = R^a\, \mathcal{P}_a + \frac{1}{2}\, R^{ab}\, \mathcal{L}_{ab} = -T^a\, \mathcal{P}_a + \frac{1}{2}\, R^{ab}\, \mathcal{L}_{ab}\,, \tag{2.254}$$

having denoted by $\{R^A\} = \{R^a,\, R^{ab}\}$ the components of the curvature 2-form. Note that we have identified, for later convenience, $\tilde{A}^a$ with minus the vierbein 1-forms V^a and R^a with minus the torsion 2-forms. We find:

$$R^A \equiv d\tilde{A}^A + \frac{1}{2}\, C_{BC}{}^A\, \tilde{A}^B \wedge \tilde{A}^C\,, \tag{2.255}$$

$$T^a = dV^a + \omega^a{}_b \wedge V^b\;,\quad R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\,. \tag{2.256}$$

We recover Eq.s (2.239), (2.242) for the torsion and the Riemann tensor. From our previous general analysis with a gauge group $\mathcal{G}$, we see that, using the Jacobi identity for the structure constants $C_{BC}{}^A$, the following Bianchi identities hold:

$$\tilde{D}R^A \equiv dR^A + C_{BC}{}^A\, \tilde{A}^B \wedge R^C = 0\,, \tag{2.257}$$

$$\tilde{D}T^a \equiv dT^a + \omega^a{}_b \wedge T^b - R^a{}_b \wedge V^b = 0\,, \tag{2.258}$$

$$\tilde{D}R^{ab} \equiv dR^{ab} + \omega^a{}_c \wedge R^{cb} - \omega^b{}_c \wedge R^{ca} = 0\,. \tag{2.259}$$

Notice that, for a Levi-Civita connection, the torsion vanishes, so that $T^a = dT^a = 0$. The Bianchi identity for T^a then implies $R^a{}_b \wedge V^b = 0$, that is:

$$R^a{}_b \wedge V^b = 0 \;\Leftrightarrow\; R_{[abc]}{}^d = 0\,. \tag{2.260}$$

Exercise: Verify this.

Example: The Schwarzschild solution. As an example let us give the vierbein, spin-connection and curvature for the Schwarzschild black hole solution:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2\, d\theta^2 - r^2 \sin^2(\theta)\, d\varphi^2 = V^a\, V^b \eta_{ab}\,, \tag{2.261}$$

The vierbein reads:

$$V^0 = \sqrt{1 - \frac{2M}{r}}\, dt\;,\quad V^1 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}\, dr\;,\quad V^2 = r\, d\theta\;,\quad V^3 = r\, \sin(\theta) d\varphi\,. \tag{2.262}$$

The independent components of the spin connection are

$$\omega^0{}_1 = \frac{M}{r^2}\, dt\;,\quad \omega^1{}_2 = -\sqrt{1 - \frac{2M}{r}}\, d\theta\;,\quad \omega^1{}_3 = -\sqrt{1 - \frac{2M}{r}}\, \sin(\theta)\, d\varphi\;,\quad \omega^2{}_3 = -\cos(\theta)\, d\varphi\,. \tag{2.263}$$

The independent components of the curvature 2-form $R_a{}^b$ are

$$\begin{aligned} R_0{}^1 &= -\frac{2MV^0 \wedge V^1}{r^3}\;,\quad R_0{}^2 = \frac{MV^0 \wedge V^2}{r^3}\;,\quad R_0{}^3 = \frac{MV^0 \wedge V^3}{r^3} \\ R_1{}^2 &= -\frac{MV^1 \wedge V^2}{r^3}\;,\quad R_1{}^3 = -\frac{MV^1 \wedge V^3}{r^3}\;,\quad R_2{}^3 = \frac{2MV^2 \wedge V^3}{r^3}\,. \end{aligned} \tag{2.264}$$

Exercise: Verify the Bianchi identities.

As we did in Subsect. (2.4.2), see Eq.s (2.199), we can extend the definitions of the 1-forms V^a, ω^{ab} to a larger manifold P which locally is the product of space-time $\mathcal{M}_4$ and of the Poincaré group G_P. The curvature 2-forms only have components along $dx^\mu \wedge dx^\nu$. The main difference with the previous discussion is that now V^a, are not orthogonal to dx^μ, as σ^A in Subsect. (2.4.2) were and as ω^{ab} are now, and the curvatures have components along them:

$$R^A = \frac{1}{2}\, R^A_{\mu\nu}\, dx^\mu \wedge dx^\nu = \frac{1}{2}\, R^A_{ab}\, V^a \wedge V^b\,. \tag{2.265}$$

Let us denote by $\{\tilde{T}_A\} = \{\tilde{\mathcal{P}}_a,\, \tilde{\mathcal{L}}_{ab}\}$ the vectors dual to V^a, ω^{ab}:

$$\tilde{A}^a(\tilde{\mathcal{P}}_b) = \delta^a_b = -V^a(\tilde{\mathcal{P}}_b)\ ,\quad \omega^{ab}(\tilde{\mathcal{L}}_{cd}) = 2\,\delta^{ab}_{cd}\ ,\quad V^a(\tilde{\mathcal{L}}_{ab}) = \omega^{ab}(\tilde{\mathcal{P}}_c) = 0\,. \tag{2.266}$$

The horizontality condition of the curvatures only holds for $\tilde{\mathcal{L}}_{cd}$ but not for $\tilde{\mathcal{P}}_b$:

$$\iota_{\tilde{\mathcal{L}}_{cd}} R^A = 0\ ,\quad \iota_{\tilde{\mathcal{P}}_b} R^A \neq 0\,. \tag{2.267}$$

The consequence of R^A not being horizontal with respect to all the vectors $\tilde{T}_A$ is that, evaluating both sides of (2.255) on $\tilde{T}_B$, $\tilde{T}_C$, see footnote 7, we find:

$$R^A_{BC} = R^A(\tilde{T}_B,\, \tilde{T}_C) = \left(d\tilde{A}^A + \frac{1}{2}\, C_{EF}{}^A \tilde{A}^E \wedge \tilde{A}^F\right)(\tilde{T}_B,\, \tilde{T}_C)\ \Leftrightarrow\ [\tilde{T}_A,\, \tilde{T}_B] = (C_{AB}{}^C - R^C_{AB})\,\tilde{T}_C\,. \tag{2.268}$$

In other words, the 1-forms $\tilde{A}^A$ are not dual to the generators T_A of the Poincaré algebra, but to vectors $\tilde{T}_A$ which close an algebra whose structure constants depend on the curvatures, which are functions over space-time, and thus are more appropriately called are *structure functions*. Since horizontality holds only for $\tilde{\mathcal{L}}_{cd}$, that is $R^A_{ab,c} = R^A_{ab,cd} = 0$, the last of Eq.s (2.268) implies that $\tilde{\mathcal{L}}_{cd}$ close the correct Lorentz algebra, so that:

$$\tilde{\mathcal{L}}_{cd} = \mathcal{L}_{cd}\,. \tag{2.269}$$

An other consequence of the horizontality condition not being complete, is that, if we compute the variation of the 1-forms $\tilde{A}^A$ due to diffeomorphisms generated by their dual vectors $\tilde{T}_A$, these *are no-longer gauge transformations.* Indeed if we compute the variation of $\tilde{A}^A$ as a Lie derivative along an infinitesimal vector $\epsilon^A \tilde{T}_A = \epsilon^a\, \tilde{\mathcal{P}}_a + \epsilon^{ab}\, \mathcal{L}_{ab}/2$, along the lines of Eq. (2.210), we find

$$\begin{aligned}\delta\tilde{A}^A &= -\ell_\epsilon \tilde{A}^A = -d\left(\iota_\epsilon \tilde{A}^A\right) - \iota_\epsilon d\tilde{A}^A = \\ &= -d\epsilon^A - \iota_\epsilon\left(R^A - \frac{1}{2}\, C_{BC}{}^A\, \tilde{A}^B \wedge \tilde{A}^C\right) = \\ &= -d\epsilon^A - C_{BC}{}^A\, \tilde{A}^B\, \epsilon^C - \iota_\epsilon R^A = -\tilde{D}\epsilon^A - \iota_\epsilon R^A\,. \end{aligned} \tag{2.270}$$

This variation is not a gauge variation since it does not contain only the covariant derivative of the local parameter. It also contains the contraction of the curvatures. Only with respect to a local Lorentz transformation $\epsilon^{ab}\, \mathcal{L}_{ab}/2$, the variation is pure gauge. For this reason it is not appropriate to define general relativity as a “gauge theory”. If space-time diffeomorphisms, generated by $\tilde{\mathcal{P}}_a$, were gauge transformations, the space-time dependence of the various fields would be the result of a gauge transformation, and this cannot be since it should be dictated by dynamics.

Since on curved space-time only local Lorentz transformations can be consistently regarded as gauge transformations, it is useful to define a Lorentz covariant derivative. On a field $\Phi(x)$ transforming in a representation D of the (local) Lorentz group we define:

$$\mathcal{D}\Phi = d\Phi + \frac{1}{2}\,\omega^{ab}\,D(\mathcal{L}_{ab}) \wedge \Phi\,. \tag{2.271}$$

Under a local Lorentz transformation $\Lambda(x)$, $\Phi \to \Phi' = D(\Lambda)\Phi$ and

$$\mathcal{D}'\Phi' = D(\Lambda)\mathcal{D}\Phi\,, \tag{2.272}$$

provided the spin-connection transforms as:

$$\omega^{ab} \;\to\; \omega'^{\,ab} = \Lambda^a{}_c\,\omega^{cd}\,\Lambda^{-1}{}_d{}^b + \Lambda^a{}_c d\Lambda^{-1\,cb}\,. \tag{2.273}$$

Exercise: Prove this. Hint: First prove, as for Eq. (2.185), that:

$$\frac{1}{2}\omega'^{\,ab}\,D(\mathcal{L}_{ab}) = \frac{1}{2}\omega^{ab}\,D(\Lambda)D(\mathcal{L}_{ab})D(\Lambda^{-1}) + D(\Lambda)dD(\Lambda^{-1})\,, \tag{2.274}$$

then use the general properties

$$D(\Lambda)D(\mathcal{L}_{ab})D(\Lambda^{-1}) = \Lambda^{-1}{}_a{}^c\Lambda^{-1}{}_b{}^d D(\mathcal{L}_{cd})\;,\quad D(\Lambda)dD(\Lambda^{-1}) = \frac{1}{2}(\Lambda d\Lambda^{-1})^{ab}D(\mathcal{L}_{ab})\,. \tag{2.275}$$

From (2.273) we derive the following transformation property of the Riemann curvature 2-form under a local Lorentz transformation:

$$R^{ab} \;\to\; R'^{\,ab} \;=\; \Lambda^a{}_c\,R^{cd}\,\Lambda^{-1}{}_d{}^b\,. \tag{2.276}$$

The reader can easily verify that:

$$\mathcal{D}^2\Phi = \frac{1}{2}\,R^{ab}\,D(\mathcal{L}_{ab}) \wedge \Phi\,,. \tag{2.277}$$

Exercise: prove this.

On the vierbein 1-forms V^a the Lorentz-covariant derivative yields the torsion tensor:

$$\mathcal{D}V^a = dV^a + \frac{1}{2}\,\omega^{bc}\,C_{bc,d}{}^a \wedge V^d = dV^a + \omega^a{}_b \wedge V^b = T^a\,, \tag{2.278}$$

where we have used the property that on a 4-vector $D(\mathcal{L}_{bc})^a{}_d = C_{bc,d}{}^a$. Deriving twice the vierbein and using (2.277) we find the Bianchi identity for the torsion:

$$\mathcal{D}T^a = \mathcal{D}^2V^a = R^a{}_b \wedge V^b\,. \tag{2.279}$$

The covariant derivative $\mathcal{D}$ on R^{ab} reads:

$$\mathcal{D}R^{ab} = \tilde{D}R^{ab} = dR^{ab} + \omega^a{}_c \wedge R^{cb} - \omega^b{}_c \wedge R^{ca} = 0\,, \tag{2.280}$$

where we have used the Bianchi identity for R^{ab}.

A general property which we shall use in the following is the transformation property of R^{ab} under a generic infinitesimal transformation of ω^{ab}:

$$\omega^{ab} \to \omega^{ab} + \delta\omega^{ab} \;\Rightarrow\quad R^{ab} \to R^{ab} + \delta R^{ab} \quad : \quad \delta R^{ab} = \mathcal{D}\delta\omega^{ab}\,, \tag{2.281}$$

Exercise: Prove this.

On a spinor ψ the covariant derivative $\mathcal{D}$ reads

$$\mathcal{D}\psi = d\psi + \frac{1}{4}\omega_{ab}\,\gamma^{ab}\psi\,, \tag{2.282}$$

where we have used the property (A.21) that $D(\mathcal{L}_{ab}) = \gamma_{ab}/2$, where γ^a are the (constant) gamma-matrices defined in the local inertial frame (see Appendix A). Equation (2.277) then implies

$$\mathcal{D}^2\psi = \frac{1}{4}\,R_{ab}\,\gamma^{ab}\psi\,. \tag{2.283}$$

To end this subsection, let us compute the variation of V^a along $\epsilon^b\,\tilde{\mathcal{P}}_b$ using (2.270), in the case of vanishing torsion $T^a = 0$:

$$\delta_\epsilon V^a = -\delta_\epsilon \tilde{A}^a = \tilde{D}\epsilon^a - \iota_\epsilon T^a = \mathcal{D}\epsilon^a - \iota_\epsilon T^a = \mathcal{D}\epsilon^a\,. \tag{2.284}$$

Defining $\epsilon^\mu = \epsilon^a\,V_a{}^\mu$ we find

$$\begin{aligned}\delta_\epsilon V_\mu{}^a &= \mathcal{D}_\mu\epsilon^a = \partial_\mu(\epsilon^\nu\,V_\nu{}^a) + \omega_\mu{}^a{}_b\,\epsilon^\nu\,V_\nu{}^b = (\nabla_\mu\epsilon^\nu)\,V_\nu{}^a + \epsilon^\nu\,(\nabla_\mu V_\nu{}^a) + \omega_\mu{}^a{}_b\,\epsilon^\nu\,V_\nu{}^a =\\ &= (\nabla_\mu\epsilon^\nu)\,V_\nu{}^a = \partial_\mu\epsilon^\nu V_\nu{}^a + \Gamma^\nu_{\mu\rho}\epsilon^\rho\,V_\nu{}^a = \partial_\mu\epsilon^\nu V_\nu{}^a + \Gamma^\rho_{\nu\mu}\epsilon^\nu\,V_\rho{}^a + 2\,\Gamma^\rho_{[\mu\nu]}\epsilon^\nu\,V_\rho{}^a =\\ &= \partial_\mu\epsilon^\nu V_\nu{}^a + \epsilon^\nu\,\partial_\nu V_\mu{}^a - \epsilon^\nu\,\nabla_\nu V_\mu{}^a + T^a_{\mu\rho}\epsilon^\rho =\\ &= \partial_\mu\epsilon^\nu V_\nu{}^a + \epsilon^\nu\,\partial_\nu V_\mu{}^a + \epsilon^\nu\,\omega_\nu{}^a{}_b\,V_\mu{}^b + T^a_{\mu\rho}\epsilon^\rho = \partial_\mu\epsilon^\nu V_\nu{}^a + \epsilon^\nu\,\partial_\nu V_\mu{}^a + \epsilon^\nu\,\omega_\nu{}^a{}_b\,V_\mu{}^b\,,\end{aligned} \tag{2.285}$$

where we have used the first of Eq.s (2.236). In the last line we have used the condition of vanishing torsion once again. Notice that the first two terms in the last line are a diffeomorphism transformation on the vierbein by a parameter ϵ^μ. The second is a local Lorentz transformation by a parameter $\epsilon^a{}_b = \epsilon^\nu\,\omega_\nu{}^a{}_b$.

2.7 Einstein Gravity in the First Order Formalism

Let us start considering pure Einstein's gravity in the absence of matter. It is useful to write Einstein-Hilbert action in the fist order (or Palatini) formalism, which consists in treating V^a, ω^{ab} as off-shell independent fields. As we shall see, the field equations will provide the torsion equation $T^a = 0$ which allows to express ω^{ab} in terms of V^a, besides yielding the Einstein equation in the vacuum.

We write the Einstein-Hilbert action (we choose $\kappa^2 = 8\pi\, G_N = 1$):

$$S = -\frac{1}{16\pi G_N}\int_{\mathcal{M}_4} d^4x e\, R = -\frac{1}{2}\int_{\mathcal{M}_4} d^4x e\, R\,, \tag{2.286}$$

as an integral over space-time $\mathcal{M}_4$ of a 4-form Lagrangian

$$S = \int_{\mathcal{M}_4} \mathcal{L}^{(4)}_{EH} = -\frac{1}{4}\int_{\mathcal{M}_4} R^{ab}\wedge V^c \wedge V^d\,\epsilon_{abcd}\,. \tag{2.287}$$

To prove this we start writing, inside $\mathcal{L}^{(4)}_{EH}$, R^{ab} in components with respect to the vielbein basis:

$$R^{ab}\wedge V^c\wedge V^d = \frac{1}{2}\,R_{ef}{}^{ab}\,V^e\wedge V^f\wedge V^c\wedge V^d\,. \tag{2.288}$$

Next we write the 4-fold exterior product of vierbeins as follows:

$$\begin{aligned} V^e\wedge V^f\wedge V^c\wedge V^d &= V_\mu{}^e V_\nu{}^f V_\rho{}^c V_\sigma{}^d\, dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma = -\epsilon^{\mu\nu\rho\sigma}\,V_\mu{}^e V_\nu{}^f V_\rho{}^c V_\sigma{}^d\, d^4x = \\ &= -d^4x\, e\,\epsilon^{efcd}\,, \end{aligned} \tag{2.289}$$

where we have used the property $dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma = -d^4x\,\epsilon^{\mu\nu\rho\sigma}$ and the definition of $e = \sqrt{|\det(g_{\mu\nu})|} = \det(V_\mu{}^a)$. The 4-form lagrangian $\mathcal{L}^{(4)}_{EH}$ can then be recast as follows:

$$\mathcal{L}^{(4)}_{EH} = -\frac{1}{4}R^{ab}\wedge V^c\wedge V^d\epsilon_{abcd} = d^4x\,\frac{e}{8}\,R_{ef}{}^{ab}\epsilon^{efcd}\epsilon_{abcd} = -d^4x\,\frac{e}{2}\,R_{ab}{}^{ab} = -d^4x\,\frac{e}{2}\,R\,. \tag{2.290}$$

The convenience with rewriting the action in the above form will become apparent below. Let us evaluate the field equations for by varying the action with respect to ω^{ab} and V^a (we recall that in the first order formalism, the two fields are regarded as independent). As we vary $\omega^{ab}\to\omega^{ab}+\delta\omega^{ab}$, only the Riemann tensor varies in the action, so that we have

$$\begin{aligned} \delta S &= -\frac{1}{4}\int_{\mathcal{M}_4}\delta R^{ab}\wedge V^c\wedge V^d\,\epsilon_{abcd} = -\frac{1}{4}\int_{\mathcal{M}_4}\mathcal{D}\delta\omega^{ab}\wedge V^c\wedge V^d\,\epsilon_{abcd} = \\ &= -\frac{1}{4}\int_{\mathcal{M}_4} d(\delta\omega^{ab}\wedge V^c\wedge V^d\,\epsilon_{abcd}) - \frac{1}{2}\int_{\mathcal{M}_4}\delta\omega^{ab}\wedge\mathcal{D}V^c\wedge V^d\,\epsilon_{abcd}\,, \end{aligned} \tag{2.291}$$

where we have used Eq. (2.281) and we have integrated by parts. Disregarding the total derivative term and using $T^a = \mathcal{D}V^a$, we find:

$$\frac{\delta S}{\delta\omega^{ab}} = 0 \;\Leftrightarrow\; 0 = \mathcal{D}V^c\wedge V^d\,\epsilon_{abcd} = \epsilon_{abcd}\,T^c\wedge V^d \;\Leftrightarrow\; T^a = 0\,, \tag{2.292}$$

that is we find the torsion equation $T^a = 0$ which makes the connection a Levi-Civita one and allows to determine ω^{ab} in terms of V^a, see Eq. (2.250).

We now vary the action with respect to V^a and find:

$$\frac{\delta S}{\delta V^d} = 0 \;\Leftrightarrow\; 0 = R^{ab} \wedge V^c \epsilon_{abcd} = 0\,. \tag{2.293}$$

To prove that this equation is the Einstein equation in the vacuum, let us multiply both sides by V^g:

$$\begin{aligned} 0 = R^{ab} \wedge V^c \wedge V^g \epsilon_{abcd} &= \frac{1}{2}\, R_{ef}{}^{ab}\, V^e \wedge V^f \wedge V^c \wedge V^g\, \epsilon_{abcd} = -d^4x\, \frac{e}{2}\, R_{ef}{}^{ab}\, \epsilon^{efcg}\, \epsilon_{abcd} = \\ = d^4x\, \frac{e}{2}\, R_{ef}{}^{ab}\, 3!\delta^{efg}_{abd} &= d^4x\, e\, R_{ef}{}^{ab}\, (\delta^{ef}_{ab}\delta^g_d - 2\,\delta^{ef}_{ad}\delta^g_b) = -2\, d^4x\, e\, \left(\mathcal{R}^g{}_d - \frac{1}{2}\,\delta^g_d\,\mathcal{R}\right)\,, \end{aligned} \tag{2.294}$$

which implies Einstein's equation in the vacuum:

$$\mathcal{R}_{ab} - \frac{1}{2}\,\eta_{ab}\,\mathcal{R} = 0\,. \tag{2.295}$$

Symmetries of the action. The action is manifestly invariant under local Lorentz transformations. Invariance of the action under local transformations generated by $\tilde{\mathcal{P}}_a$ follows from the fact that these transformations are described by diffeomorphisms on the potentials V^a, ω^{ab}, and thus implemented at the infinitesimal level by Lie derivatives, ad by the fact that the Lagrangian 4-form is written just in terms of exterior products and exterior derivatives of forms. By the properties of Lie derivatives with respect to the exterior product of forms and exterior derivatives, the infinitesimal variation of the whole Lagrangian under a corresponding variation of its elementary fields, amounts of a Lie derivative:

$$\delta_\epsilon \mathcal{L}^{(4)}_{EH} = \ell_\epsilon \mathcal{L}^{(4)}_{EH} = d(\iota_\epsilon \mathcal{L}^{(4)}_{EH}) + \iota_\epsilon\, d\mathcal{L}^{(4)}_{EH}\,. \tag{2.296}$$

The latter term vanishes being $\mathcal{L}^{(4)}_{EH}$ a top-rom on the 4-dimensional space-time, while the former is a total derivative, which vanishes if all fields are taken to vanish at the boundary of $\mathcal{M}_4$.

2.8 Supergravity

Let us now extend the above discussion to the construction of a theory which is invariant under the *local* $\mathcal{N} = 1$ super-Poincaré group. Historically $\mathcal{N} = 1$ pure supergravity was constructed first by Ferrara, Freedman and van Nieuwenhuizen in 1976 [25]. It was first derived in the second order formalism, i.e. writing ω^{ab} in terms of the other fields by imposing the vanishing torsion equation (actually the vanishing *supersorsion* equation, as we shall see) from the very start. Eventually the same results were derived by Deser and Zumino in the first order formalism [26].

Just as we did for pure gauge theories and for Einstein gravity, we start working on an extended space P. This time however P is locally the product of $\mathcal{N} = 1$ superspace $\mathcal{M}^{(4|1)}$ and the super-Poincaré group. We define on it the connection 1-forms, curvatures and local super-Poincaré transformations thereof as effected by diffeomorphisms on P. In particular supersymmetry transformations will be viewed in this framework as diffomorphisms along the base space $\mathcal{M}^{(4|1)}$ acting on the fermionic coordinates, just as general coordinate transformations were implemented by diffeomorphisms on the x^μ coordinates. Then we reduce everything back to $\mathcal{M}_4$ by setting all fermionic coordinates θ, as well ad the components of the forms along $d\theta$ in $\mathcal{M}^{(4|1)}$ to zero. The reason why the base space is superspace and not simply space-time is that while horizontality will still hold for the Lorentz generators, the same will not be true for the supersymmetry generators. In other words the dependence on θ of the superfields is not the effect of a gauge transformation, and thus the restriction to $\theta = d\theta = 0$ is not a gauge choice. Indeed if supersymmetry transformations were of gauge type, also their square, namely space-time diffeomorphisms would be gauge transformations, which, as pointed our earlier, cannot be the case.

This approach to supergravity, called the geometric or rheonomic approach, was developed by the authors of [5] and is thoroughly explained in these references. It is a powerful technique in order to construct supergravity theories from simple geometric principles. In what follows however, we shall use this geometric picture only to recover the interpretation of the vierbein V^a, the gravitino field Ψ and ω^{ab} as the "gauge fields" of the super-Poincaré algebra. The construction of the supergravity action will be done using a purely space-time perspective, namely writing down a combination of the Einstein-Hilbert action and the action for the gravitino field. The supersymmetry transformation laws of the elementary fields leaving the action invariant will be guessed and a posteriori interpreted as the result of a super-diffeomorphisms over P.

The generators T_A of the $\mathcal{N} = 1$ super-Poincaré group comprise, aside from the Poincaré generators $\mathcal{P}_a$, $\mathcal{L}_{ab}$, the supersymmetry generators Q. The super-Lie-algebra valued 1-form $\tilde{\Omega}_g$ on P now has the following form (let us restore κ):

$$\tilde{\Omega}_g = \tilde{A}^A\, T_A = -V^a\, \mathcal{P}_a + \frac{1}{2}\, \omega^{ab}\, \mathcal{L}_{ab} - \frac{i}{\sqrt{2}}\, \kappa\, \bar{\Psi}\, Q\,, \tag{2.297}$$

where Ψ is a spinor-valued 1-form on P, whose components Ψ_μ along $T^*\mathcal{M}_4$ have one space-time index and one spinor index. They describe a spin-3/2 field called the *gravitino* which completes, together with the graviton field V^a, an $\mathcal{N} = 1$ massless supermultiplet. We shall deal below with the dynamical description of spin-3/2 fields. Let us just anticipate that the dimension of Ψ_μ is length$^{-\frac{3}{2}}$ and thus that of Ψ is length$^{-\frac{1}{2}}$.

Just as the Q-generators, also Ψ satisfies the Majorana condition:

$$\Psi = C\bar{\Psi}^T\,. \tag{2.298}$$

We denote by $\tilde{Q}$ the vector in TP which is dual to Ψ. Due to the normalization factor in the definition of $\tilde{\Omega}_g$ we choose the duality relations to be

$$\begin{aligned} &V^a(\tilde{\mathcal{P}}_b) = -\delta^a_b\ ,\quad \omega^{ab}(\mathcal{L}_{cd}) = 2\,\delta^{ab}_{cd}\ ,\quad \kappa\,\Psi_\alpha(\bar{\tilde{Q}}^\beta) = -i\,\sqrt{2}\delta^\beta_\alpha\ ,\quad \kappa\,\bar{\Psi}^\alpha(\tilde{Q}_\beta) = i\,\sqrt{2}\delta^\alpha_\beta\,, \\ &V^a(\tilde{\mathcal{L}}_{ab}) = \omega^{ab}(\tilde{\mathcal{P}}_c) = \Psi(\mathcal{L}_{ab}) = \Psi(\tilde{\mathcal{P}}_c) = V^a(\tilde{Q}) = \omega^{ab}(\tilde{Q}) = 0\,. \end{aligned} \tag{2.299}$$

Out of (2.297) we construct, as usual, the Lie-algebra-valued curvature 2-form:

$$\tilde{\mathcal{F}} \equiv d\tilde{\Omega}_g + \tilde{\Omega}_g \wedge \tilde{\Omega}_g = R^A\, T_A = -\tilde{T}^a\, \mathcal{P}_a + \frac{1}{2}\, R^{ab}\, \mathcal{L}_{ab} - \frac{i}{\sqrt{2}}\, \kappa\, \bar{\rho}\, Q\,, \tag{2.300}$$

having denoted by $\{R^A\} = \{-\tilde{T}^a,\, R^{ab},\, \rho\}$ the components of the curvature 2-forms. Using the commutation/anti-commutation rules for the super-Poincaré algebra given at the beginning of Sect. 2.1, we can compute the exterior product $\tilde{\Omega}_g \wedge \tilde{\Omega}_g$:

$$\begin{aligned}\tilde{\Omega}_g \wedge \tilde{\Omega}_g &= \frac{1}{2}\omega^a{}_c \wedge \omega^{cb}\,\mathcal{L}_{ab} - \frac{1}{2}\omega^a{}_c \wedge V^c\,\mathcal{P}_a + \frac{\kappa}{2}\,\frac{\omega^{ab}}{2} \wedge \left(-\frac{i}{\sqrt{2}}\,\bar{\Psi}\right)\,[\mathcal{L}_{ab},\, Q] + \\ &+ \frac{\kappa^2}{2}\left(-\frac{i}{\sqrt{2}}\right)\left(-\frac{i}{\sqrt{2}}\right)\bar{\Psi}\{Q,\,\bar{Q}\}\,\Psi = \\ &= \frac{1}{2}\omega^a{}_c \wedge \omega^{cb}\,\mathcal{L}_{ab} - \frac{1}{2}\omega^a{}_c \wedge V^c\,\mathcal{P}_a - \frac{i}{\sqrt{2}}\kappa\omega^{ab}\,\bar{Q} \wedge \frac{\gamma_{ab}}{4}\,\Psi - \frac{i}{2}\,\kappa^2\,\bar{\Psi}\gamma^a\Psi\,\mathcal{P}_a\,, \end{aligned} \tag{2.301}$$

where we have used the property that $C\gamma^{ab}$ is symmetric, so that $\bar{\Psi}\gamma^{ab}Q = -\bar{Q}\gamma^{ab}\Psi$. From (2.300) we then find the definition of curvatures:

$$\tilde{T}^a = \mathcal{D}V^a + \frac{i}{2}\,\kappa^2\,\bar{\Psi}\gamma^a\Psi\,, \tag{2.302}$$

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\,, \tag{2.303}$$

$$\rho = \mathcal{D}\Psi = d\Psi + \frac{1}{4}\,\omega^{ab} \wedge \gamma_{ab}\Psi\,. \tag{2.304}$$

The curvature associated with V^a was denoted by $\tilde{T}^a$ and is called *super-torsion*. It differs from the torsion T^a by the gravitino bilinear $\frac{i}{2}\,\bar{\Psi}\gamma^a\Psi$.

We can also derive the super-Bianchi identities from:

$$d\tilde{\mathcal{F}} + \tilde{\Omega}_g \wedge \tilde{\mathcal{F}} - \tilde{\mathcal{F}} \wedge \tilde{\Omega}_g = 0 \;\;\Leftrightarrow\;\; dR^A + C_{BC}{}^A\,\tilde{A}^B \wedge R^C = 0\,.$$

$$\mathcal{D}\tilde{T}^a = R^a{}_b \wedge V^b - i\,\kappa^2\,\bar{\Psi} \wedge \gamma^a\mathcal{D}\Psi\,, \tag{2.305}$$

$$\mathcal{D}R^a{}_b = dR^{ab} + \omega^a{}_c \wedge R^{cb} - \omega^b{}_c \wedge R^{ca} = 0\,, \tag{2.306}$$

$$\mathcal{D}\rho = \mathcal{D}^2\Psi = R^{ab}\frac{\gamma_{ab}}{4} \wedge \Psi\,. \tag{2.307}$$

Let us set now κ back to one.

2.8.1 The Gravitino Field

The field $\Psi_a = V_a{}^\mu\Psi_\mu$ transforms, with respect to the local Lorentz group, in the product $(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}) \times [(\mathbf{\frac{1}{2}}, \mathbf{0}) + (\mathbf{0}, \mathbf{\frac{1}{2}})]$, that is, with respect to the spin-group, in the product of the spin-1 times the spin 1/2 representation. The irreducible spin-3/2 component is selected by imposing the constraint:

$$\gamma^a\,\Psi_a = \gamma^\mu\,\Psi_\mu = 0\,, \tag{2.308}$$

which sets the $(\mathbf{\frac{1}{2}}, \mathbf{0}) + (\mathbf{0}, \mathbf{\frac{1}{2}})$ component in the product (i.e. the spin-1/2 one) to zero.

The dimension of Ψ_μ is that of spinor field, that is, in natural units:

$$[\Psi_\mu] = (\text{length})^{-\frac{3}{2}}\,. \tag{2.309}$$

In flat space-time, the field equation for a massless spin-3/2 field Ψ_μ is the *RaritaSchwinger (RS) equation*:

$$\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\partial_\rho\Psi_\sigma = 0\,. \tag{2.310}$$

A property if this equation is its invariance under the gauge transformation:

$$\Psi_\mu \;\to\; \Psi_\mu + \partial_\mu\lambda\,, \tag{2.311}$$

which implies that the *longitudinal modes* represented by the spin-1/2 field λ, are unphisical, and thus that the only physical components are the two helicity $\pm 3/2$ states.

In curved space-time, on a field as the gravitino which has both spinor and space-time indices, in principle we should replace ordinary derivatives by the covariant one ∇ which includes the affine connection Γ and the spin connection ω, needed for the covariance with respect to diffeomorphisms and to the local Lorentz group, respectively:

$$\partial_\mu\Psi \;\longrightarrow\; \nabla_\mu\Psi_\nu = \mathcal{D}_\mu\Psi_\nu - \Gamma^\rho_{\mu\nu}\,\Psi_\rho = \partial_\mu\Psi_\nu + \frac{1}{4}\,\omega^{ab}\gamma_{ab}\Psi_\nu - \Gamma^\rho_{\mu\nu}\,\Psi_\rho\,. \tag{2.312}$$

However, we use the following generalization of equation (2.310) to curved space-time:

$$\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\mathcal{D}_\rho\Psi_\sigma = 0 \;\;\Leftrightarrow\;\; \gamma_{[\mu}\mathcal{D}_\nu\Psi_{\rho]} = 0\,, \tag{2.313}$$

where $[\nu\rho\sigma]$ indicates the complete antisymmetrization in the three indices. We did not include the affine connection Γ in the covariant derivative because it would make it inconsistent with supersymmetry [6]. Moreover equation (2.313) is diffeomorphism-invariant since the Christoffel symbol, due to the antisymmetrization in the indices, would contribute a term depending on the torsion tensor, which is separately covariant under diffeomorphisms:

$$0 = \gamma_{[\mu}\mathcal{D}_\nu\Psi_{\rho]} = \gamma_{[\mu}\nabla_\nu\Psi_{\rho]} + \frac{1}{2}\,\gamma_{[\mu}T^\sigma_{\nu\rho]}\Psi_\sigma\,, \tag{2.314}$$

Next we show that (2.313) implies that each space-time component of the field Ψ satisfies the Dirac equation:

$$\gamma^\nu\mathcal{D}_\nu\Psi_\mu = 0\,. \tag{2.315}$$

A way for deriving equation (2.315) is then to contract the last of eq.s (2.313) by $\gamma^{\mu\nu}$. After some gamma-matrix algebra, and using the properties (A.33), we find:

$$\gamma^\nu\mathcal{D}_\nu\Psi_\mu = \gamma^\nu\mathcal{D}_\mu\Psi_\nu\,. \tag{2.316}$$

Exercise: Prove this.

Using then (2.316), we find:

$$\begin{aligned}
\gamma^\nu\mathcal{D}_\nu\Psi_\mu &= \gamma^\nu\mathcal{D}_\mu\Psi_\nu = \gamma^\nu\partial_\mu\Psi_\nu + \frac{1}{4}\,\omega_\mu{}^{ab}\gamma^\nu\gamma_{ab}\Psi_\nu = \\
&= \partial_\mu(\gamma^\nu\Psi_\nu) - (\partial_\mu\gamma^\nu)\,\Psi + \frac{1}{4}\,\omega_\mu{}^{ab}[\gamma^c,\,\gamma_{ab}]\,V_c{}^\nu\,\Psi_\nu = \\
&= -(\partial_\mu V_a{}^\nu)\,\gamma^a\,\Psi + \omega_\mu{}^c{}_a\,V_c{}^\nu\gamma^a\Psi_\nu = -\omega_\mu{}^c{}_a\,V_c{}^\nu\gamma^a\,\Psi_\nu + \omega_\mu{}^c{}_a\,V_c{}^\nu\gamma^a\Psi_\nu = 0\,,
\end{aligned} \tag{2.317}$$

by virtue of (2.308) and the second of (2.236).

Let us derive from the RS equation an other property, namely that:

$$g^{\mu\nu}\mathcal{D}_\mu\Psi_\nu = 0\,. \tag{2.318}$$

To this end we write the left hand side as follows:

$$\begin{aligned} g^{\mu\nu}\,\mathcal{D}_\mu\Psi_\nu &= \frac{1}{2}(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)\,\mathcal{D}_\mu\Psi_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu\mathcal{D}_\mu\Psi_\nu+\gamma^\nu\gamma^\mu\mathcal{D}_\mu\Psi_\nu) = \frac{1}{2}\gamma^\mu\gamma^\nu\mathcal{D}_\mu\Psi_\nu = \\ &= \frac{1}{2}\gamma^\mu\gamma^\nu\mathcal{D}_\nu\Psi_\mu = 0\,, \end{aligned} \tag{2.319}$$

where we have used Eq.s (2.315) and (2.316). By the same token, the RS equation also implies $\gamma^{\mu\nu}\mathcal{D}_\mu\Psi_\nu = 0$.

The RS equation can be derived from the Lagrangian density:

$$\mathcal{L}_{RS} = \epsilon^{\mu\nu\rho\sigma}\,\bar{\Psi}_\mu\gamma_5\gamma_\nu\mathcal{D}_\rho\Psi_\sigma\,, \tag{2.320}$$

which can also be written in the following equivalent form:[10]

$$\mathcal{L}_{RS} = -i\,e\,\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\Psi_\rho\,. \tag{2.321}$$

Exercise: Check this.

Just as we did for the Einstein-Hilbert action, it is useful to write the action in terms of a Lagrangian 4-form written just in terms of exterior derivatives and exterior products of the elementary fields $\{V^a,\,\omega^{ab},\,\Psi\}$ (we include here ω^{ab} as an elementary field since we shall work in the first order formalism in which this field is fixed in terms of the others only through one of the field equations). We can write the following RS 4-form Lagrangian:

$$\mathcal{L}^{(4)}_{RS} = \bar{\Psi}\wedge\gamma_5\gamma_a\,\mathcal{D}\Psi\wedge V^a\,. \tag{2.322}$$

The reader can prove that:

$$\mathcal{L}^{(4)}_{RS} = d^4x\,\mathcal{L}_{RS}\,. \tag{2.323}$$

A consistent definition of the gravitino field on a curved space-time would require the decoupling of its longitudinal $\pm 1/2$ helicity modes in a local-Lorentz invariant way, that is the action should be invariant under the following local-Lorentz-covariant version of (2.311)

$$\Psi_\mu \;\to\; \Psi_\mu + \mathcal{D}_\mu\lambda\,. \tag{2.324}$$

The RS action $\mathcal{L}^{(4)}_{RS}$ alone does not exhibit such invariance but, as we shall see below, a theory describing Ψ coupled to gravity does. This is supergravity and (2.324) will describe a local supersymmetry transformation of Ψ.

[10] There is no factor κ.

2.8.2 The Supergravity Action

Let us couple Ψ to gravity by writing the following Lagrangian 4-form:

$$\mathcal{L}^{(4)} = \mathcal{L}^{(4)}_{EH} + \mathcal{L}^{(4)}_{RS} = -\frac{1}{4\,\kappa^2}\,R^{ab}\wedge V^c\wedge V^d - \bar{\Psi}\wedge\gamma_5\gamma_a\,\mathcal{D}\Psi\wedge V^a = \\ = d^4x\,e\,\left[-\frac{\mathcal{R}}{2\kappa^2} + \frac{1}{e}\,\epsilon^{\mu\nu\rho\sigma}\,\bar{\Psi}_\mu\gamma_5\gamma_\nu\mathcal{D}_\rho\Psi_\sigma\right]\,. \quad (2.325)$$

We set again $\kappa = 1$ and we shall restore it in the sequel when needed.

In the first-order (Palatini) formalism, we vary the action with respect to ω^{ab}, treated as independent:

$$\delta\mathcal{L}^{(4)} = \delta\omega^{ab}\wedge\left[-\frac{1}{2}\,DV^c\wedge V^d\epsilon_{abcd} + \frac{1}{4}\bar{\Psi}\wedge\gamma_5\gamma_c\gamma_{ab}\Psi\wedge V^c\right]\,. \quad (2.326)$$

Now use the property that $\Psi^\alpha\wedge\Psi^\beta$, $\alpha,\beta = 1,2,3,4$ being the spinor labels, is symmetric in the two indices, since switching the position of the two fields implies one minus sign from the fact that the fields are Grassman valued, and an other minus sign being them 1-forms. For this reason a bilinear $\bar{\Psi}\gamma^{a_1\dots a_k}\Psi = \Psi^T C\gamma^{a_1\dots a_k}\Psi$ is non-vanishing only if $C\gamma^{a_1\dots a_k}$ is symmetric, and thus:

$$\bar{\Psi}\gamma^{abc}\Psi = \bar{\Psi}\gamma_5\gamma^a\Psi = \bar{\Psi}\Psi = 0\,. \quad (2.327)$$

Next, in the second term on the right hand side of (2.326), we write $\gamma_5\gamma_c\gamma_{ab} = \gamma_5\gamma_{cab} + 2\eta_{c[a}\gamma_5\gamma_{b]}$. The second matrix does not contribute to the gravitino bilinear, so that we can write (2.326) as follows:

$$\delta\mathcal{L}^{(4)} = \delta\omega^{ab}\wedge\left[-\frac{1}{2}\,DV^c\wedge V^d\epsilon_{abcd} + \frac{1}{4}\bar{\Psi}\wedge\gamma_5\gamma_{cab}\Psi\wedge V^c\right] = \\ = \delta\omega^{ab}\wedge\left[-\frac{1}{2}\,DV^c\wedge V^d\epsilon_{abcd} - \frac{i}{4}\epsilon_{abcd}\bar{\Psi}\wedge\gamma^c\Psi\wedge V^d\right]\,. \quad (2.328)$$

The equation of motion from the variation of ω^{ab} reads:

$$\frac{\delta S}{\delta\omega^{ab}}\ \Rightarrow,\ \tilde{T}^a = \mathcal{D}V^a + \frac{i}{2}\,\bar{\Psi}\wedge\gamma^a\Psi = T^a + \frac{i}{2}\,\bar{\Psi}\wedge\gamma^a\Psi = 0\,. \quad (2.329)$$

In contrast to the pure gravity case, the equation implies the vanishing of the *super-torsion* instead of the torsion tensor. As a consequence of this, *the connection ω^{ab} is torsionful*, the torsion being:

$$T^a = \mathcal{D}V^a = -\frac{i}{2}\,\bar{\Psi}\wedge\gamma^a\Psi\ \Rightarrow\ T^a_{\mu\nu} = 2\,V_\rho{}^a\,\Gamma^\rho_{[\mu\nu]} = -i\,\bar{\Psi}_{[\mu}\gamma^a\Psi_{\nu]}\,. \quad (2.330)$$

We now compute the other field equations by varying the action with respect to V^a and Ψ:[11]

$$\begin{aligned}
\delta_V \mathcal{L}^{(4)} &= \delta V^a \, \wedge \left[-\frac{1}{2} R^{bc} \wedge V^d \epsilon_{abcd} + \bar{\Psi} \wedge \gamma_5 \gamma_a \mathcal{D}\Psi\right] = 0 \,, \\
-\delta_\Psi \mathcal{L}^{(4)} &= \delta\bar{\Psi} \wedge \gamma_5\gamma_a \mathcal{D}\Psi \wedge V^a + \bar{\Psi} \wedge \gamma_5\gamma_a \mathcal{D}\delta\Psi \wedge V^a = \\
&= \delta\bar{\Psi} \wedge \gamma_5\gamma_a \mathcal{D}\Psi \wedge V^a + \mathcal{D}\bar{\Psi} \wedge \gamma_5\gamma_a \delta\Psi \wedge V^a + \bar{\Psi} \wedge \gamma_5\gamma_a \delta\Psi \wedge \mathcal{D}V^a = \\
&= 2\,\delta\bar{\Psi} \wedge \gamma_5\gamma_a \mathcal{D}\Psi \wedge V^a + \bar{\Psi} \wedge \gamma_5\gamma_a \delta\Psi \wedge \mathcal{D}V^a = \\
&= 2\delta\bar{\Psi} \left[\gamma_5\gamma_a \mathcal{D}\Psi \wedge V^a - \frac{1}{2}\gamma_5\gamma_a \Psi \wedge \mathcal{D}V^a\right] = 0 \,,
\end{aligned} \tag{2.331}$$

where in going from the first to the second line of $\delta_\Psi \mathcal{L}^{(4)}$ we have performed an integration by parts, using:

$$d(\bar{\Psi} \wedge \gamma_5\gamma_a \delta\Psi \wedge V^a) = \mathcal{D}\bar{\Psi} \wedge \gamma_5\gamma_a \delta\Psi \wedge V^a - \bar{\Psi} \wedge \gamma_5\gamma_a \mathcal{D}\delta\Psi \wedge V^a + \bar{\Psi} \wedge \gamma_5\gamma_a \delta\Psi \wedge \mathcal{D}V^a \,. \tag{2.332}$$

We then find the two equations:

$$R^{ab} \wedge V^c \epsilon_{abcd} = -2\bar{\Psi} \wedge \gamma_5\gamma_d \mathcal{D}\Psi \,, \tag{2.333}$$

$$\gamma_a \mathcal{D}\Psi \wedge V^a = \frac{1}{2}\gamma_a \Psi \wedge \mathcal{D}V^a \,. \tag{2.334}$$

Let us now use the property (A.32):

$$\gamma_a \Psi \, \bar{\Psi}\gamma^a \Psi = 0 \,, \tag{2.335}$$

to rewrite on the left hand side of (2.334):

$$\gamma_a \Psi \wedge \mathcal{D}V^a = \gamma_a \Psi \wedge \left(\tilde{T}^a - \frac{i}{2}\bar{\Psi} \wedge \gamma^a \Psi\right) = \gamma_a \Psi \wedge \tilde{T}^a \tag{2.336}$$

Exercise: Prove the identity (A.32) using the basic Fierz identity for the product of two Ψ (A.31).

The field equations can then be recast in the following equivalent form:

$$\tilde{T}^a = 0 \,, \tag{2.337}$$

$$R^{ab} \wedge V^c \epsilon_{abcd} = -2\bar{\Psi} \wedge \gamma_5\gamma_d \mathcal{D}\Psi \,, \tag{2.338}$$

$$\gamma_a \mathcal{D}\Psi \wedge V^a = \frac{1}{2}\gamma_a \Psi \wedge \tilde{T}^a = 0 \,. \tag{2.339}$$

Einstein's equation in the torsionless connection. We can write everything in terms of a *torsionless* connection $\hat{\omega}^{ab}$ by writing:

$$\begin{aligned}
\omega^{ab} &= \hat{\omega}^{ab} + \Delta\omega^{ab} \,, \\
\hat{\mathcal{D}}V^a &\equiv dV^a + \hat{\omega}^a{}_b \wedge V^b = 0 \,.
\end{aligned} \tag{2.340}$$

[11]In varying with respect to Ψ we recall that, due to the Majorana condition, $\bar{\Psi}$ and Ψ are not independent.

The component $\Delta\omega$ can be evaluated as follows. From Eq.s (2.340) and (2.330) we find:

$$\Delta\omega_{[c}{}^{a}{}_{b]} = -\frac{i}{2}\,\bar{\Psi}_c\gamma^a\Psi_b\,. \tag{2.341}$$

We then write the following three equations:

$$\begin{aligned}\Delta\omega_{c,ab} - \Delta\omega_{b,ac} &= -i\,\bar{\Psi}_c\gamma_a\Psi_b\,,\\ \Delta\omega_{a,bc} - \Delta\omega_{c,ba} &= -i\,\bar{\Psi}_a\gamma_b\Psi_c\,,\\ \Delta\omega_{b,ca} - \Delta\omega_{a,cb} &= -i\,\bar{\Psi}_b\gamma_c\Psi_a\,.\end{aligned}$$

summing the first two and subtracting the third, we find:

$$\Delta\omega_{c,ab} = -\frac{i}{2}\left(\bar{\Psi}_c\gamma_a\Psi_b + \bar{\Psi}_a\gamma_b\Psi_c - \bar{\Psi}_b\gamma_c\Psi_a\right)\,. \tag{2.342}$$

Next we rewrite the field equations in terms of the torsionless connection $\hat{\omega}$, denoting by hatted symbolsquantities expressed in terms of it. The Riemann tensor then reads:

$$\begin{aligned}R_{cd}{}^{ab} &= \hat{R}_{cd}{}^{ab} + 2\hat{\mathcal{D}}_{[c}\Delta\omega_{d]}{}^{ab} + 2\,\Delta\omega_{[c}{}^{ae}\,\Delta\omega_{d]\,e}{}^{b}\,,\\ \mathcal{R}_c{}^{a} &= \hat{\mathcal{R}}_c{}^{a} + 2\hat{\mathcal{D}}_{[c}\Delta\omega_{b]}{}^{ab} + 2\,\Delta\omega_{[c}{}^{ae}\,\Delta\omega_{b]\,e}{}^{b}\,,\\ \mathcal{R} &= \hat{\mathcal{R}} + 2\hat{\mathcal{D}}_{[a}\Delta\omega_{b]}{}^{ab} + 2\,\Delta\omega_{[a}{}^{ae}\,\Delta\omega_{b]\,e}{}^{b}\,.\end{aligned} \tag{2.343}$$

Recall now that (we suppress the symbol $\wedge$ for the sake of notational simplicity):

$$\begin{aligned}R^{ab}\wedge V^c\wedge V^g\epsilon_{abcd} &= -2\,d^4x\,e\,\left(\mathcal{R}_d{}^{g} - \frac{1}{2}\,\delta^g_d\,\mathcal{R}\right)\,,\\ \bar{\Psi}\gamma_5\gamma_d\mathcal{D}\Psi\,V^g &= -d^4x\,\epsilon^{\mu\rho\sigma\delta}\,\bar{\Psi}_\mu\gamma_5\gamma_\nu\mathcal{D}_\rho\Psi_\sigma V_d{}^{\nu}\,V_\delta{}^{g}\,.\end{aligned} \tag{2.344}$$

This allows to rewrite Einstein's equation in space-time components:

$$\mathcal{R}^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}\,\mathcal{R} = -\frac{1}{e}\,\epsilon^{\rho\sigma\delta\nu}\bar{\Psi}_\rho\gamma_5\gamma^\mu\mathcal{D}_\sigma\Psi_\delta\,. \tag{2.345}$$

Using (2.343) we can write this equation in terms of quantities defined with the torsionless connection:

$$\begin{aligned}&\hat{\mathcal{R}}^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}\,\hat{\mathcal{R}} = \mathcal{T}^{\mu\nu}\,,\\ &\quad\mathcal{T}^{\mu\nu} \equiv -\frac{1}{e}\,\epsilon^{\rho\sigma\delta\nu}\bar{\Psi}_\rho\gamma_5\gamma^\mu\mathcal{D}_\sigma\Psi_\delta - 2\,\Big[\hat{\mathcal{D}}_{[c}\Delta\omega_{b]}{}^{ab} + \Delta\omega_{[c}{}^{ae}\,\Delta\omega_{b]\,e}{}^{b} -\\ &\quad -\frac{1}{2}\left(\hat{\mathcal{D}}_{[b}\Delta\omega_{d]}{}^{bd} + \Delta\omega_{[b}{}^{be}\,\Delta\omega_{d]\,e}{}^{d}\right)\delta^a_c\Big]\,V_a{}^{\nu}\,V^{\mu c}\,.\end{aligned} \tag{2.346}$$

Integrability of the RS equation. Let us now work again with the torsionful connection and work out the integrability condition for the RS equation (2.339)under the condition $\tilde{T}^a = 0 = \mathcal{D}\tilde{T}^a$ implied by (2.337):

$$0 = \mathcal{D}\left(\gamma^a \mathcal{D}\Psi\, V_a\right) = \gamma^a \mathcal{D}^2\Psi\, V_a + \gamma^a \mathcal{D}\Psi\, \mathcal{D}V_a = \gamma^a\, R_{bc}\frac{\gamma^{bc}}{4}\Psi\, V_a - \frac{i}{2}\gamma^a \mathcal{D}\Psi\, \bar{\Psi}\gamma_a\Psi\,, \qquad (2.347)$$

having used Eq.s (2.283) and (2.330). Next we use the general property:

$$\gamma^a\gamma^{bc} = i\epsilon^{abcd}\gamma_5\gamma_d + 2\eta^{a[b}\gamma^{c]}\,, \qquad (2.348)$$

to rewrite the integrability condition in the form:

$$0 = \frac{i}{4}\,\epsilon^{abcd} R_{bc}\gamma^5\gamma_d\Psi\, V_a + \frac{1}{2}\, R^a{}_b\gamma^b\Psi\, V_a - \frac{i}{2}\gamma^a \mathcal{D}\Psi\, \bar{\Psi}\gamma_a\Psi\,. \qquad (2.349)$$

Using equations $\tilde{T}^a = 0 = \mathcal{D}\tilde{T}^a$, the super-Bianchi identity (2.305) yields:

$$R^a{}_b \wedge V^b = i\, \bar{\Psi}\gamma^a \mathcal{D}\Psi\,, \qquad (2.350)$$

which allows to rewrite the second term on the right hand side of (2.349) and as follows:

$$0 = \frac{i}{4}\,\epsilon^{abcd} R_{bc}\gamma^5\gamma_d\Psi\, V_a - \frac{i}{2}\,\gamma^b\Psi\, \bar{\Psi}\gamma_b\mathcal{D}\Psi - \frac{i}{2}\gamma^a \mathcal{D}\Psi\, \bar{\Psi}\gamma_a\Psi\,. \qquad (2.351)$$

The first term on the right hand side can now be rewritten using Einstein's equation (2.338) so to obtain:

$$\begin{aligned} 0 &= \frac{i}{2}\,\gamma^5\gamma_d\Psi\, \bar{\Psi}\gamma_5\gamma^d\mathcal{D}\Psi + \frac{i}{2}\,\gamma^b\Psi\, \bar{\Psi}\gamma_b\mathcal{D}\Psi + \frac{i}{2}\gamma^a\mathcal{D}\Psi\, \bar{\Psi}\gamma_a\Psi = \\ &= \frac{i}{2}\,\gamma^5\gamma_d\Psi\, \bar{\Psi}\gamma_5\gamma^d\mathcal{D}\Psi + \frac{i}{2}\,\gamma^b\Psi\, \bar{\Psi}\gamma_b\mathcal{D}\Psi + \frac{i}{2}\mathcal{D}(\gamma^a\Psi\, \bar{\Psi}\gamma_a\Psi) - i\,\gamma^a\Psi\, \bar{\Psi}\gamma_a\mathcal{D}\Psi = \\ &= \frac{i}{2}\,\left(\gamma^5\gamma_d\Psi\, \bar{\Psi}\gamma_5\gamma^d\mathcal{D}\Psi - \gamma^b\Psi\, \bar{\Psi}\gamma_b\mathcal{D}\Psi\right)\,, \end{aligned} \qquad (2.352)$$

where we have used (A.32) in the total covariant derivative. That equality (2.352) is identically satisfied can be easily verified using the Fierz identity (A.31).

Exercise: Prove this.(Hint: Use the properties (A.33) $\gamma_d\gamma^{ab}\gamma^d = 0$*,* $\gamma_d\gamma^a\gamma^d = -2\gamma^a$*.)*

We shall use in the following the identity (2.352) in the equivalent form:

$$\gamma_d\Psi\, \bar{\Psi}\gamma_5\gamma^d\mathcal{D}\Psi = \gamma^5\,\gamma^b\Psi\, \bar{\Psi}\gamma_b\mathcal{D}\Psi\,. \qquad (2.353)$$

Supersymmetry. The action (2.325) is not invariant under the local transformation of the form (2.324):

$$\Psi_\mu \;\rightarrow\; \Psi_\mu + \mathcal{D}_\mu\epsilon\,. \qquad (2.354)$$

In order for the action to be off-shell invariant in the first order formalism, we would have to devise a corresponding transformation property of the other fields which are off-shell independent, namely of V^a and ω^{ab}. In order to make things simpler, we can go partly on-shell and only require $\tilde{T}^a = \mathcal{D}\tilde{T}^a = 0$. In this way ω^{ab} is no longer independent and we only need to define a transformation property of V^a. After defining this local invariance of the action we shall prove that it *realizes the supersymmetry transformations* on the fields. This latter property, namely the closure of the supersymmetry algebra on the fields, only holds on-shell, that is upon using the field equations. For the sake of simplicity, we shall only verify it on the vierbein V^a for which the field equations are not needed.

Consider then the variation of the action S deriving from a variation of the three fields V^a, ω^{ab}, Ψ:

$$\delta S = \delta\omega^{ab}\,\frac{\delta S}{\delta\omega^{ab}} + \delta V^a\,\frac{\delta S}{\delta V^a} + \delta\Psi\,\frac{\delta S}{\delta\Psi}\,. \tag{2.355}$$

We then impose the condition $\tilde{T}^a = 0$ which allows to express $\omega^{ab} = \omega^{ab}(V,\Psi)$. This implies $\frac{\delta S}{\delta\omega^{ab}} = 0$ so that

$$\delta S = \delta V^a\,\frac{\delta S}{\delta V^a} + \delta\Psi\,\frac{\delta S}{\delta\Psi}\,. \tag{2.356}$$

We wish to define a transformation property for V^a for which, modulo a total derivative, under a transformation $\delta\Psi = \mathcal{D}\epsilon$,

$$\delta\mathcal{L}^{(4)} = \delta_V\mathcal{L}^{(4)} + \delta_\Psi\mathcal{L}^{(4)} = 0\,. \tag{2.357}$$

Let us compute $\delta_\Psi\mathcal{L}^{(4)}$:

$$\delta_\Psi\mathcal{L}^{(4)} = -2\,\bar{\Psi}\gamma_5\gamma_a\mathcal{D}^2\epsilon\,V^a = -\frac{1}{2}\,\bar{\Psi}\gamma_5\gamma_a\gamma_{bc}\epsilon\,R^{bc}\,V^a = -\frac{i}{2}\,\epsilon_{abcd}\bar{\Psi}\gamma^d\epsilon\,R^{bc}\,V^a + \bar{\Psi}\gamma_5\gamma_a\epsilon\,R^{ab}\,V_b\,. \tag{2.358}$$

Now use $\mathcal{D}\tilde{T}^a = 0$ which implies $R^{ab}\,V_b = i\,\bar{\Psi}\gamma^a\mathcal{D}\Psi$, according to Eq. (2.305):

$$\delta_\Psi\mathcal{L}^{(4)} = -\frac{i}{2}\,\epsilon_{abcd}\bar{\Psi}\gamma^d\epsilon\,R^{bc}\,V^a + i\,\bar{\Psi}\gamma_5\gamma_a\epsilon\,\bar{\Psi}\gamma^a\mathcal{D}\Psi\,. \tag{2.359}$$

If we apply the identity (2.353) to the last term (after rewriting $\bar{\Psi}\gamma_5\gamma_a\epsilon = \bar{\epsilon}\gamma_5\gamma_a\Psi$), we find:

$$\begin{aligned}\delta_\Psi\mathcal{L}^{(4)} &= \frac{i}{2}\,R^{ab}\,V^c\epsilon_{abcd}\bar{\Psi}\gamma^d\epsilon + i\,\bar{\epsilon}\gamma_a\Psi\,\bar{\Psi}\gamma_5\gamma^a\mathcal{D}\Psi = \\ &= -\frac{1}{2}\,R^{ab}\,V^c\epsilon_{abcd}(i\,\bar{\epsilon}\gamma^d\Psi) - \bar{\Psi}\gamma_5\gamma^a\mathcal{D}\Psi\,(i\bar{\epsilon}\gamma_a\Psi)\,.\end{aligned} \tag{2.360}$$

Notice that $\delta_\Psi\mathcal{L}^{(4)}$ is precisely canceled by a variation $\delta_V\mathcal{L}^{(4)}$ of the Lagrangian corresponding to the following transformation of the vierbein:

$$\delta V^a = -i\,\bar{\epsilon}\gamma^a\Psi\,. \tag{2.361}$$

Restoring the κ-factors:

$$\delta V^a = -i\,\kappa\,\bar{\epsilon}\gamma^a\Psi\,. \tag{2.362}$$

The action is therefore invariant under the following local transformations:

$$\delta\Psi = \frac{1}{\kappa}\,\mathcal{D}\epsilon\ ,\quad \delta V^a = -i\,\kappa\,\bar{\epsilon}\gamma^a\Psi\,. \tag{2.363}$$

Let us set κ back to one.

These transformation properties are the result of diffeomorphisms in P generated by the vector $v = -i\,\bar{\epsilon}\tilde{Q}/\sqrt{2} = -i\,\overline{\tilde{Q}}\epsilon/\sqrt{2}$:

$$\delta_\epsilon\Psi = -\ell_v\Psi = -d(\iota_v\Psi) - \iota_v\,d\Psi = -d(\iota_v\Psi) - \iota_v\,(\rho - \omega_{ab}\frac{\gamma^{ab}}{4}\,\Psi) = \mathcal{D}\epsilon\,, \tag{2.364}$$

provided $\iota_v\rho = 0$.[12] In the above derivation we have used the properties (2.299) and, in particular:

$$\iota_v\Psi_\alpha = \Psi_\alpha(v) = \Psi_\alpha(-i\,\overline{\tilde{Q}}^\beta\,\epsilon_\beta/\sqrt{2}) = -\frac{1}{\sqrt{2}}\,\Psi_\alpha(i\overline{\tilde{Q}}^\beta)\epsilon_\beta = -\epsilon_\alpha\,. \tag{2.365}$$

As for V^a we find:

$$\begin{aligned}\delta_\epsilon V^a &= -d(\iota_v V^a) - \iota_v\,dV^a = -\iota_v\,dV^a = -\iota_v\,(\tilde{T}^a - \omega^a{}_b\,V^b - \frac{i}{2}\bar{\Psi}\gamma^a\Psi) =\\ &= i\,(\iota_v\bar{\Psi})\gamma^a\Psi = -i\,\bar{\epsilon}\gamma^a\Psi\,,\end{aligned} \tag{2.366}$$

where we have used once again the property (2.299) in writing:

$$\iota_v\bar{\Psi}^\alpha = \bar{\Psi}^\alpha(v) = \bar{\Psi}^\alpha(-\frac{i}{\sqrt{2}}\,\bar{\epsilon}^\beta Q_\beta) = \frac{1}{\sqrt{2}}\,\bar{\epsilon}^\beta\,\bar{\Psi}^\alpha(i\,Q_\beta) = -\bar{\epsilon}^\alpha\,. \tag{2.367}$$

Let us evaluate now the effect of two consecutive transformations (2.363) on V^a. We evaluate it in the passive description (see Appendix A):

$$\begin{aligned}&\delta_{\epsilon_1}\delta_{\epsilon_2}V^a = -\delta_{\epsilon_1}(i\,\bar{\epsilon}_2\gamma^a\Psi) = -i\,\bar{\epsilon}_2\gamma^a\delta_{\epsilon_1}\Psi = -i\,\bar{\epsilon}_2\gamma^a\mathcal{D}\epsilon_1\ \Rightarrow\\ &\quad\Rightarrow\ [\delta_{\epsilon_1},\,\delta_{\epsilon_2}]V^a = -i\,(\bar{\epsilon}_2\gamma^a\mathcal{D}\epsilon_1 - \bar{\epsilon}_1\gamma^a\mathcal{D}\epsilon_2) = -i\,(\bar{\epsilon}_2\gamma^a\mathcal{D}\epsilon_1 + \mathcal{D}\bar{\epsilon}_2\gamma^a\epsilon_1) = -\mathcal{D}(i\,\bar{\epsilon}_2\gamma^a\epsilon_1)\,.\end{aligned} \tag{2.368}$$

This result is precisely $-\delta_w V^a$, where $w = i\,\bar{\epsilon}_2\gamma^a\epsilon_1\,\tilde{\mathcal{P}}_a = w^a\,\tilde{\mathcal{P}}_a$ is the parameter of the space-time translation resulting from the two subsequent supersymmetries according to the supersymmetry algebra:

$$[v_1,\,v_2] = [-\frac{i}{\sqrt{2}}\bar{\epsilon}_1\,\tilde{Q},\,-\frac{i}{\sqrt{2}}\overline{\tilde{Q}}\,\epsilon_2] = -\frac{1}{2}\,\bar{\epsilon}_1\{Q,\,\overline{\tilde{Q}}\}\epsilon_2 = -i\,\bar{\epsilon}_1\gamma^a\epsilon_2\,\tilde{\mathcal{P}}_a = x_0^a\,\tilde{\mathcal{P}}_a = w\,, \tag{2.369}$$

and,

$$\delta_w V^a = -d(\iota_w V^a) - \iota_w(\tilde{T}^a - \omega^a{}_b\,V^b - \frac{i}{2}\bar{\Psi}\gamma^a\Psi) = \mathcal{D}(i\,\bar{\epsilon}_2\gamma^a\epsilon_1) = -[\delta_{\epsilon_1},\,\delta_{\epsilon_2}]V^a\,, \tag{2.370}$$

[12] This property can be understood using the rheonomic analysis [5].

where we have used $\tilde{T}^a = 0$. The minus sign is due to the passive description of the consecutive variations (see (A.17)).

We have therefore proven that the local invariance realizes, at least on the vierbein field, the supersymmetry algebra. The closure of supersymmetry on the gravitino field requires the use of the field equations and we do not prove it. This represents a general feature: while the lagrangian is off-shell invariant under local super-Poincaré transformations, these close on the fields only on-shell, namely upon use of the field equations.

Let us end this part by expressing the local variation of the vierbein resulting from the commutator of two supersymmetries in terms of diffeomorphisms, along the lines of the derivation (2.285). We define $\epsilon^\mu = w^a\, V_a{}^\mu$, where $w^a = i\,\bar{\epsilon}_2\gamma^a\epsilon_1$ and find:

$$\delta_w V^a = \mathcal{D}_\mu(w^a) = \partial_\mu\epsilon^\nu V_\nu{}^a + \epsilon^\nu\,\partial_\nu V_\mu{}^a + \epsilon^\nu\,\omega_\nu{}^a{}_b\,V_\mu{}^b + T^a_{\mu\rho}\epsilon^\rho\,, \tag{2.371}$$

where we have taken into account the fact that the connection defining $\mathcal{D}$ is the torsionful one. Thus, in contract to the case (2.285) of a local transformation generated by $\hat{\mathcal{P}}_a$ in the absence of the gravitino field, we now find, aside from the effect of a local space-time diffeomorphism parametrized by ϵ^μ (i.e. the term $\partial_\mu\epsilon^\nu V_\nu{}^a + \epsilon^\nu\,\partial_\nu V_\mu{}^a$) and of a local Lorentz transformation ($\epsilon^\nu\,\omega_\nu{}^a{}_b\,V_\mu{}^b$), an extra term $T^a_{\mu\rho}\epsilon^\rho = i\,(\epsilon^\rho\bar{\Psi}_\rho)\gamma^a\Psi_\mu$, depending on the torsion, can be viewed as a supersymmetry transformation with parameter $\epsilon^\rho\bar{\Psi}_\rho$.

3 Part II: Extended Supergravities and Black Holes

3.1 Matter Coupled and Extended Supergravities

We have described in detail the pure $\mathcal{N} = 1$ supergravity. The above construction is generalized [27] by coupling the supergravity multiplet to a number n of *chiral* or Wess-Zumino multiplets, each consisting of a chiral fermion and two scalar fields and a number n_v of vector multiplets, each consisting of a vector field and a chiral fermion:

$$[1\times(2), 1\times(\frac{3}{2})] \quad n_v\times[1\times(1), 1\times(\frac{1}{2})] \quad n\times[1\times(\frac{1}{2}),\, 2\times(0)]\,. \tag{3.1}$$

The vector multiplets define the *gauge sector*, with the vector fields possibly gauging a suitable local internal symmetry group, while the chiral multiplets define the *matter sector*. The former consist of one vector field and one Majorana fermion, the latter of one chiral fermion and two scalar fields: one scalar and the other pseudo-scalar. This couple of scalar fields a, b in each chiral multiplet enter the Lagrangian and the supersymmetry transformation laws only in a certain complex combination $z = a + i\,b$.

We can also consider extended supergravity theories ($\mathcal{N} > 1$) describing the supergravity multiplet, consisting of the graviton and of $\mathcal{N}$ gravitino fields Ψ^i, $i = \ldots, \mathcal{N}$, coupled to a number of vector and matter multiplets. As previously emphasized, the consistent definition of a number $\mathcal{N}$ of massless gravitino fields on a curved space-time requires, for each of them, the decoupling of the spin-1/2 longitudinal modes, which in turn follows from the invariance of the theory under a transformation of the form (2.324):

$$\Psi^i_\mu \;\to\; \Psi^i_\mu + \mathcal{D}_\mu\epsilon^i\,, \tag{3.2}$$

that is under $\mathcal{N}$-independent supersymmetries. Thus a consistent theory containing $\mathcal{N}$ massless gravitinos is an $\mathcal{N}$-extended supergravity.

In the $\mathcal{N}=2$ theory[13] for example we can have, besides the supergravity multiplet, $n_{v.m.}$ vector multiplets and n_H hyper-multiplets:

$$[1\times(2), 2\times(\frac{3}{2}),\, 1\times(1)] \quad n_{v.m.}\times[1\times(1), 2\times(\frac{1}{2}),\, 2\times(0)] \quad n_H\times[2\times(\frac{1}{2}),\, 4\times(0)]\,. \tag{3.3}$$

As for the chiral multiplets, the two scalar fields in each vector multiplet appear in complex combinations $z_k = a_k + i\, b_k$, while the four scalar fields in each hyper-multiplet combine in *quaternionic numbers*. The two chiral spinors in the hypermultiplet merge in a single Dirac one.

The most general $\mathcal{N}=3$ theory describes the supergravity multiplet coupled to n_v vector multiplets :

$$[1\times(2), 3\times(\frac{3}{2}),\, 3\times(1),\, 1\times(\frac{1}{2})] \quad n_v\times[1\times(1), (3+1)\times(\frac{1}{2}),\, (3+3)\times(0)]\,, \tag{3.4}$$

where the $3+3$ scalar fields arrange themselves in three complex scalars.

In the $\mathcal{N}=4$ supergravity the graviton multiplet is coupled to n_v vector multiplets:

$$[1\times(2), 4\times(\frac{3}{2}),\, 6\times(1),\, 4\times(\frac{1}{2}),\, 2\times(0)] \quad n_v\times[1\times(1), 4\times(\frac{1}{2}),\, 6\times(0)]\,, \tag{3.5}$$

where only the two scalar fields in the graviton multiplet arrange themselves in a single complex one.

The $\mathcal{N}=5$ supergravity (as well as the $\mathcal{N}>5$ theories) only describes the graviton multiplet:

$$[1\times(2), 5\times(\frac{3}{2}),\, 10\times(1),\, (10+1)\times(\frac{1}{2}),\, (5+5)\times(0)],\,, \tag{3.6}$$

where the scalar fields arrange themselves in 5 complex ones.

Similarly also the field content of the $\mathcal{N}=6$ theory consists of the only graviton multiplet:

$$[1\times(2), 6\times(\frac{3}{2}),\, (15+1)\times(1),\, (20+6)\times(\frac{1}{2}),\, (15+15)\times(0)]\,, \tag{3.7}$$

where the scalar fields arrange themselves in 15 complex ones.

The $\mathcal{N}=7$ theory coincides with the maximal $\mathcal{N}=8$ one describing a single supergravity multiplet of the form:

$$[1\times(2), 8\times(\frac{3}{2}),\, 28\times(1),\, 56\times(\frac{1}{2}),\, 70\times(0)]\,, \tag{3.8}$$

Scalars have an important role in the construction of any phenomenologically viable model, since they define, though non-vanishing v.e.v., vacua in which the internal symmetry is spontaneously broken and a Higgs mechanism occur. This includes supersymmetry which ought

[13] The first $\mathcal{N}=2$ supergravity describing the only graviton multiplet, was constructed in [28].

to be ultimately broken since we know it is not realized in Nature: superpartners (selectron, squarks, photino, gluinos etc...) of the known fermions and bosons (electron, quarks, photon, gluons etc...) with the same masses as their counterparts are not observed. Spontaneous supersymmetry breaking occurs when the vacuum, characterized by certain v.e.v. of the scalar fields, preserves at most part [14] of the off-shell supersymmetry of the theory. Aside from the ordinary Higgs mechanism, a super-Higgs mechanism is at work through which all or part of the gravitinos Ψ^i, $i = \ldots, \mathcal{N}$, namely those corresponding to the broken supersymmetries, acquire mass. A mass for a gravitino is clearly inconsistent with supersymmetry since gravitinos are the superpartners of the massless graviton field. The longitudinal modes of these massive gravitinos are provided by spin-1/2 fields called *Goldstinos*, superpartners of the scalar fields whose v.e.v break supersymmetry, which are then "eaten" by the spin- 3/2 gauge fields of supersymmetry just as spin-0 Glodstone bosons are "eaten" by the ordinary gauge fields through the Higgs mechanism. We shall not discuss here the interesting issue of spontaneous (local) supersymmetry breaking in phenomenological model building, for which the literature is vast and we refer the reader to some excellent reviews (see, for instance, [4, 3] or the more recent [6]).

Supersymmetry constrains the form of the Lagrangian, i.e. the structure of its kinetic terms, mass terms, couplings and scalar potential. The larger the amount $\mathcal{N}$ of supersymmetry, the more stringent these constraints. The theory is characterized by a bosonic sector and a fermionic one. Once the former is given, the latter is completely fixed by supersymmetry. Here are some general common features of the bosonic sector of a supergravity Lagrangian. It consists of:

$$\text{The graviton field } \; V_\mu{}^a \;\; , \;\; n_v \text{ vector fields } \; A^\Lambda_\mu \; (\Lambda = 1, \ldots, n_v) \, , \;\; n_s \text{ scalar fields } \; \phi^s \; (s = 1, \ldots, n_s) \, . \tag{3.9}$$

Let us consider the simpler case of an *ungauged* supergravity, namely of a supergravity model in which the vector fields are not minimally coupled to any other field. This is the class of models we shall be dealing with in the following, when discussing black hole solutions. The general form of the supergravity action describing the only bosonic sector is:

$$\begin{aligned} S_B = \int d^4x \, \mathcal{L}_B = \int d^4x \, e \left[-\frac{\mathcal{R}}{2} + \frac{1}{2} G_{st}(\phi) \, \partial_\mu \phi^s \partial^\mu \phi^t + \right. \\ \left. + \frac{1}{4} F^\Lambda_{\mu\nu} I_{\Lambda\Sigma}(\phi) \, F^{\Sigma\,\mu\nu} + \frac{1}{8e} \, \epsilon_{\mu\nu\rho\sigma} F^{\Lambda\,\mu\nu} R_{\Lambda\Sigma}(\phi) \, F^{\Sigma\,\rho\sigma} - V(\phi) \right] \, , \end{aligned} \tag{3.10}$$

where $F^\Lambda_{\mu\nu} = \partial_\mu A^\lambda_\nu - \partial_\nu A^\lambda_\mu$. Let us comment on the general characteristics of the above action:

- The scalar fields ϕ^s are described by a non-linear σ-model, that is they are coordinates of a non-compact, *Riemannian* n_s-dimensional differentiable manifold (target space), named *scalar manifold* and to be denoted by $\mathscr{M}_{scal}$. The positive definite metric on

[14] For $\mathcal{N} = 1$, supersymmetry breaking is clearly complete, while for extended theories, $\mathcal{N} > 1$, a fraction of $\mathcal{N}$ may be preserved.

the manifold is $G_{st}(\phi)$.The corresponding kinetic part of the Lagrangian density reads:

$$\mathcal{L}_{scal} = \frac{e}{2}\, G_{st}(\phi)\, \partial_\mu \phi^s \partial^\mu \phi^t \,. \tag{3.11}$$

The σ-model action is clearly invariant under the action of global (i.e. space-time independent) isometries of the scalar manifold. Indeed, if G is the isometry group of $\mathscr{M}_{scal}$, a generic element of it will map the scalar fields $\phi = (\phi^s)$ in new ones, to be denoted by $g \star \phi$ or $\phi' = (\phi'^s)$, which are in general non-linear functions of the original ones $\phi'^s = \phi'^s(\phi^t)$ such that:

$$\forall g \in G \,:\, \phi \stackrel{g}{\longrightarrow} g \star \phi = \phi'(\phi) \quad : \quad G_{s't'}(\phi'(\phi)) \frac{\partial \phi'^{s'}}{\partial \phi^s} \frac{\phi'^{t'}}{\partial \phi^t} = G_{st}(\phi)\,. \tag{3.12}$$

As we shall discuss below, the group G can be promoted to a global symmetry group of the field equations and Bianchi identities (i.e. *on-shell global symmetry group*) provided its (non-linear) action on the scalar fields (3.12) is combined with an electric-magnetic duality transformation on the vector field strengths and their magnetic duals.

- The two terms containing the vector field strengths will be called vector kinetic terms. A general feature of supergravity theories is that the scalar fields are non-minimally coupled to the vector fields as they enter these terms through symmetric matrices $I_{\Lambda\Sigma}(\phi)$, $R_{\Lambda\Sigma}(\phi)$ which contract the vector field strengths (not to be confused with the real and imaginary parts R_{ij}, I_{ij} of the central charge matrix Z_{ij}, for which we have used the same symbols R, I). The former $I_{\Lambda\Sigma}(\phi)$ is negative definite and generalizes the $-1/g^2$ factor in the Yang-Mills kinetic term. The latter $R_{\Lambda\Sigma}(\phi)$ generalizes the θ-term.

- The presence of a scalar potential. In an ungauged supergravity a scalar potential is allowed only for $\mathcal{N} = 1$ (called the *F-term potential*). In extended supergravities a non-trivial scalar potential can be introduced without explicitly breaking supersymmetry only through the *gauging procedure*, which consists in promoting a suitable global symmetry group (a subgroup of the isometry group G) to local symmetry to be gauged by the vector fields of the theory. This is effected, as usual, by replacing ordinary derivatives and vector field strengths by covariant ones. Supersymmetry of the action further requires the introduction of additional terms in the supersymmetry transformation rules of the gravitino and fermion fields, together with gravitino and fermion mass terms in the Lagrangian and a scalar potential. These new ingredients (extra terms in the supersymetry transformation rules, mass matrices and the scalar potential) all have a well defined expression in terms of the scalar fields and the newly introduced gauge group. This procedure is the only way for introducing in an extended supergravity either minimal couplings of the vector fields to the other fields, or a scalar potential. Since a scalar potential is an essential ingredient for having spontaneous supersymmetry breaking, the latter phenomenon in extended supergravities ultimately depends on the choice of the internal gauge symmetry.

The fermion part of the action is totally determined by supersymmetry once the bosonic one is given.

Minimal supergravity. In the $\mathcal{N}=1$ case, the scalar manifold $\mathscr{M}_{scal}$ describes the scalar fields in the chiral multiplets. Strictly speaking this is a complex manifold of *Hodge-Kähler* type (see for instance [29, 6]), which is a particular kind of Kähler manifold [15] in which the Kähler transformations act on the fermion fields as U(1)-transformations, which are the $\mathcal{N}=1$ U(1) R-symmetry transformations. Consistency of such transformations on the fermion and gravitino fields (similar to that yielding the Dirac quantization of the electric charge) imposes a constraint on the geometry of the Kähler manifold. The structure of the bosonic Lagrangian is completely fixed by the following independent data: the Kähler potential $\mathcal{K}(z,\bar{z})$ associated with the manifold , a holomorphic superpotential $W(z)$,[16] and of the matrices $I_{\Lambda\Sigma}$, $R_{\Lambda\Sigma}$ defining the vector kinetic part and which are constrained by supersymmetry to be holomorphic functions of the complex scalar fields: $I_{\Lambda\Sigma}(z)$, $R_{\Lambda\Sigma}(z)$. The σ-model action reads:

$$\mathcal{L}_{scal} = e\, G_{\alpha\bar{\beta}}(z,\bar{z})\, \partial_\mu z^\alpha \partial^\mu \bar{z}^{\bar{\beta}}\,. \tag{3.13}$$

If the theory is gauged, that is a subgroup of the isometry group G of the scalar manifold is promoted to local internal symmetry, additional terms, as mentioned above, appear in the supersymmetry transformation laws and in the Lagrangian, which also affect the scalar potential (through additional *D-terms*). For the sake of completeness we write the most general $\mathcal{N}=1$ potential:

$$V(z,\bar{z}) = e^{\mathscr{G}}\left(G^{\alpha\bar{\beta}}\,\frac{\partial}{\partial z^\alpha}\mathscr{G}\frac{\partial}{\partial \bar{z}^{\bar{\beta}}}\mathscr{G} - 3\right) + \frac{1}{4}\, I^{-1\,\Lambda\Sigma}\mathcal{P}_\Lambda \mathcal{P}_\Sigma\,, \tag{3.14}$$

where $\mathscr{G}(z,\bar{z}) \equiv \mathcal{K}(z,\bar{z}) + \log(|W(z)|^2)$ and $\mathcal{P}_\Lambda(z,\bar{z})$ are real quantities depending on the choice of the gauged isometries.[17]

Extended supergravities. In $\mathcal{N}>1$ supergravities, multiplets start becoming large enough as to accommodate both the scalar fields and the vector fields. As we increase $\mathcal{N}$ from $\mathcal{N}=1$, the first instance of scalar and vector fields connected by supersymmetry is in the $\mathcal{N}=2$ vector multiplet. This feature has profound implications on the mathematical structure of the models. In particular it poses strong constraints on the (non-minimal) scalar-vector couplings in the Lagrangian, that is on the matrices $I_{\Lambda\Sigma}(\phi)$, $R_{\Lambda\Sigma}(\phi)$. Given the scalar manifold, supersymmetry fixes $I_{\Lambda\Sigma}(\phi)$, $R_{\Lambda\Sigma}(\phi)$.[18] Moreover global isometry transformations

[15] Let us recall here the definition of a Kähler manifold [30]. A Kähler manifold is a *hermitian* complex manifold with metric $ds^2 = 2\,G_{\alpha\bar{\beta}}\,dz^\alpha \otimes d\bar{z}^{\bar{\beta}}$ in which the *Kähler 2-form* $K = iG_{\alpha\bar{\beta}}\,dz^\alpha \wedge d\bar{z}^{\bar{\beta}}$ is closed: $dK=0$. In such manifolds the metric can be locally expressed in terms of a *Kähler potential* $\mathcal{K}(z,\bar{z})$ as follows: $G_{\alpha\bar{\beta}} = \frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial \bar{z}^{\bar{\beta}}}\mathcal{K}(z,\bar{z})$. The Kähler potential is defined modulo a *Kähler transformation* connecting its expressions in two overlapping parches: $\mathcal{K}(z,\bar{z}) \to \mathcal{K}(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$. A *Hodge-Kähler* manifold is a Kähler manifold with an additional structure defined over it: a *holomorphic line bundle* $\mathcal{L}$. Associated with this line bundle there is a U(1)-bundle in which the fermion fields and the gravitino field have values. The formal defining condition of a Hodge-Kähler manifold is that the first Chern class of the bundle equals the cohomology class of the Kähler 2-form: $c_1(\mathcal{L}) = [K]$. This condition is nothing but the consistency condition mentioned below (see [6] for a discussion on this point).

[16] The superpotential $W(z)$ in an $\mathcal{N}=1$ model is a section of this line bundle.

[17] They are the *moment maps* associated with these isometries, in terms of which the holomorphic Killing vectors $k_\Lambda^\alpha(z)$ are expressed as follows: $k_\Lambda^\alpha(z) = i\,G^{\alpha\bar{\beta}}\,\frac{\partial}{\partial \bar{z}^{\bar{\beta}}}\mathcal{P}_\Lambda$.

[18] This is true up to a choice of the *symplectic frame*, see below.

(3.12) on the scalar fields induce, by supersymmetry, global transformations on the vector fields. These act as electric-magnetic transformations on the vector field strengths and their magnetic duals and define the on-shell global symmetries of the theory, as mentioned above. Before discussing the issue of the global symmetry of these models, to be dealt with in the next Section, let us first discuss the general features of the scalar manifolds.

3.1.1 Scalar Manifolds of Extended Supergravities

While the $\mathcal{N} = 2$ models allow for a class of homogeneous scalar manifolds, in all $\mathcal{N} > 2$ models supersymmetry constrains the scalar manifold to be *homogeneous symmetric* (see Table 3.1.1).

A homogeneous manifold $\mathscr{M}$ is a manifold in which any couple of points are connected by an isometry. As a consequence of this, any point p can be reached from a given reference one O, called the origin, through an element (in general not unique) of the isometry group G. The isometry group G is said to have a *transitive action* on $\mathscr{M}$. We define this action to be a *left action* and denote it by a star symbol:

$$\forall p \in \mathscr{M}\ \exists g_p \in G \ \ : \ \ p = g_p \star O\,. \tag{3.15}$$

By left action we mean that for any $g_1\, g_2 \in G$ and $p \in \mathscr{M}$, we have $g_1 \star (g_2 \star p) = (g_1\, g_2) \star p$ (see the second of [24]). The action of G on $\mathscr{M}$ may not be *free*. This means that the element g_p in (3.15) is not unique or, equivalently, that for any $p \in \mathscr{M}$ there may be a subgroup H_p of G which leaves p invariant: $H_p \star p = p$. This group is called the *isotropy (or stabilizer) group* of p. It can be shown that *the isotropy groups of any two points of a homogeneous space are isomorphic*. Let us denote by H the isotropy group of the origin O: $H \star O = O$. Given a point p in $\mathscr{M}$ and an element g_p of G mapping O to p as in (3.15), any other element differing from g_p by the *right multiplication* by an element of H will still map O to p:

$$\forall g' \in G \ \ ; \ \ g' = g_p\, h \ \ (h \in H) \ \ : \ \ g' \star O = (g_p\, h) \star O = g_p \star (h \star O) = g_p \star O = p\,. \tag{3.16}$$

If we denote by $g\, H = \{g\, h \in G \mid \ h \in H\}$ the *left coset* of H in G, there is a one-to-one correspondence between the points of the homogeneous manifold $\mathscr{M}$ and left cosets $g\, H$:

$$p \in \mathscr{M} \ \ \leftrightarrow, \ \ \ g_p\, H \subset G\,. \tag{3.17}$$

Denoting by G/H the set of all left cosets of H in G, there is therefore a bijection (or diffeomorphism) between $\mathscr{M}$ and G/H so that the two can be identified:

$$\mathscr{M} \sim G/H\,, \tag{3.18}$$

where $\sim$ means that the two manifolds are *diffeomorphic*. G/H is called a *coset manifold* and thus homogeneous spaces can be described as coset manifolds. Actually, being $\mathscr{M}$ a metric manifold and G its isometry group, $\mathscr{M}$ and G/H are isometric: We can compute all geometric quantities of $\mathscr{M}$ (connection, curvature, geodesics etc...) on G/H. Note that the coset space G/H is not a group since in general H is not a normal subgroup of G. A generic

$\mathcal{N}$	$\frac{G}{H}$	n_s
8	$\frac{E_{7(7)}}{SU(8)}$	70
6	$\frac{SO^*(12)}{U(6)}$	30
5	$\frac{SU(5,1)}{U(5)}$	10
4	$\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(6,n)}{SO(6)\times SO(n)}$	6n+2
3	$\frac{SU(3,n)}{S[U(3)\times U(n)]}$	6n
2	$\frac{SU(1,n+1)}{U(n+1)}$	2(n+1)
	$\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(2,n+2)}{SO(2)\times SO(n+2)}$	2(n+2)+2
	$\frac{Sp(6)}{U(3)}$	12
	$\frac{SU(3,3)}{S[U(3)\times U(3)]}$	18
	$\frac{SO^*(12)}{U(6)}$	30
	$\frac{E_{7(-25)}}{U(1)\times E_6}$	54

Table 1: Homogeneous symmetric scalar manifolds in extended supergravities and their real dimensions n_s.

element g of G is defined by $\dim(G)$ continuous parameters. Through right multiplication by an element of H we may fix $\dim(H)$ of these parameters, so that the minimum number of parameters a representative of each left-coset depends on is $\dim(G) - \dim(H)$. This is the dimension of $\mathscr{M}$:

$$\dim(\mathscr{M}) = \dim(G) - \dim(H)\,. \tag{3.19}$$

Let ϕ^s denote the $\dim(G) - \dim(H)$ parameters obtained upon fixing the right-action of H. The corresponding representative of each coset is denoted by $\mathbb{L}(\phi^s) \in G$. We therefore describe each point of $\mathscr{M}$ in terms of a *coset representative* $\mathbb{L}(\phi^s)$:

$$p \in \mathscr{M} \quad \leftrightarrow \quad \mathbb{L}(\phi^s) \in g_p\, H \subset G\,. \tag{3.20}$$

They provide a parametrization of $\mathscr{M}$ and depend on how this fixing is performed, namely which representative $\mathbb{L}(\phi^s)$ of each coset $g_p H$ is taken to represent the corresponding point p of $\mathscr{M}$. Let $g \in G$ be an isometry of $\mathscr{M}$, p a point of coordinates $\phi = (\phi^s)$ and $p' = g \star p$ the transformed of p though g, of coordinates $\phi' = g \star \phi = (\phi'^s(\phi^t))$. Since both $g\,\mathbb{L}(\phi)$ and $\mathbb{L}(g \star \phi)$ represent the same point p', they must belong to the same left-coset, so that:

$$g\,\mathbb{L}(\phi) = \mathbb{L}(g \star \phi)\, h(\phi, g)\,, \tag{3.21}$$

where the element $h(\phi, g)$ of H is called *compensator* and in general depends on g and the point p (ϕ).[19]

Example 1. An example of homogeneous manifold is the n-dimensional sphere S^n defined the subspace of points of $\mathbb{R}^{n+1}$, of coordinates $(x_1, \dots x_{n+1})$, satisfying the condition: $x_1^2 + \dots + x_{n+1}^2 = 1$. The metric on S^n is the one induced by the Euclidean one on $\mathbb{R}^{n+1}$: $ds^2 = dx_1^2 + \dots + dx_{n+1}^2$. The group $\mathrm{O}(n+1)$ acts linearly on the coordinate vector $\mathbf{x} = (x_i)$ and transitively on S^n. It moreover leaves the metric on S^n invariant. Its action however is not free since, if we take the point $\mathbf{x} = (1, 0, \dots, 0)$, that is clearly invariant under the action of the subgroup $\mathrm{O}(n)$ acting only on the n coordinates $\{x_2, \dots, x_{n+1}\}$. This subgroup $\mathrm{O}(n)$ is the isotropy group and we can then write $S^n \sim \mathrm{O}(n+1)/\mathrm{O}(n)$.

Example 2. An other example is the n-dimensional anti-de Sitter space AdS_n defined as the subspace of points of $\mathbb{R}^{2,n-1}$ whose coordinates $(y_{0_1}, y_0, y_1, \dots y_{n-1})$ satisfy the condition:

$$y_a^2 \bar{\eta}^{ab}\, y_b^2 = y_{0_1}^2 + y_0^2 - \sum_{i=1}^{n-1} y_i^2 = R^2\,. \tag{3.22}$$

The metric on AdS_n is induced by the pseudo-Euclidean one on $\mathbb{R}^{2,n-1}$ with metric tensor $\bar{\eta}_{ab} = \mathrm{diag}(+1, +1, -1, \dots, -1)$. The isometry group which acts transitively on this space is $\mathrm{O}(2, n-1)$ and the isotropy group $\mathrm{O}(1, n-1)$, so that we can write:

$$AdS_n = \frac{\mathrm{O}(2, n-1)}{\mathrm{O}(1, n-1)}\,. \tag{3.23}$$

[19]Here we have described homogeneous spaces as left-cosets. They might as well be described as right-cosets, just as we did for Minkowski, or superspace earlier.

In a local patch the AdS_n metric can be written in the form:

$$ds^2 = \rho^2\, dx^\mu \eta_{\mu\nu}\, dx^\nu + R^2\, \frac{d\rho^2}{\rho^2}\,, \tag{3.24}$$

where $\rho > 0$ and $\eta_{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$, $x^\mu = (x^0, x^1, \dots, x^{n-2})$, R is called the *radius of the anti-de Sitter space.* The above metric can be obtained by restricting the pseudo-Euclidean one on $\mathbb{R}^{2,n-1}$ through the following embedding of AdS_n in $\mathbb{R}^{2,n-1}$:

$$y_- \equiv y_{0_1} - y_{n-1} = \rho\ ,\quad y_+ \equiv y_{0_1} + y_{n-1} = \frac{R^2 - \rho^2\,(x^\mu \eta_{\mu\nu}\, x^\nu)}{\rho}\ ,\quad y^\mu = \rho\, x^\mu\,. \tag{3.25}$$

Exercise: Prove this.
Exercise: Prove that the Ricci tensor reads:

$$\mathcal{R}_{\mu\nu} = \Lambda\, g_{\mu\nu} = \frac{n-1}{R^2}\, g_{\mu\nu}\,, \tag{3.26}$$

where $\Lambda = \frac{n-1}{R^2}$ *is the cosmological constant which, in the mostly minus convention for the metric, is positive. Prove also that this metric is solution to the Einstein-Hilbert action:*

$$S_\Lambda = -\int d^n x\, \frac{e}{2}\,(\mathcal{R} - (n-2)\Lambda)\,. \tag{3.27}$$

Example 3. An other example is the *lower half plane* $\mathscr{M} \equiv \{z \in \mathbb{C}\,|\,\text{Im}(z) < 0\}$. We may define on it the metric:

$$ds^2 = 2\, g_{z\bar{z}} dz\, d\bar{z} = \frac{1}{2\text{Im}(z)^2}\, dz\, d\bar{z}\,. \tag{3.28}$$

The group $\text{SL}(2, \mathbb{R})$ acting on z as follows:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})\ ,\ \ (ab - cd = 1)\ \ :\ \ z \xrightarrow{\ g\ } z' = \frac{az + b}{cz + d}\,, \tag{3.29}$$

is an isometry group and acts transitively on $\mathscr{M}$.

Exercise: Prove that $\text{SL}(2, \mathbb{R})$ *is an isometry group, namely that* $\frac{1}{2\text{Im}(z)^2}\, dz\, d\bar{z} = \frac{1}{2\text{Im}(z')^2}\, dz'\, d\bar{z}'$. *Prove also that it has a transitive action on* $\mathscr{M}$

The reader can also verify that the point $z = -i$ is left invariant by the action of the $\text{SO}(2)$ group:

$$\text{SO}(2) = \{\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\}\,, \tag{3.30}$$

which is the isotropy group of $\mathscr{M}$. The lower-half plane can thus be identified with the following coset:

$$\mathscr{M} = \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}\,. \tag{3.31}$$

In general G need not be a semisimple Lie group. Homogeneous manifolds occurring in supergravity theories are non-compact, simply-connected, negative-curvature spaces. Let $\mathfrak{g}$ and $\mathfrak{H}$ denote the Lie algebras of the groups G and H, respectively. We can split the former as follows:

$$\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{K} . \tag{3.32}$$

Being $\mathfrak{H}$ a Lie algebra we have:

$$[\mathfrak{H},\, \mathfrak{H}] \subseteqq \mathfrak{H} . \tag{3.33}$$

We can always define $\mathfrak{K}$ in such a way that:

$$[\mathfrak{H},\, \mathfrak{K}] \subseteqq \mathfrak{K} . \tag{3.34}$$

We see that the above adjoint action of $\mathfrak{H}$ on $\mathfrak{K}$ defines a *representation* of H. Indeed, if we denote by (H_u) a basis of $\mathfrak{H}$ and by $(K_{\underline{s}})$ a basis of $\mathfrak{K}$, we have:

$$[H_u,\, K_{\underline{s}}] = C_{u\underline{s}}{}^{\underline{t}}\, K_{\underline{t}} = -(H_u)_{\underline{s}}{}^{\underline{t}}\, K_{\underline{t}} , \tag{3.35}$$

where the matrices $(H_u)_{\underline{s}}{}^{\underline{t}} = C_{su}{}^{t}$ define a representation $\mathscr{K}$ of the generators (H_u).

Exercise: Prove this using the Jacobi identity.

From this it follows that, if h is an element of H, we have:

$$h^{-1}\, K_{\underline{s}}\, h = h_{\underline{s}}{}^{\underline{t}}\, K_{\underline{t}} , \tag{3.36}$$

where the matrix $(h_{\underline{s}}{}^{\underline{t}})$ represents the element h in the representation $\mathscr{K}$.

The space $\mathfrak{K}$ can be viewed as the *tangent space* to G/H at the origin.

In general, however, we have:

$$[\mathfrak{K},\, \mathfrak{K}] \subseteqq \mathfrak{K} \oplus \mathfrak{H} . \tag{3.37}$$

It can be proven that, if we can define a $\mathfrak{K}$ so that:

$$[\mathfrak{K},\, \mathfrak{K}] \subseteqq \mathfrak{H} , \tag{3.38}$$

the homogeneous space is *symmetric*. A symmetric space is defined in general as a space whose curvature is covariantly constant (i.e. it is invariant under parallel translations). Symmetric, simply-connected spaces are also homogeneous. For non-compact, simply-connected symmetric spaces with negative curvature (i.e. those which are relevant to supergravity) there exists a transitive *semisimple*, non-compact isometry group G and H is its *maximal compact subgroup*. In any given matrix representation of G, one can choose a basis in which $\mathfrak{H}$ is represented by anti-hermitian matrices ($H \in \mathfrak{H} \;\Rightarrow\; H^\dagger = -H$) and $\mathfrak{K}$ by hermitian ones ($K \in \mathfrak{K} \;\Rightarrow\; K^\dagger = K$). This basis is called the *Cartan basis*. Properties (3.34) and (3.38) clearly follow because the commutator of an anti-hermitian with an hermitian generator is hermitian while that of two hermitian generators is anti-hermitian. In the corresponding basis $(T_A) = (H_u,\, K_{\underline{s}})$ of generators of $\mathfrak{g}$, equation (3.38) reads:

$$[K_{\underline{s}},\, K_{\underline{t}}] = C_{\underline{s}\,\underline{t}}{}^{u}\, H_u . \tag{3.39}$$

We can define in this basis the coset representative $\mathbb{L}(\phi)$ as follows. Let $\{K_s\}$ denote a basis of $\mathfrak{K}$ consisting of hermitian matrices, we define:

$$\mathbb{L}(\phi^s) = \exp(\phi^s\, K_s)\,. \tag{3.40}$$

This parametrization, defined by the coordinates ϕ^s, will be called *Cartan parametrization.* Its relevant feature is that the coordinates ϕ^s transform under H (isotropy group of the origin $\phi^s = 0$) in a linear way, namely in the representation $\mathscr{K}$ defined by the adjoint action of H on $\mathfrak{K}$:

$$\mathbb{L}(\phi) \;\longrightarrow\; h\,\mathbb{L}(\phi) = h\,\mathbb{L}(\phi)\,h^{-1}\,h = \mathbb{L}(h\star\phi)\,h\,, \tag{3.41}$$

being

$$\mathbb{L}(h\star\phi) = h\,\mathbb{L}(\phi)\,h^{-1} = h\,e^{\phi^s\,K_s}\,h^{-1} = e^{\phi^s\,h\,K_s\,h^{-1}} = e^{\phi'^t\,K_s} \quad\Rightarrow\quad \phi'^t = (h\star\phi)^t = \phi^s\,(h^{-1})_s{}^t\,,$$

where we have used Eq. (3.36).

As mentioned earlier, $\mathcal{N} = 2$ supergravity admits non-homogeneous, homogeneous and homogeneous-symmetric scalar manifolds, while the scalar manifolds of $\mathcal{N} > 2$ supergravities are only of homogeneous symmetric type. All homogeneous scalar manifolds (symmetric or not) are of *normal type*, namely they admit a transitive *solvable* Lie group of isometries G_S whose action on $\mathscr{M}$ is free[20]. This means that we can choose a representative $\mathbb{L}_s(\phi_p)$ in each left coset $g_p\,H$, by suitably fixing the right-action of H, so that

$$\{\mathbb{L}_s(\phi_p)\}_{p\in\mathscr{M}} = G_S\,.$$

In other words the manifold $\mathscr{M}$ is *isometric* to a solvable Lie group

$$\mathscr{M} \sim G_S\,,$$

once we fix on the tangent space to G_S at the origin the metric of $\mathscr{M}$ on the tangent space at the corresponding point. This description defines a parametrization $\phi = (\phi^s)$ called the *solvable parametrization* of $\mathscr{M}$.

Both the solvable and (for symmetric cosets) the Cartan parametrizations are global parametrizations of the scalar manifold. For symmetric manifolds the solvable Lie group G_S is defined by the Iwasawa decomposition of the non-compact semisimple group G with respect to H according to which there is a unique decomposition of a generic element g of G as the product of an element s of G_S and an element h of H: $g = s\,h$. This defines a unique coset representative $\mathbb{L}_s$ for each point of $\mathscr{M}$. The solvable parametrization is useful when the four dimensional supergravity is described as resulting from the Kaluza-Klein reduction of a higher dimensional supergravity on some internal compact manifold.

[20] A solvable Lie group G_S can be described (locally) as a the Lie group generated by *solvable Lie algebra Solv*: $G_S = \exp(Solv)$. A Lie algebra $Solv$ is solvable iff, for some $k > 0$, $\mathbf{D}^k Solv = 0$, where the *derivative* $\mathbf{D}$ of a Lie algebra $\mathfrak{g}$ is defined as follows: $\mathbf{D}\mathfrak{g} \equiv [\mathfrak{g},\mathfrak{g}]$, $\mathbf{D}^n\mathfrak{g} \equiv [\mathbf{D}^{n-1}\mathfrak{g},\mathbf{D}^{n-1}\mathfrak{g}]$. In a suitable basis of a given representation, elements of a solvable Lie group or a solvable Lie algebra are all described by upper (or lower) triangular matrices.

The solvable coordinates directly describe dimensionally reduced fields and moreover this parametrization makes the shift symmetries of the metric manifest. The drawback of such description is that *Solv* does not define the carrier of a representation of H as $\mathfrak{K}$ does, namely Eq. (3.34) does not hold for *Solv*: $[\mathfrak{H},\, Solv] \nsubseteq Solv$.

In what follows we shall restrict ourselves to *symmetric cosets* of which we can give a description either in terms of Cartan coordinates or of solvable coordinates.

Let us see how vielbeine and connection can be defined on a symmetric coset. Let $\mathbb{L}(\phi)$ be a coset representative corresponding to a generic parametrization. We can construct the *left-invariant one form* $\Omega = L^{-1}dL$ which is a 1-form on G/H with value in $\mathfrak{g}$. Let us expand it in the Cartan basis $(T_A) = (H_u,\, K_{\underline{s}})$

$$\Omega(\phi) = \sigma^A(\phi)\, T_A = \mathbb{L}(\phi)^{-1} d\mathbb{L}(\phi) = V^{\underline{s}}(\phi)\, K_{\underline{s}} + \omega^u(\phi)\, H_u = P(\phi) + \omega(\phi)\,, \tag{3.42}$$

where $\Omega(\phi) = \Omega_s(\phi)\, d\phi^s$, $V^{\underline{t}}(\phi) = V_s{}^{\underline{t}}(\phi)\, d\phi^s$, $P(\phi) = V^{\underline{s}}(\phi)\, K_{\underline{s}}$, $\omega(\phi) = \omega^u(\phi)\, H_u$ and we use the underlined indices $\underline{s}, \underline{t}, \dots$ as *rigid indices* to label the basis $(K_{\underline{s}})$ of the tangent space to the group manifold defining a representation $\mathscr{K}$ of H, and should not be confused with the *curved* indices $s, t, \dots$ labeling the coordinates (ϕ^t), i.e. the scalar fields. Only in the Cartan parametrization the scalar fields carry rigid indices. Let us see how this quantities transform under the action of G. For any $g \in G$, using Eq. (3.21), we can write $\mathbb{L}(g \star \phi) = g\, \mathbb{L}(\phi)\, h^{-1}$, so that:

$$\begin{aligned}\Omega(g \star \phi) &= h\, \mathbb{L}(\phi)^{-1}\, g^{-1} d(g\, \mathbb{L}(\phi)\, h^{-1}) = h\, \mathbb{L}(\phi)^{-1}\, g^{-1} g d(\mathbb{L}(\phi)\, h^{-1}) = h\, \mathbb{L}(\phi)^{-1} (d\mathbb{L}(\phi))\, h^{-1} + \\ &\quad + h\, dh^{-1}\,.\end{aligned} \tag{3.43}$$

From (3.42) we find:

$$\begin{aligned}P(g \star \phi) + \omega(g \star \phi) &= V^{\underline{s}}(g \star \phi)\, K_{\underline{s}} + \omega^u(g \star \phi)\, H_u = h\, (V^{\underline{s}}(\phi)\, K_{\underline{s}})\, h^{-1} + h\, (\omega^u(\phi)\, H_u) h^{-1} + \\ &\quad + h\, dh^{-1} = h\, P(\phi)\, h^{-1} + h\, \omega(\phi) h^{-1} + h\, dh^{-1}\,.\end{aligned} \tag{3.44}$$

Since $h\, dh^{-1}$ is the left-invariant 1-form on $\mathfrak{H}$, it has value in this algebra. Projecting the above equation over $\mathfrak{K}$ and $\mathfrak{H}$, we find:

$$P(g \star \phi) = h\, P(\phi)\, h^{-1} \;\; \Leftrightarrow \;\; V^{\underline{s}}(g \star \phi) = V^{\underline{s}}(\phi)\, h^{-1}{}_{\underline{t}}{}^{\underline{s}} = h^{\underline{s}}{}_{\underline{t}} V^{\underline{s}}(\phi)\,, \tag{3.45}$$

$$\omega(g \star \phi) = h\, \omega(\phi)\, h^{-1} + h dh^{-1}\,. \tag{3.46}$$

Note the analogy with the description of space-time that we gave in Sect. 2.6. In particular compare Eq.s (2.235) and (2.273) with (3.45) and (3.46): $V^{\underline{s}}$ have the role here of the vielbein, H of the (local) Lorentz group and ω^u of the spin connection. In Sect. 2.2 in particular Minkowski space was described as the coset (in that case with respect to the left-action of H) in which the isometry group G is the Poincaré one and H the Lorentz group. $V^{\underline{s}}$ are then identified with the vielbein 1-forms and ω the H-connection. We shall see below how the G-invariant metric on $\mathscr{M}$ is constructed in terms of $V^{\underline{s}}$.

Just as we did in curved space-time with respect to the local Lorentz group, we define the H*-covariant derivative* $\mathcal{D}^{(H)}$ of the vielbein and the curvature as follows:

$$\mathcal{D}^{(H)}P \equiv dP + Q\,P - P\,Q = \mathcal{D}^{(H)}V^{\underline{s}}\,K_{\underline{s}} \;\Rightarrow\; \mathcal{D}^{(H)}V^{\underline{s}} = dV^{\underline{s}} + \omega^u\,(H_u)^{\underline{s}}{}_{\underline{t}} \wedge V^{\underline{t}} = \\ = dV^{\underline{s}} + \omega^u\,C_{u\underline{t}}{}^{\underline{s}} \wedge V^{\underline{t}}\,, \tag{3.47}$$

$$R = d\omega + \omega\wedge\omega = R^u\,H_u \;\Rightarrow\; R^u = d\omega^u + \frac{1}{2}C_{vw}{}^u\,\omega^v\wedge\omega^w\,, \tag{3.48}$$

where we have used $(H_u)^{\underline{s}}{}_{\underline{t}} = -(H_u)_{\underline{t}}{}^{\underline{s}} = C_{u\underline{t}}{}^{\underline{s}}$. The reader can verify that:

$$D^{(H)\,2}V^{\underline{s}} = R^u\wedge C_{u\underline{t}}{}^{\underline{s}}\,V^{\underline{t}}\,. \tag{3.49}$$

Exercise: Verify the above equation. Prove moreover that $\mathcal{D}^{(H)}$ *is the covariant derivative with respect to the* H*-transformations (3.45) and (3.46), namely that* $(\mathcal{D}^{(H)}P)(g\star\phi) = h\,\mathcal{D}^{(H)}P(\phi)\,h^{-1}$.

Let us compute the exterior derivative of Ω:

$$d\Omega = dL^{-1}\wedge dL = dL^{-1}\,L\,L^{-1}\wedge dL = -L^{-1}dL\wedge L^{-1}dL = -\Omega\wedge\Omega \;\Leftrightarrow\; d\Omega + \Omega\wedge\Omega = 0\,. \tag{3.50}$$

In components this is nothing but the Maurer-Cartan equations for G:

$$d\sigma^A + \frac{1}{2}C_{BC}{}^A\,\sigma^B\wedge\sigma^C = 0\,. \tag{3.51}$$

Splitting σ^A into $V^{\underline{s}}$, ω^u the above equations read:

$$dV^{\underline{s}} + \omega^u\,C_{u\underline{t}}{}^{\underline{s}}\wedge V^{\underline{t}} = 0 \;\Leftrightarrow\; \mathcal{D}^{(H)}P = 0 = \mathcal{D}^{(H)}V^{\underline{s}}\,, \tag{3.52}$$

$$d\omega^u + \frac{1}{2}C_{vw}{}^u\,\omega^v\wedge\omega^w + \frac{1}{2}C_{\underline{st}}{}^u\,V^{\underline{s}}\wedge V^{\underline{t}} = 0 \;\Leftrightarrow\; R = -P\wedge P \;\Leftrightarrow\; R^u = -\frac{1}{2}C_{\underline{st}}{}^u\,V^{\underline{s}}\wedge V^{\underline{t}}\,. \tag{3.53}$$

Notice that the components of the curvature 2-form in the vielbein basis are *constant and fixed in terms of the structure constants* of $\mathfrak{g}$. This is a general feature not just of symmetric spaces, but in general of homogeneous spaces. The following Bianchi identity follows directly from the Jacobi identity for $\mathfrak{H}$ generators:

$$\mathcal{D}^{(H)}R^u = dR^u + \omega^v\wedge R^w\,C_{vw}{}^u = 0\,,$$

Just as we defined on the tangent space of a curved space-time a (local) Lorentz invariant metric η_{ab}, here we define on the tangent space to $\mathscr{M}$ an H-invariant (positive definite) metric $\kappa_{\underline{st}}$. With reference to matrix representation of G we define $\kappa_{\underline{st}}$ as the restriction of the Cartan-Killing metric of $\mathfrak{g}$ to $\mathfrak{K}$:

$$\kappa_{\underline{st}} \equiv k\,\mathrm{Tr}(K_{\underline{s}}\,K_{\underline{t}})\,, \tag{3.54}$$

where k is a representation-dependent normalization constant. The metric on $\mathscr{M}$ is defined as follows:

$$G_{st}(\phi) = V_s{}^{\underline{s}}(\phi)V_t{}^{\underline{t}}(\phi)\,\kappa_{\underline{st}} \;\Leftrightarrow\; ds^2(\phi) = G_{st}(\phi)d\phi^s\,d\phi^t = k\,\mathrm{Tr}(P(\phi)^2)\,. \tag{3.55}$$

The G-invariance of this metric immediately follows from (3.45):

$$\forall g \in G \ : \quad ds^2(g\star\phi) = k\,\mathrm{Tr}(P(g\star\phi)^2) = k\,\mathrm{Tr}(h\,P(\phi)^2\,h^{-1}) = k\,\mathrm{Tr}(P(\phi)^2) = ds^2(\phi)\,. \quad (3.56)$$

The σ-model Lagrangian density can be written in the following form

$$\mathcal{L}_{scal} = e\,\frac{k}{2}\,\mathrm{Tr}(P_s(\phi)\,P_t(\phi))\,\partial_\mu\phi^s\,\partial^\mu\phi^t\,, \quad (3.57)$$

where $P = P_s\,d\phi^s$.

A worked out example. Consider the lower-half plane of Example 3. We can take the following basis of generators of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$:

$$\mathfrak{sl}(2,\mathbb{R}) = \{\sigma^1,\, i\,\sigma^2, \sigma^3\} = \{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\}\,. \quad (3.58)$$

In the Cartan basis the space $\mathfrak{K}$ is spanned by the following matrices:

$$\mathfrak{K} = \{K_{\underline{s}}\} = \{\sigma^1,\, \sigma^3\}\,, \quad (3.59)$$

while $Solv$ is the subalgebra of upper triangular generators:

$$Solv = \{\sigma^3,\, \sigma^+\}\ ,\quad \sigma^+ \equiv \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\,. \quad (3.60)$$

Exercise: Prove that Solv is a solvable Lie algebra, using the definition given in footnote 18. In the Cartan parametrization we denote the coordinates by $\phi^s = (\xi,\,\alpha)$ and define the coset representative as:

$$\mathbb{L}(\xi,\alpha) = \exp(\frac{\xi\sin(\alpha)\,\sigma^1 + \xi\sin(\alpha)\,\sigma^3}{2}) = \begin{pmatrix}\cosh\left(\frac{\xi}{2}\right) + \sin(\alpha)\sinh\left(\frac{\xi}{2}\right) & \cos(\alpha)\sinh\left(\frac{\xi}{2}\right)\\ \cos(\alpha)\sinh\left(\frac{\xi}{2}\right) & \cosh\left(\frac{\xi}{2}\right) - \sin(\alpha)\sinh\left(\frac{\xi}{2}\right)\end{pmatrix}\,. \quad (3.61)$$

The reader can verify that the adjoint action of a generic element $h \in H = \mathrm{SO}(2)$, of the form (3.30) on $\mathfrak{K}$ defines the following matrix representation $\mathscr{K}$ of h:

$$h^{-1}K_{\underline{s}}h = h_{\underline{s}}{}^{\underline{t}}\,K_{\underline{s}}\ ;\quad h_{\underline{s}}{}^{\underline{t}} = \{\begin{pmatrix}\cos(2\theta) & \sin(2\theta)\\ -\sin(2\theta) & \cos(2\theta)\end{pmatrix}\}\,. \quad (3.62)$$

We can alternatively define the solvable parametrization $\phi^s = (\varphi,\,\chi)$, in which the coset representative has the following form:

$$\mathbb{L}_s(\varphi,\,\chi) \equiv e^{\chi\sigma^+}\,e^{\frac{\varphi}{2}\sigma^3} = \begin{pmatrix}1 & \chi\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{\varphi/2} & 0\\ 0 & e^{-\varphi/2}\end{pmatrix} \in\ e^{Solv}\,. \quad (3.63)$$

The relation between the two parametrizations is defined by the condition:

$$\mathbb{L}_s(\varphi,\,\chi)\,h(\theta) = \mathbb{L}(\xi,\alpha)\,, \quad (3.64)$$

which is solved by the following relations:

$$
\begin{aligned}
e^{-\varphi} &= \cosh(\xi) - \sin(\alpha)\sinh(\xi)\ ;\ \ \chi = \frac{\cos(\alpha)\sinh(\xi)}{\cosh(\xi) - \sin(\alpha)\sinh(\xi)}\,,\\
\sin(\theta) &= -\frac{\cos(\alpha)\sinh\left(\frac{\xi}{2}\right)}{\sqrt{\cosh(\xi) - \sin(\alpha)\sinh(\xi)}}\ ;\ \ \cos(\theta) = \frac{\cosh\left(\frac{\xi}{2}\right) - \sin(\alpha)\sinh\left(\frac{\xi}{2}\right)}{\sqrt{\cosh(\xi) - \sin(\alpha)\sinh(\xi)}}\,.
\end{aligned} \tag{3.65}
$$

Let us now compute P and ω in the solvable parametrization:

$$
\mathbb{L}_s^{-1} d\mathbb{L}_s = P + \omega\ ,\ \ \omega = \frac{d\chi e^{-\varphi}}{2}\, i\sigma^2\ \ ;\ \ P = \frac{d\varphi}{2}\,\sigma^3 + \frac{d\chi e^{-\varphi}}{2}\,\sigma^1\,. \tag{3.66}
$$

Choosing the normalization factor $k = 1$, the metric in the solvable parametrization reads:

$$
ds^2 = \frac{d\varphi^2}{2} + \frac{1}{2} d\chi^2 e^{-2\varphi}\,, \tag{3.67}
$$

which coincides with (3.28) if we identify:

$$
z = \chi - i\, e^{\varphi}\,. \tag{3.68}
$$

Exercise: Verify that in the Cartan parametrization: $ds^2 = \frac{d\xi^2}{2} + \frac{1}{2} d\alpha^2 \sinh^2(\xi)$.

With some algebra the reader can also verify that the identification (3.68) is also consistent with the $\mathrm{SL}(2,\mathbb{R})$ action (3.29) on z:

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{L}_s(\varphi, \chi) = \mathbb{L}_s(\varphi', \chi') h\ \ \Rightarrow\ \ z' = \chi' - i\, e^{\varphi'} = \frac{a\, z + b}{c\, z + d}\,. \tag{3.69}
$$

Exercise: Verify for this space the Eq.s (3.52) and (3.53).

3.2 On-Shell Duality Invariance

We shall focus from now on extended ungauged supergravities with homogeneous-symmetric scalar manifold. As mentioned earlier, supersymmetry does not allow for a scalar potential at the ungauged level. Let us derive the bosonic field equations from the action (3.10) and then discuss their global symmetries, restricting ourselves to the bosonic terms only (the presence of additional terms containing fermion bilinears in the field equations for the bosonic fields is of course understood).

It is useful to introduce the dual field strengths $G_{\Lambda\,\mu\nu}$ defined as:

$$
G_{\Lambda\,\mu\nu} \equiv -\epsilon_{\mu\nu\rho\sigma} \frac{\partial \mathcal{L}_4}{\partial F^\Lambda_{\rho\sigma}} = R_{\Lambda\Sigma}\, F^\Sigma_{\mu\nu} - I_{\Lambda\Sigma}\, {}^*F^\Sigma_{\mu\nu}\,, \tag{3.70}
$$

where

$$
{}^*F^\Sigma_{\mu\nu} \equiv \frac{e}{2}\,\epsilon_{\mu\nu\rho\sigma}\, F^{\Lambda\,\rho\sigma}\,. \tag{3.71}
$$

In ordinary Maxwell Theory $I_{\Lambda\Sigma} = -1$, $R_{\Lambda\Sigma} = 0$ and $G_{\mu\nu} = {}^*F_{\mu\nu}$.

The equations of motion for the scalar and vector fields read:

$$\mathscr{D}_\mu(\partial^\mu \phi^s) = \frac{1}{4} G^{st} \left[F^\Lambda_{\mu\nu}\, \partial_t\, I_{\Lambda\Sigma}\, F^{\Sigma\,\mu\nu} + F^\Lambda_{\mu\nu} \partial_t\, R_{\Lambda\Sigma}\, {}^*F^{\Sigma\,\mu\nu}\right] , \tag{3.72}$$

$$\nabla_\mu \left({}^*F^{\Lambda\,\mu\nu}\right) = 0 \ ; \ \ \nabla_\mu \left({}^*G^{\Lambda\,\mu\nu}\right) = 0 \, , \tag{3.73}$$

where $\partial_s \equiv \frac{\partial}{\partial\phi^s}$, ∇_μ is the covariant derivative containing the Levi-Civita connection on space-time, while $\mathscr{D}_\mu$ also contains the Levi-Civita connection $\tilde{\Gamma}$ on $\mathscr{M}_{scal}$:

$$\mathscr{D}_\mu(\partial_\nu\phi^s) \equiv \nabla_\mu(\partial_\nu\phi^s) + \tilde{\Gamma}^s_{t_1 t_2} \partial_\mu\phi^{t_1}\, \partial_\nu\phi^{t_2} \, . \tag{3.74}$$

Using (3.70) and the property that ${}^{**}F^\Lambda = -F^\Lambda$, we can express ${}^*F^\Lambda$ and ${}^*G_\Lambda$ as linear functions of F^Λ and G_Λ:

$${}^*F^\Lambda = I^{-1\,\Lambda\Sigma} \left(R_{\Sigma\Gamma}\, F^\Gamma - G_\Sigma\right) \ ; \ \ {}^*G_\Lambda = (RI^{-1}R + I)_{\Lambda\Sigma}\, F^\Sigma - (RI^{-1})_\Lambda{}^\Sigma\, G_\Sigma \, , \tag{3.75}$$

where, for the sake of simplicity, we have omitted the space-time indices. It is useful to arrange F^Λ and G_Λ in a single $2n_V$-dimensional vector $\mathbb{F} \equiv (\mathbb{F}^M)$ of two-forms:

$$\mathbb{F}_{\mu\nu} \equiv \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Lambda\mu\nu} \end{pmatrix} , \tag{3.76}$$

in terms of which eq.s (3.75) are easily rewritten in the following compact form:

$${}^*\mathbb{F} = -\mathbb{C}\mathcal{M}(\phi^s)\, \mathbb{F} \, , \tag{3.77}$$

where

$$\mathbb{C} = (\mathbb{C}^{MN}) \equiv \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} , \tag{3.78}$$

$\mathbf{1}$, $\mathbf{0}$ being the $n_V \times n_V$ identity and zero-matrices, respectively, and

$$\mathcal{M}(\phi) = (\mathcal{M}(\phi)_{MN}) \equiv \begin{pmatrix} (RI^{-1}R + I)_{\Lambda\Sigma} & -(RI^{-1})_\Lambda{}^\Gamma \\ -(I^{-1}R)^\Delta{}_\Sigma & I^{-1\,\Delta\Gamma} \end{pmatrix} , \tag{3.79}$$

is a symmetric, negative-definite matrix, function of the scalar fields.

The Maxwell equations can then be recast in the following equivalent forms:

$$\nabla_\mu({}^*\mathbb{F}^{\mu\nu}) = 0 \ \Leftrightarrow \ \ \nabla_\mu(\mathbb{C}\mathcal{M}(\phi)\mathbb{F}^{\mu\nu}) = 0 \ \Leftrightarrow \ \ d\mathbb{F} = 0 \, , \tag{3.80}$$

where we have used the matrix notation and suppressed the indices $M, N, \dots$.

Since the matrix $\mathcal{M}(\phi)$ will play an important role in the discussion of the global symmetries of the field equations and Bianchi identities (on-shell global symmetries), it is useful to express the part of the field equations depending on the vector field strengths in terms of it and of its derivatives. Let us start with the scalar field equations (3.72) and compute the following expression:

$$\mathbb{F}^T_{\mu\nu} \partial_s \mathcal{M}(\phi)\, \mathbb{F}^{\mu\nu} \equiv \mathbb{F}^M_{\mu\nu} \partial_s \mathcal{M}(\phi)_{MN}\, \mathbb{F}^{N\,\mu\nu} \, , \tag{3.81}$$

Using the definition of $\mathcal{M}$ and suppressing the space-time indices together with the Λ,Σ ones, we find:

$$
\begin{aligned}
\mathbb{F}^T\partial_s\mathcal{M}(\phi)\,\mathbb{F} &= (F^T,\,G^T)\begin{pmatrix}\partial_s(RI^{-1}R+I) & -\partial_s(RI^{-1})\\ -\partial_s(I^{-1}R) & \partial_s I^{-1}\end{pmatrix}\begin{pmatrix}F\\ G\end{pmatrix}=\\
&= F^T\,\partial_s I\,F + F^T\partial_s RI^{-1}RF + F^T R\partial_s I^{-1}RF + F^T RI^{-1}\partial_s RF-\\
&-2\,F^T\partial_s R\,I^{-1}G - 2\,F^T R\,\partial_s I^{-1}G + G^T\partial_s I^{-1}G =\\
&= F^T\,\partial_s I\,F + F^T\partial_s RI^{-1}RF + F^T R\partial_s I^{-1}RF + F^T RI^{-1}\partial_s RF-\\
&-2\,F^T\partial_s R\,I^{-1}\,(RF - I^*F) - 2\,F^T R\,\partial_s I^{-1}(RF-I^*F)+\\
&+(F^T R - {}^*F^T I)\partial_s I^{-1}(RF-I^*F) = F^T\,\partial_s I\,F - {}^*F^T\,\partial_s I\,{}^*F + 2\,F^T\,\partial_s R\,{}^*F =\\
&= 2\,(F^T\,\partial_s I\,F + F^T\,\partial_s R\,{}^*F)\,.
\end{aligned}
\tag{3.82}
$$

We can then rewrite the scalar field equations in the following form:

$$
\mathscr{D}_\mu(\partial^\mu\phi^s) = \frac{1}{8}\,G^{st}\,\mathbb{F}^T_{\mu\nu}\partial_s\mathcal{M}(\phi)\,\mathbb{F}^{\mu\nu}\,, \tag{3.83}
$$

Let us now compute the Einstein equations:

$$
\mathcal{R}_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,\mathcal{R} = T^{(S)}_{\mu\nu} + T^{(V)}_{\mu\nu}\,, \tag{3.84}
$$

where the energy-momentum tensors for the scalar and vector fields can be cast in the following general form

$$
T^{(S)}_{\mu\nu} = G_{rs}(\phi)\,\partial_\mu\phi^r\partial_\nu\phi^s - \frac{1}{2}\,g_{\mu\nu}\,G_{rs}(\phi)\,\partial_\rho\phi^r\partial^\rho\phi^s\,, \tag{3.85}
$$

$$
T^{(V)}_{\mu\nu} = F^T_{\mu\rho}\,I\,F_\nu{}^\rho - \frac{1}{4}\,g_{\mu\nu}\,(F^T_{\rho\sigma}IF^{\rho\sigma})\,. \tag{3.86}
$$

In order to make the global symmetries of the field equations, to be discussed below, manifest, we rewrite $T^{(V)}_{\mu\nu}$ in terms of the matrix $\mathcal{M}(\phi)$. We start noticing that, using Eq. (3.77) we can write (we suppress the indices $M,N,\dots$ and Λ,Σ):

$$
\begin{aligned}
&\mathbb{F}^T_{\mu\rho}\,\mathcal{M}(\phi)\,\mathbb{F}^{\nu\rho} = \mathbb{F}^T_{\mu\rho}\,\mathbb{C}\,{}^*\mathbb{F}^{\nu\rho} = F^T_{\mu\rho}\,{}^*G^{\nu\rho} - G^T_{\mu\rho}\,{}^*F^{\nu\rho} =\\
&= F^T_{\mu\rho}\,I\,F^{\nu\rho} + F^T_{\mu\rho}\,R\,{}^*F^{\nu\rho} - (F^T_{\mu\rho}\,R - {}^*F^T_{\mu\rho}\,I)\,{}^*F^{\nu\rho} = F^T_{\mu\rho}\,I\,F^{\nu\rho} + {}^*F^T_{\mu\rho}\,I\,{}^*F^{\nu\rho} =\\
&= F^T_{\mu\rho}\,I\,F^{\nu\rho} + \frac{1}{4}\epsilon_{\mu\rho\mu_1\mu_2}\epsilon^{\nu\rho\nu_1\nu_2}\,F^{T\,\mu_1\mu_2}\,I\,F_{\nu_1\nu_2} = F^T_{\mu\rho}\,I\,F^{\nu\rho} - \frac{3!}{4}\delta^{\nu\nu_1\nu_2}_{\mu\mu_1\mu_2}\,F^{T\,\mu_1\mu_2}\,I\,F_{\nu_1\nu_2} =\\
&= F^T_{\mu\rho}\,I\,F^{\nu\rho} - \frac{1}{2}(\delta^\nu_\mu\delta^{\nu_1\nu_2}_{\mu_1\mu_2} + 2\,\delta^\nu_{\mu_1}\delta^{\nu_1\nu_2}_{\mu_2\mu})\,F^{T\,\mu_1\mu_2}\,I\,F_{\nu_1\nu_2} = 2\,F^T_{\mu\rho}\,I\,F^{\nu\rho} - \frac{1}{2}\delta^\nu_\mu\,F^T_{\rho\sigma}IF^{\rho\sigma}\,.
\end{aligned}
\tag{3.87}
$$

We can then write:

$$
T^{(V)}_{\mu\nu} = \frac{1}{2}\,\mathbb{F}^T_{\mu\rho}\,\mathcal{M}(\phi)\,\mathbb{F}_\nu{}^\rho\,. \tag{3.88}
$$

Since in (3.84) $\mathcal{R} = G_{st}(\phi)\,\partial_\rho\phi^s\partial^\rho\phi^t$, the equation can be finally recast in the following form:

$$
\mathcal{R}_{\mu\nu} = G_{rs}(\phi)\,\partial_\mu\phi^r\partial_\nu\phi^s + \frac{1}{2}\,\mathbb{F}^T_{\mu\rho}\,\mathcal{M}(\phi)\,\mathbb{F}_\nu{}^\rho\,, \tag{3.89}
$$

While the Maxwell equations $\nabla_\mu({}^*\mathbb{F}^{M\,\mu\nu}) = 0$ are invariant with respect to a generic linear transformation on $\mathbb{F}$, the definition of G_Λ or, equivalently, eq. (3.77) is not. On the other hand the isometry group G is a global symmetry of the scalar kinetic term, but it will in general alter the action for the vector fields as a consequence of the scalar field-dependence of the matrices I and R.

One of the most intriguing features of extended supergravities is the fact that the global invariance of the scalar kinetic term, described by G, can be extended to a global symmetry of the full set of equations of motion and Bianchi identities [31] (though not in general of the whole action). This is possible because in extended supergravities there are scalar fields which are connected by supersymmetry to vector fields and, as a consequence of this, that transformations on the former imply to transformations on the latter (more precisely transformations on the vector field strengths F^Λ and their duals G_Λ). From the mathematical point of view this follows from the definition on the scalar manifold (at least on the manifold spanned by the scalar fields sitting in the same supermultiplet as the vector ones) of a geometric structure (called *symplectic structure*) which associates with each point ϕ on the manifold a symmetric, symplectic $(2n_V) \times (2n_V)$ matrix $\mathcal{M}(\phi)_{MN}$ and with each isometry transformation $g \in G$ on the same manifold a corresponding constant symplectic $(2n_V) \times (2n_V)$ matrix $\mathbf{S}[g] = (\mathbf{S}[g]^M{}_N)$ such that:

$$\mathcal{M}(g \star \phi) = \mathbf{S}[g]^{-T} \mathcal{M}(\phi) \mathbf{S}[g]^{-1} \,. \tag{3.90}$$

Recall that a symplectic matrix M in $\mathrm{Sp}(2n_V, \mathbb{R})$ is defined by the property: $M^T \mathbb{C} M = M \mathbb{C} M^T = \mathbb{C}$, where the symplectic invariant matrix $\mathbb{C}$ is defined in (3.78). Thus the symmetric matrix $\mathcal{M}(\phi)_{MN}$ satisfies the properties

$$\mathcal{M}(\phi)_{MP} \mathbb{C}^{PL} \mathcal{M}(\phi)_{LN} = \mathbb{C}_{MN} \;\Leftrightarrow\; \mathcal{M}(\phi)^{-1} = -\mathbb{C} \mathcal{M}(\phi) \mathbb{C} \,, \tag{3.91}$$

$(\mathbb{C}_{MN})$ having the same matrix form as the matrix $(\mathbb{C}^{MN})$ in (3.78), while the correspondence between $g \in G$ and $\mathbf{S}[g]$ defines a *symplectic representation* of the group G, i.e. an embedding $\mathbf{S}$ of G inside $\mathrm{Sp}(2n_V, \mathbb{R})$

$$\begin{aligned} G \overset{\mathbf{S}}{\hookrightarrow} \mathrm{Sp}(2n_V, \mathbb{R}) \;\Leftrightarrow\; g \in G \to \mathbf{S}[g] \in \mathrm{Sp}(2n_V, \mathbb{R}) \;;\quad \mathbf{S}[g_1\, g_2] = \mathbf{S}[g_1] \mathbf{S}[g_2] \,, \\ \mathbf{S}[g]^M{}_N \mathbb{C}^{NP} \mathbf{S}[g]^L{}_P = \mathbb{C}^{ML} \;\Leftrightarrow\; \mathbf{S}[g]^M{}_N \mathbb{C}_{ML} \mathbf{S}[g]^L{}_P = \mathbb{C}_{NP} \,, \end{aligned} \tag{3.92}$$

where in the second line we have written the general property defining a symplectic matrix: $\mathbf{S}[g] \mathbb{C} \mathbf{S}[g]^T = \mathbf{S}[g]^T \mathbb{C} \mathbf{S}[g] = \mathbb{C}$. We learn then that the definition of the matrix $\mathcal{M}(\phi)_{MN}$ is built-in the mathematical structure of the scalar manifold (and below we shall illustrate this explicitly for the homogeneous manifolds). The matrices $I(\phi)$ and $R(\phi)$ entering the action are then defined in terms of $\mathcal{M}(\phi)$ by Eq. (3.79). The only freedom which is left consists in the choice of the basis of the symplectic representation (*symplectic frame*) which amounts to a change in the definition of $\mathcal{M}(\phi)$ by a constant symplectic transformation E:

$$\mathcal{M}(\phi) \to \mathcal{M}'(\phi) = E^T \mathcal{M}(\phi) E \,. \tag{3.93}$$

This affects the form of the action, in particular the coupling of the scalar fields to the vectors. However, at the ungauged level, it only amounts to a (non-perturbative) redefinition of the vector field strengths and their duals which has no physical implication. In the presence of a gauging, namely if vectors are minimally coupled to the other fields, the symplectic frame becomes physically relevant and may lead to different vacuum-structures of the scalar potential.

We emphasize here that the existence of this symplectic structure on the scalar manifold is a general feature of all extended supergravites, including those $\mathcal{N}=2$ models in which the scalar manifold is not even homogeneous (i.e. the isometry group, if it exists, does not act transitively on the manifold itself). In the $\mathcal{N}=2$ case, only the scalar fields belonging to the vector multiplets are non-minimally coupled to the vector fields, namely enter the matrices $I(\phi)$, $R(\phi)$, and they span a *special Kähler* manifold. On this manifold a flat symplectic bundle is defined [21], which fixes the scalar dependence of the matrices $I(\phi)$, $R(\phi)$ and the matrix $\mathcal{M}(\phi)$ defined in (3.79), satisfies the properties (3.91), (3.90).

For homogeneous manifolds, the isometry group G has a symplectic, $2n_V$-dimensional representation $\mathbf{S}$ and we can express $\mathcal{M}(\phi)$ in terms of the coset representative:

$$\mathcal{M}(\phi)_{MN}=\mathbb{C}_{MP}\mathbb{L}(\phi)^P{}_L\mathbb{L}(\phi)^R{}_L\,\mathbb{C}_{RN}\;\Leftrightarrow\;\mathcal{M}(\phi)=\mathbb{C}\mathbf{S}[\mathbb{L}(\phi)]\,\mathbf{S}[\mathbb{L}(\phi)]^T\,\mathbb{C}\,, \tag{3.94}$$

where summation over the index L is understood and $\mathbb{L}^P{}_L$ are the entries of the symplectic matrix $\mathbf{S}[\mathbb{L}(\phi)]$ associated with $\mathbb{L}(\phi)$ as an element of G. Since $\mathbf{S}$ is a homomorphism, Eq. (3.21) can also be written in terms of symplectic matrices as follows:

$$\mathbf{S}[g]\,\mathbf{S}[\mathbb{L}(\phi)]=\mathbf{S}[\mathbb{L}(g\star\phi)]\,\mathbf{S}[h(g,\,\phi)]\,. \tag{3.95}$$

We see that from (3.94) and (3.95), properties (3.91) and (3.90) easily follow. Let us derive (3.90):

$$\begin{aligned}\mathcal{M}(g\star\phi)&=\mathbb{C}\mathbf{S}[\mathbb{L}(g\star\phi)]\,\mathbf{S}[\mathbb{L}(g\star\phi)]^T\,\mathbb{C}=\\&=\mathbb{C}\mathbf{S}[g]\,\mathbf{S}[\mathbb{L}(\phi)]\mathbf{S}[h]^{-1}\,\mathbf{S}[h]^{-T}\mathbf{S}[\mathbb{L}(\phi)]^T\mathbf{S}[g]^T\,\mathbb{C}=\\&=\mathbf{S}[g]^{-T}\,\mathbb{C}\mathbf{S}[\mathbb{L}(\phi)]\,\mathbf{S}[\mathbb{L}(\phi)]^T\,\mathbb{C}\,\mathbf{S}[g]^{-1}=\mathbf{S}[g]^{-T}\,\mathcal{M}(\phi)\,\mathbf{S}[g]^{-1}\,,\end{aligned} \tag{3.96}$$

where we have used the property that $\mathbf{S}[g]$ is symplectic, $\mathbb{C}\mathbf{S}[g]=\mathbf{S}[g]^{-T}\,\mathbb{C}$, and that $\mathbf{S}[h]\equiv\mathbf{S}[h(g,\,\phi)]$ is orthogonal, being in a real representation of $\mathrm{U}(n_V)$: $\mathbf{S}[h]^T=\mathbf{S}[h]^{-1}$. The latter property in particular implies that $\mathcal{M}(\phi)$, as defined in (3.94), is H-invariant, namely it does not depend on the choice of the coset representative, but only on the point ϕ of the manifold, as it should be.

We can now easily verify that the simultaneous action of G on the scalar fields and on the field strength vector $\mathbb{F}^M_{\mu\nu}$:

$$g\in G:\;\begin{cases}\phi^r\rightarrow g\star\phi^r\\ \mathbb{F}^M_{\mu\nu}\rightarrow\mathbb{F}'^M_{\mu\nu}=\mathbf{S}[g]^M{}_N\mathbb{F}^N_{\mu\nu}\end{cases}\,, \tag{3.97}$$

[21] A special Kähler manifold is in general characterized by the product of a U(1)-bundle, associated with its Kähler structure (with respect to which the manifold is Hodge Kähler), and a flat symplectic bundle. See for instance [32] for an in depth account of this issue.

is a symmetry of the field equations. The Maxwell equations are clearly invariant under (3.3.4). We must however show that the above transformation leaves (3.77) invariant, namely that is holds in the transformed fields as well. Using (3.3.4), eq. (3.77) can indeed be written in the new quantities as follows:

$$\mathbf{S}[g]^{-1}{}^*\mathbb{F}' = -\mathbb{C}\mathbf{S}[g]^T\mathcal{M}(g\star\phi)\mathbf{S}[g]\,\mathbf{S}[g]^{-1}\mathbb{F}' = -\mathbf{S}[g]^{-1}\mathbb{C}\mathcal{M}(g\star\phi)\mathbb{F}'\,, \tag{3.98}$$

which is equivalent to ${}^*\mathbb{F}' = -\mathbb{C}\mathcal{M}(g\star\phi)\mathbb{F}'$.

The invariance of the scalar and Einstein equations is manifest if we write them in the forms (3.83) and (3.89), respectively, and follows from the invariance of the quantity:

$$\mathbb{F}^T_{\mu\nu}\mathcal{M}(\phi)\mathbb{F}_{\rho\sigma}\,, \tag{3.99}$$

which can be easily proven as follows:

$$\mathbb{F}^T_{\mu\nu}\,\mathcal{M}(\phi)\,\mathbb{F}_{\rho\sigma} = \mathbb{F}'^T_{\mu\nu}\mathbf{S}[g]^{-T}\,\mathbf{S}[g]^T\mathcal{M}(g\star\phi)\mathbf{S}[g]\,\mathbf{S}[g]^{-1}\mathbb{F}'_{\rho\sigma} = \mathbb{F}'^T_{\mu\nu}\mathcal{M}(g\star\phi)\mathbb{F}'_{\rho\sigma}\,. \tag{3.100}$$

This directly implies the invariance of $T^{(V)}$ and the covariance of the scalar field equation. The duality invariance of the space-time metric and the scalar action imply the same property for the Einstein tensor and $T^{(S)}_{\mu\nu}$.

The action of G on the field strengths and their magnetic duals, defined by the symplectic embedding $\mathbf{S}$, is a *generalized electric-magnetic duality transformation*, which promotes the isometry group of the scalar manifold to a global symmetry group of the full set of field equations and Bianchi identities. It generalizes the known duality invariance of ordinary Maxwell theory:

$$\begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} \longrightarrow \begin{pmatrix} F'_{\mu\nu} \\ {}^*F'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix}. \tag{3.101}$$

For this reason G is also referred to as the *duality group* of the classical theory. In the presence of electric and magnetic sources, just as in ordinary Maxwell theory, the symplectic action of G is extended to the charges themselves.

Note however that G will contain transformations g whose duality action $\mathbf{S}[g]$ is *non-perturbative*, namely under which $F^\Lambda \to F'^\Lambda = A^\Lambda{}_\Sigma\,F^\Sigma + B^{\Lambda\Sigma}\,G_\Sigma$ and $G_\Lambda \to G'_\Lambda = C_{\Lambda\Sigma}\,F^\Sigma + D_\Lambda{}^\Sigma\,G_\Sigma$, with $C_{\Lambda\Sigma}$, $B^{\Lambda\Sigma} \neq 0$. These are not a symmetry of the action but only of the field equations and Bianchi identities (on-shell symmetry).

The relevance of the (quantum) duality group resides in the existence of important evidence that it (or a suitable extension of it) might encode all the known string/M-theory dualities [33].

Let us end this section by collecting the bosonic equations derived above in their manifestly

G-invariant form:

Scalar:

$$\mathscr{D}_\mu(\partial^\mu\phi^s) = \frac{1}{8}\, G^{st}\, \mathbb{F}^T_{\mu\nu}\partial_s\mathcal{M}(\phi)\, \mathbb{F}^{\mu\nu}\,, \tag{3.102}$$

Einstein:

$$\mathcal{R}_{\mu\nu} = G_{rs}(\phi)\, \partial_\mu\phi^r\partial_\nu\phi^s + \frac{1}{2}\, \mathbb{F}^T_{\mu\rho}\, \mathcal{M}(\phi)\, \mathbb{F}_\nu{}^\rho\,, \tag{3.103}$$

Maxwell:

$$d\mathbb{F} = 0 \;\;,\quad {}^*\mathbb{F} = -\mathbb{C}\mathcal{M}(\phi^s)\, \mathbb{F}\;, \tag{3.104}$$

where we have omitted the terms containing the fermion fields. We shall comment on them in the next Subsection.

On a charged dyonic solution, we define the electric and magnetic charges as the integrals:

$$e_\Lambda \equiv \frac{1}{4\pi}\int_{S^2} G_\Lambda = \frac{1}{8\pi}\int_{S^2} G_{\Lambda\,\mu\nu}\, dx^\mu\wedge dx^\nu \;\;,\quad m^\Lambda \equiv \frac{1}{4\pi}\int_{S^2} F^\Lambda = \frac{1}{8\pi}\int_{S^2} F^\Lambda{}_{\mu\nu}\, dx^\mu\wedge dx^\nu\;, \tag{3.105}$$

where S^2 is a spatial two-sphere. In our conventions, the electric and magnetic charges (e, m) are related to those (e', m') in the (rationalized) Heaviside-Lorentz units by a factor 4π:[22]

$$e_\Lambda = \frac{1}{4\pi}\, e'_\Lambda \;\;,\;\; m^\Lambda = \frac{1}{4\pi}\, m'^\Lambda\,. \tag{3.107}$$

They define a symplectic vector Γ^M:

$$\Gamma = (\Gamma^M) = \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} = \frac{1}{4\pi}\int_{S^2}\mathbb{F}\,. \tag{3.108}$$

These are the *quantized charges*, namely they satisfy the Dirac-Schwinger-Zwanziger quantization condition for dyonic particles [36]:

$$(4\pi)^2\, \Gamma_2^T\mathbb{C}\Gamma_1 = m_2'^\Lambda\, e'_{1\Lambda} - m_1'^\Lambda\, e'_{2\Lambda} = 2\pi\,\hbar\, c\, n \;\;;\quad n\in\mathbb{Z}\,. \tag{3.109}$$

[22] In the rationalized-Heaviside-Lorentz (RHL) system of units, the charge unit is defined so that $\varepsilon_0 = 1$. In the non-rationalized-Heaviside-Lorentz (HL) system of units, $\varepsilon_0 = 1/4\pi$. We make a choice of units such that $8\pi\, G\, \varepsilon_0 = 1$. Further choosing $8\pi\, G = 1$ then implies the adoption of the rationalized-HL convention $\varepsilon_0 = 1$. This fixes the choice of electric/magnetic charge units. The further rescaling by a factor 4π to define e, m is just for later convenience in the calculations, though e', m' should always be intended to be the true charges of the solution. Denoting by Q_{RHL} and Q_{HL} the charges in the rationalized and non-rationalized-Heaviside-Lorentz system of units, respectively, in all our formulas the quantities expressed in our charges Q (generically denoting by Q either e or m) or central charges $\mathscr{Z}$ are expressed in terms of the corresponding quantities in the two systems through the replacement:

$$Q = \frac{1}{4\pi}\, Q_{RHL} = \frac{1}{\sqrt{4\pi}}\, Q_{HL}\,,$$
$$\mathscr{Z} = \frac{1}{4\pi}\, \mathscr{Z}_{RHL} = \frac{1}{\sqrt{4\pi}}\, \mathscr{Z}_{HL}\,. \tag{3.106}$$

At the quantum level the dyonic charges therefore belong to a symplectic lattice and this breaks the duality group G to a suitable discrete subgroup $G(\mathbb{Z})$ which leaves this symplectic lattice invariant.

Due to the non-minimal coupling of the scalar fields to the vector fields, the electric and magnetic fields that one would actually measure at spatial infinity on a solution (and thus the electric and magnetic charges), are not given simply by the field strengths F^Λ and G_Λ (and thus by the quantized charges e, m). They also depend on the scalar fields at infinity and are expressed in terms of composite fields depending on the scalar fields as well as on F^Λ, to be defined in the next Subsection.

3.2.1 The Fermion Fields

In the previous Subsection we have dealt with the description of the bosonic sector of an extended supergravity and its on-shell global symmetry. Let us discuss the general features of the fermionic sector, its symmetries and the couplings of the fermions fields to the bosons. We have seen that the vector fields and the scalar fields transform under the action of the group G, isometry group of the scalar manifold. More precisely this group has a global symplectic (duality) action on the vector of electric field strengths and their magnetic duals, while it acts on the scalar fields as an isometry group, according to Eq. (3.3.4). Just as the fermion fields (including the graviton), transform covariantly with respect to the isotropy group of space-time (local Lorentz transformations), they have a well defined transformation property only with respect to the isotropy group H of the scalar manifold. In all extended supergravities this group has the following form [34]:

$$H = G_R \times H_{matter}\,, \tag{3.110}$$

where G_R is the automorphism of the supersymmetry algebra (the R-symmetry group), while H_{matter} is a compact Lie group acting on the matter multiplets. Let us use the chiral (or Weyl) basis for the fermion fields, discussed in 2.1, in which the full (S)U($\mathcal{N}$) G_R group is manifest. The super-Poincaré-algebra- valued 1-form (2.297) now contains a term of the form

$$\tilde{\Omega}_g = \dots - \frac{i}{\sqrt{2}}\,(\bar{\Psi}^i\,\mathbf{Q}_i + \bar{\Psi}_i\,\mathbf{Q}^i)\,, \tag{3.111}$$

Consistently with our conventions, we then define $\Psi_{\mu\,i}$ to be a Weyl-spinor 1-form with positive chirality:

$$\gamma^5\,\Psi_i = \Psi_i = \begin{pmatrix} 0 \\ \bar{\Psi}_i^{\dot\alpha} \end{pmatrix} \quad\Rightarrow\quad \Psi^i \equiv (\Psi_i)_c = C\overline{(\Psi_i)}^T = \begin{pmatrix} \Psi^i_\alpha \\ 0 \end{pmatrix}\ ,\ \ (\alpha,\,\dot\alpha = 1,2)\,. \tag{3.112}$$

The same convention will be used for the supersymmetry parameter: ϵ_i, ϵ^i. Aside from the gravitino, the other fermion fields consist in *dilatinos* χ_{ijk} which are spin-1/2 fields belonging to the gravitational supermultiplet for $\mathcal{N} \geq 3$, and spin-1/2 fields $\lambda_{i\,A}$ (where $A = 1, \dots, n$ labels the vector fields in the vector multiplets) belonging to the vector multiplets (i.e. super multiplets in which the highest spin field has spin 1), which are called *gauginos*. In the

$\mathcal{N}=2$ we also have spin-1/2 fields ζ^a in the hypermultiplets (*hyperinos*). The most general scalar manifold of an $\mathcal{N}=2$ model is described by the product of a *special Kähler manifold* $\mathscr{M}_{SK}$ spanned by the complex scalars z^α ($\alpha=1,\dots,n$) in the vector multiplets, times a *quaternionic Kähler manifold* $\mathscr{M}_{QK}$ spanned by the scalar fields q^u in the hypermultiplets (see [35, 32] for a mathematical definition of the two kinds of manifolds):

$$\mathscr{M}_{scal}=\mathscr{M}_{SK}\times\mathscr{M}_{QK}\,. \tag{3.113}$$

The symplectic structure is defined only over the first factor, since only the scalars z^α enter the matrices $I_{\Lambda\Sigma}$, $R_{\Lambda\Sigma}$. As we shall see, the coupling of the bosons to the fermionic fields is also fixed by the geometry of the scalar manifold, in particular, in the models with a homogeneous scalar manifold, by the coset representative $\mathbb{L}(\phi)$ representing the coset representative. To understand the general structure let us recall that, by (3.21), the matrix $\mathbb{L}(\phi)$ is acted to the left by G and to the right by the compensator in H

$$G\rightarrow\ \mathbb{L}(\phi)\ \leftarrow\ H\,. \tag{3.114}$$

The matrix $\mathbb{L}(\phi)$ therefore "mediates" between objects, like the bosonic fields, transforming directly under G and other objects, like the fermionic fields, transforming only under H. This means that we can construct G-invariant quantities coupling (in suitable ways) the bosonic fields (including their derivatives) to the fermions through $\mathbb{L}(\phi)$, that is, symbolically, considering the contraction

$$(\partial Bosons)\cdot\mathbb{L}(\phi)\cdot(Fermions)=\mathbf{f}(\phi,\,\partial Bosons)\cdot(Fermions)\,. \tag{3.115}$$

In the Lagrangian and in the equations of motion bosons and fermions are indeed coupled through this scalar-dependent matrix. The fermions in other words couple to composite objects (denoted above by the symbol $\mathbf{f}(\phi,\,\partial Bosons)$) obtained by "dressing" the derivatives of bosonic fields by scalar fields through the matrix $\mathbb{L}(\phi)$ and which thus transform, as the scalar fields and vector fields transform under G, only though the corresponding compensating transformations $h(\phi,g)$ in H, see (3.21). We then transform all fermion fields by means of $h(\phi,g)$, namely define the action of G over all the fields as follows:

$$g\in G:\quad\begin{cases}\phi^r\rightarrow g\star\phi^r\\ \mathbb{F}^M_{\mu\nu}\rightarrow\mathbb{F}'^M_{\mu\nu}=\mathbf{S}[g]^M{}_N\mathbb{F}^N_{\mu\nu}\\ \text{fermions}\rightarrow\text{fermions}'=h(\phi,g)\star\text{fermions}\end{cases}\,. \tag{3.116}$$

All the Lagrangian is then constructed in a manifestly H-invariant way using the fermion fields and the composite fields $\mathbf{f}(\phi,\,\partial Bosons)$. Moreover, H-covariance of the supersymmetry transformation laws implies that the supersymmetry variations for the fermion fields be symbolically expressed as follows:

$$\delta_\epsilon(Fermions)=\mathbf{f}(\phi,\,\partial Bosons)\epsilon\,. \tag{3.117}$$

The fields transforming in representations of G_R, as determined in our construction of the Poincaré supermultiplets are therefore either the fermions (including the gravitino) or the composite fields $\mathbf{f}(\phi,\, \partial Bosons)$, and *not* the scalar fields ϕ^s and vector fields A^Λ_μ directly, the latter being always real fields. One can view these composite objects $\mathbf{f}(\phi,\, \partial Bosons)$ as the actual bosonic fields that one would measure at spatial infinity on a solution.

Let us review the general structure of the fermion supersymmetry transformation laws:

$$\delta\Psi_{\mu\, i} = \mathcal{D}_\mu \epsilon_i - \frac{1}{8}\, T^-_{\rho\sigma\, ij}\, \gamma^{\rho\sigma}\gamma_\mu \epsilon^j\,, \tag{3.118}$$

$$\delta\chi_{ijk} = a_1\, P_{ijkl,\, s}\, \partial_\mu\phi^s\, \gamma^\mu \epsilon^l + a_2\, T^-_{\rho\sigma\,[ij}\gamma^{\rho\sigma}\epsilon_{k]}\,, \tag{3.119}$$

$$\delta\lambda_{iA} = a_3 P_{A\, ij,\, s}\, \partial_\mu\phi^s\, \gamma^\mu\epsilon^i + a_4\, T^-_{\rho\sigma\, A}\gamma^{\rho\sigma}\, \epsilon_i\,, \tag{3.120}$$

$$\delta\zeta^{\underline{m}\, i} = a_5\, P^{\underline{m}}_m \partial_\mu q^m\, \gamma^\mu \epsilon^i\,, \tag{3.121}$$

where a_k are constants to be fixed by requiring (no-shell) closure of the super-algebra and the invariance of the action. The quantities

$$P_{ijkl,\, s}(\phi)\, \partial_\mu\phi^s,\; P_{A\, ij,\, s}(\phi)\, \partial_\mu\phi^s\;\; , P^{\underline{m}}_m(q)\partial_\mu q^m\;\; , \;\; T^-_{\rho\sigma\, ij}(\phi, F^\Lambda)\,,$$

are examples of derivatives of the bosonic fields dressed by the scalar fields through the coset representative:

$$P_{ijkl,\, s}(\phi)\, \partial_\mu\phi^s,\; P_{A\, ij,\, s}(\phi)\, \partial_\mu\phi^s\;\; , P^{\underline{m}}_m(q)\partial_\mu q^m\,, \tag{3.122}$$

are components along the Cartan basis $K_{\underline{s}}$ of $\mathfrak{K}$ of the vielbein matrix P (pulled back on space-time by the scalar fields) defined in (3.47). They clearly have the general form $\mathbf{f}(\phi,\, \partial\phi)$ and transform, as the scalar fields and vector fields are acted on by G, through the compensating transformation in H, see Eq. (3.45). Let us define the quantity $T^-_{\rho\sigma,\, ij}(\phi, F^\Lambda)$. This has the general form $\mathbf{f}(\phi,\, \partial A)$. To construct it we need to introduce, for a symplectic $(2n_V)$-vector $V^M = (V^\Lambda,\, V_\Lambda)$ a complex representation $V^{\underline{M}}$ defined through the Cayley matrix $\mathcal{A}$:

$$V^{\underline{M}} = \begin{pmatrix} V^{\underline{\Lambda}} \\ V_{\underline{\Lambda}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^\Lambda + i\, V_\Lambda \\ V^\Lambda - i\, V_\Lambda \end{pmatrix} = \mathcal{A}^{\underline{M}}{}_M\, V^M\,, \tag{3.123}$$

$$\mathcal{A}^{\underline{M}}{}_M \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & i\,\mathbf{1} \\ \mathbf{1} & -i\,\mathbf{1} \end{pmatrix}\,. \tag{3.124}$$

The usefulness of this basis is the fact that, in the duality (symplectic) representation $\mathbf{S}$, the matrices representing H are block-diagonal. To see this consider the matrices $\mathbf{S}(\mathfrak{g})$ representing infinitesimal generators of G. The symplectic condition on generators reads:

$$\mathbf{g} \in \mathbf{S}(\mathfrak{g})\;,\;\; \mathbf{g}^T\, \mathbb{C} + \mathbb{C}\, \mathbf{g} = \mathbf{0}\,. \tag{3.125}$$

This implies that the most general matrix form of $\mathbf{g}$ have the following block structure:

$$\mathbf{g} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\;\;;\;\; \mathbf{D} = -\mathbf{A}^T\;,\;\; \mathbf{C}^T = \mathbf{C}\;,\;\; \mathbf{B}^T = \mathbf{B}\,. \tag{3.126}$$

Exercise: Prove the above relations.

If $\mathbf{g}$ is an element $\mathbf{h}$ of $\mathbf{S}(\mathfrak{H})$, on top of the symplectic condition, it should also be, in a suitable basis, anti-hermitian. Being the symplectic representation *real*, the generator $\mathbf{h}$ of H is represented by an anti-symmetric matrix and therefore has the general form:

$$\mathbf{h} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B} & \mathbf{A} \end{pmatrix} \quad ; \quad \mathbf{A} = -\mathbf{A}^T \ , \ \mathbf{B}^T = \mathbf{B} \, . \tag{3.127}$$

Let us now change basis to the complex one, so that the matrix $\mathbf{h} = (\mathbf{h}^M{}_N)$ becomes $\mathbf{h}^c = (\mathbf{h}^{\underline{M}}{}_{\underline{N}})$ of the form:

$$\mathbf{h}^c = \mathcal{A}\,\mathbf{h}\,\mathcal{A}^\dagger = \begin{pmatrix} \mathbf{A} - i\,\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} + i\,\mathbf{B} \end{pmatrix} \, . \tag{3.128}$$

Exercise: Prove the above relations.

Notice that $\mathbf{A} \pm i\,\mathbf{B}$, being $\mathbf{A}$ antisymmetric and $\mathbf{B}$ symmetric, represent a generator of $\mathrm{U}(n_V)$ in two representations of which one is the complex conjugate of the other. Therefore the upper-half $V^{\underline{\Lambda}}$ and the lower half $V_{\underline{\Lambda}} = (V^{\underline{\Lambda}})^*$ of the complex vector $V^{\underline{M}}$ in (3.123) transform in two conjugate representations $\mathbf{R}_V$, $\overline{\mathbf{R}}_V$ of H.

Recall now the general form of H in Eq. (3.110) and the fact that in the gravitational multiplet there are spin-1 states in the 2-times antisymmetric representation $[\mathcal{N}]_{\mathbf{2}} = \mathcal{N} \wedge \mathcal{N}$ of $(\mathrm{S})\mathrm{U}(\mathcal{N}) = G_R$ while the spin-1 states in the vector multiplets (being top-spin states) are singlets of G_R while transform in general in a representation $\mathbf{n}$ of H_{matter}, so that:

$$\mathbf{R}_V \overset{G_R \times H_{matter}}{\longrightarrow} ([\mathcal{N}]_{\mathbf{2}}, \mathbf{1}) + (\mathbf{1}, \mathbf{n}) \ \Leftrightarrow \ (V_{\underline{\Lambda}}) = (V_{ij},\, V_A) \ ; \ V^{\underline{\Lambda}} = (V_{\underline{\Lambda}})^* = (V^{ij},\, V^A) \, , \tag{3.129}$$

where $V^{ij} = -V^{ji}$, $V_{ij} = -V_{ji}$. Written in the complex basis, a generator of $\mathfrak{H}$ is, as we have seen, block-diagonal, while a generator of $\mathfrak{K}$ in the Cartan basis is block-off-diagonal:

$$\mathbf{k} \in \mathbf{S}[\mathfrak{K}] \ , \ \ \mathbf{k}^c = \mathcal{A}\,\mathbf{k}\,\mathcal{A}^\dagger = (\mathbf{k}^{\underline{M}}{}_{\underline{N}}) = \begin{pmatrix} \mathbf{0} & K^{\underline{\Lambda\Sigma}} \\ K_{\underline{\Lambda\Sigma}} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \begin{matrix} K^{ij,kl} & K^{ij,B} \\ K^{A,kl} & K^{AB} \end{matrix} \\ \begin{matrix} K_{ij,kl} & K_{ij,B} \\ K_{A,kl} & K_{AB} \end{matrix} & \mathbf{0} \end{pmatrix} , \tag{3.130}$$

where $K_{\underline{\Lambda\Sigma}} = (K^{\underline{\Lambda\Sigma}})^* = K_{\underline{\Sigma\Lambda}}$. Correspondingly the $\mathfrak{K}$-valued vielbein one-form P in the representation $\mathbf{S}$, in the complex basis, reads

$$P^c = \mathcal{A}\,P\,\mathcal{A}^\dagger = (P^{\underline{M}}{}_{\underline{N}}) = \begin{pmatrix} \mathbf{0} & P^{\underline{\Lambda\Sigma}} \\ P_{\underline{\Lambda\Sigma}} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \begin{matrix} P^{ij,kl} & P^{ij,B} \\ P^{A,kl} & P^{AB} \end{matrix} \\ \begin{matrix} P_{ij,kl} & P_{ij,B} \\ P_{A,kl} & P_{AB} \end{matrix} & \mathbf{0} \end{pmatrix} . \tag{3.131}$$

This defines the first three quantities in (3.122). In particular the scalar states in (2.86) belonging to the gravitational multiplet $\lambda_0 = 2$ and to the vector multiplets $\lambda_0 = 1$, in supergravity are described by composite fields which are nothing but components of the vielbein of the scalar manifold:

$$\text{scalars in the supergravity multiplet:}\ \ |E,\,2-\frac{4}{2},\,[ijkl]\rangle\ \leftrightarrow\quad P_{ijkl,\,s}\partial_\mu\phi^s=P_{ij,kl,\,s}\partial_\mu\phi^s\,,$$

$$\text{scalars in the }A^{th}\text{ vector multiplet:}\ \ |E,\,1-\frac{2}{2},\,[ij],\,A\rangle\ \leftrightarrow\quad P_{ij,A,\,s}\partial_\mu\phi^s\,. \tag{3.132}$$

These are the actual fields entering the supersymmetry transformation rules (3.118)-(3.120). Consistency then requires:

$$P_{ijkl}=P_{ij,kl}=P_{[ikjl]}\ ,\quad P^{ijkl}=P^{ij,kl}=P^{[ikjl]}\,. \tag{3.133}$$

We also write ω as a matrix in the complex basis:

$$\omega^c=\mathcal{A}\,\omega\,\mathcal{A}^\dagger=(\omega^{\underline{M}}{}_{\underline{N}})=\begin{pmatrix}\omega^{\underline{\Lambda}}{}_{\underline{\Sigma}} & \mathbf{0}\\ \mathbf{0} & \omega_{\underline{\Lambda}}{}^{\underline{\Sigma}}\end{pmatrix}=\begin{pmatrix}\begin{matrix}\omega^{ij}{}_{kl} & 0\\ 0 & \omega^A{}_B\end{matrix} & \mathbf{0}\\ \mathbf{0} & \begin{matrix}\omega_{ij}{}^{kl} & 0\\ 0 & \omega_A{}^B\end{matrix}\end{pmatrix}\,. \tag{3.134}$$

Since the coset representative $\mathbb{L}(\phi)$ contracts to the right against fermion fields (see (3.114)), which belong to complex representations, and to the left against bosonic fields, which can be real (as the vector fields are), it is useful to express the corresponding symplectic matrix $\mathbf{S}[\mathbb{L}(\phi)]$ changing only the right index to a complex one and thus defining the following *hybrid matrix*:

$$\mathbb{L}_c(\phi)=(\mathbb{L}^N{}_{\underline{M}})\equiv\mathbf{S}[\mathbb{L}(\phi)]\mathcal{A}^\dagger=(L^M{}_{ij},\,L^M{}_A,\,L^{M\,ij},\,L^{M\,A})=\begin{pmatrix}L^\Lambda{}_{ij} & L^\Lambda{}_A & L^{\Lambda\,ij} & L^{\Lambda\,A}\\ L_{\Lambda\,ij} & L_{\Lambda\,A} & L_\Lambda{}^{ij} & L_\Lambda{}^A\end{pmatrix}\,. \tag{3.135}$$

The reader can verify that this matrix satisfies the following properties (which derive from the symplectic property of $\mathbf{S}[\mathbb{L}(\phi)]$):

$$\mathbb{L}_c(\phi)^\dagger\mathbb{C}\,\mathbb{L}_c(\phi)=\varpi\ ,\quad \mathbb{L}_c(\phi)\,\varpi\,\mathbb{L}_c(\phi)^\dagger=\mathbb{C}\,, \tag{3.136}$$

$$\varpi\equiv\mathcal{A}\mathbb{C}\mathcal{A}^\dagger=-i\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & -\mathbf{1}\end{pmatrix} \tag{3.137}$$

Exercise: Verify the above properties.

From the definition (3.94) of $\mathcal{M}$ we can express this matrix in terms of the hybrid matrix $\mathbb{L}_c$:

$$\mathcal{M}(\phi)=\mathbb{C}\,\mathbb{L}_c(\phi)\mathbb{L}_c(\phi)^\dagger\,\mathbb{C}\,. \tag{3.138}$$

Next we define the following complex $(2n_V)$-vector of 2-forms by "dressing" $\mathbb{F}^M$ in (3.76) with scalar fields by using $\mathbb{L}_c$:

$$\mathbb{T}_{\mu\nu}(\phi,\partial A^\Lambda)\equiv-\mathbb{L}_c(\phi)^\dagger\mathbb{C}\mathbb{F}_{\mu\nu}=-(L^{\underline{M}\,N}\,\mathbb{C}_{NP}\,\mathbb{F}^P_{\mu\nu})=\begin{pmatrix}T^{ij}_{\mu\nu}\\ T^A_{\mu\nu}\\ T_{\mu\nu\,ij}\\ T_{\mu\nu\,A}\end{pmatrix}\,. \tag{3.139}$$

Let us see how the composite field $\mathbb{T}$ transforms under a transformation in G of the elementary fields it depends on:

$$\mathbb{T}(\phi', \partial A'^{\Lambda}) = -\mathbb{L}_c(g\star\phi)^{\dagger}\mathbb{C}\mathbb{F}' = -h_c(\phi, g)\,\mathbb{L}_c(\phi)^{\dagger}\mathbf{S}[g]^T\mathbb{C}\mathbf{S}[g]\mathbb{F} = h_c(\phi, g)\,\mathbb{T}(\phi, \partial A^{\Lambda})\,, \quad (3.140)$$

where we have used (3.95) and the property of the compensating transformation in the complex basis that $(h_c)^{-1\,\dagger} = h_c$. The composite field transforms only by the compensating transformation in H, so that, transforming the fermion fields under the same transformation, the equations (3.118) and (3.120) retain the same form in the transformed quantities.

In the transformation laws (3.118)-(3.120) we see that the chiral fermions are connected by supersymmetry to the anti-self-dual component of the field strengths, entering the definition of $T^-_{\mu\nu}$. The self-dual and anti-self-dual components of a field strength $F_{\mu\nu}$ are defined as follows:

$$F^{\pm}_{\mu\nu} \equiv \frac{F_{\mu\nu} \pm i\,{}^*F_{\mu\nu}}{2} \quad \Rightarrow \quad {}^*F^{\pm}_{\mu\nu} = \mp i\,F^{\pm}_{\mu\nu}\,. \quad (3.141)$$

Using Eq. (3.77) we can write these components as the result of a projection on the symplectic vector $\mathbb{F}^M_{\mu\nu}$:

$$\mathbb{F}^{\pm}_{\mu\nu} = \mathbb{P}^{\pm}\,\mathbb{F}_{\mu\nu}\ ;\ \ \mathbb{P}^{\pm} = \frac{1}{2}\left(\mathbf{1} \mp i\,\mathbb{C}\mathcal{M}(\phi)\right)\,. \quad (3.142)$$

Exercise: Verify using the symplectic property of the matrix $\mathcal{M}$ that $\mathbb{P}^{\pm}$ are projectors, namely that:

$$\mathbb{P}^{\pm}\,\mathbb{P}^{\pm} = \mathbb{P}^{\pm}\ ;\ \ \mathbb{P}^{\pm}\,\mathbb{P}^{\mp} = 0\,. \quad (3.143)$$

From the definition (3.139) and using Eq. s (3.138), (3.136) we find:

$$\mathbb{T}^{\pm} \equiv -\mathbb{L}_c^{\dagger}\,\mathbb{C}\,\mathbb{F}^{\pm} = -\mathbb{L}_c^{\dagger}\,\mathbb{C}\,\mathbb{P}^{\pm}\,\mathbb{F} = -\frac{1}{2}\mathbb{L}_c^{\dagger}\mathbb{C}\,\left(\mathbf{1} \pm i\,\mathbb{L}_c\mathbb{L}_c^{\dagger}\,\mathbb{C}\right)\mathbb{F} = -\frac{1}{2}\,\left(\mathbb{L}_c^{\dagger}\mathbb{C} \pm i\,\varpi\,\mathbb{L}_c^{\dagger}\,\mathbb{C}\right)\mathbb{F} =$$
$$= \frac{\mathbf{1} \pm i\,\varpi}{2}\left(-\mathbb{L}_c^{\dagger}\,\mathbb{C}\,\mathbb{F}\right) = \frac{\mathbf{1} \pm i\,\varpi}{2}\,\mathbb{T}\,. \quad (3.144)$$

Using the expression of ϖ we then find:

$$\mathbb{T}^+_{\mu\nu} = \frac{\mathbf{1} + i\,\varpi}{2}\,\mathbb{T} = \begin{pmatrix} T^{ij}_{\mu\nu} \\ T^{A}_{\mu\nu} \\ 0 \\ 0 \end{pmatrix} \quad ; \quad \mathbb{T}^-_{\mu\nu} = \frac{\mathbf{1} - i\,\varpi}{2}\,\mathbb{T} = \begin{pmatrix} 0 \\ 0 \\ T_{\mu\nu\,ij} \\ T_{\mu\nu\,A} \end{pmatrix}\,. \quad (3.145)$$

In other words we find:

$$T^+_{\mu\nu\,ij} = T^+_{\mu\nu\,A} = T^{ij\,-}_{\mu\nu} = T^{A\,-}_{\mu\nu} = 0 \ \Leftrightarrow\ T_{\mu\nu\,ij} = T^-_{\mu\nu\,ij}\ ;\ T_{\mu\nu\,A} = T^-_{\mu\nu}\,. \quad (3.146)$$

that is the upper or lower position if the complex H indices are related to the chirality of the fermion fields and to the (anti-) self-duality property of the field strengths. The reason why chiral spinors transform into anti-self-dual composite tensors $\mathbb{T}^-$ can be understood by noticing that:

$$T^{\pm}_{\rho\sigma}\gamma^{\rho\sigma} = T^{\pm}_{\rho\sigma}\gamma^{\rho\sigma}\,\frac{1}{2}(\mathbf{1} \mp \gamma^5)\,, \quad (3.147)$$

so that

$$T^+_{\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\epsilon^j = T^+_{\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\,\frac{1}{2}(\mathbf{1}+\gamma^5)\epsilon^j = 0\,. \tag{3.148}$$

Just as we did for the quantized charges, we define on a solution the *central* and *matter* charges as the following integrals over a sphere S^2_∞ at spatial infinity:

$$\mathscr{Z}_{ij}(\phi, e, m) \equiv \frac{1}{4\pi}\int_{S^2_\infty} T_{ij} = -L^M{}_{ij}(\phi)\;\mathbb{C}_{MN}\,\Gamma^N = L_{\Lambda\,ij}(\phi)\,m^\Lambda - L^\Lambda{}_{ij}(\phi)\,e_\Lambda\,, \tag{3.149}$$

$$\mathscr{Z}_{A}(\phi, e, m) \equiv \frac{1}{4\pi}\int_{S^2_\infty} T_{A} = -L^M{}_{A}(\phi)\;\mathbb{C}_{MN}\,\Gamma^N = L_{\Lambda\,A}(\phi)\,m^\Lambda - L^\Lambda{}_{A}(\phi)\;e_\Lambda\,, \tag{3.150}$$

where we assume that the scalar fields at spatial infinity are constant over S^2_∞. These can be thought of as the physical charges measured on a solution at radial infinity. Together with their complex conjugates, they can be arranged in a vector $\mathscr{Z}^{\underline{M}}$ in the complex symplectic basis

$$\mathscr{Z}(\phi, e, m) = (\mathscr{Z}^{\underline{M}}(\phi, e, m)) = \begin{pmatrix}\mathscr{Z}^{ij}\\ \mathscr{Z}^{A}\\ \mathscr{Z}_{ij}\\ \mathscr{Z}_{A}\end{pmatrix} = -\mathbb{L}_c^\dagger(\phi)\mathbb{C}\Gamma\,. \tag{3.151}$$

Just as $\mathbb{T}_{\mu\nu}$, this vector transforms under G through the compact compensator $h_c(\phi,\, g)$ in H:

$$\mathscr{Z}(g\star\phi; g\,\Gamma) = h_c(\phi,\, g)\;\mathscr{Z}(\phi;\Gamma)\,, \tag{3.152}$$

where we have written Γ instead of (e, m) and $g\,\Gamma$ instead of $\mathbf{S}[g]\,\Gamma$, for the sake of notational simplicity.

To make contact with our initial treatment of the supersymmetry algebra, we notice that $\mathscr{Z}_{ij}$ are topological charges associated with composite fields $T_{\mu\nu\,ij}$ entering the supersymmetry transformation rules (3.118) of the gravitino. These charges $\mathscr{Z}_{ij}$, $\mathscr{Z}_A$ are not carried by the elementary fields of the theory [38]. They are rather associated with non-trivial massive configurations of elementary fields which solve the field equations and are known as *solitons*. On these solitonic solutions $\mathscr{Z}_{ij}$ can be identified with the *central charges* Z_{ij} of the supersymmetry algebra (2.10) realized on the solution (see [11] in rigid-supersymmetric gauge theories). The precise relation is:[23][24]

$$\mathscr{Z}_{ij} = -i\,Z_{ij}\,. \tag{3.155}$$

[23] To see this, following [12], one should compute the supersymmetry generators Q_i on the solution as an integral over space of the (time component of the) conserved supersymmetry currents (i.e. the Noether currents associated with supersymmetry). Anticommuting them gives rise to *boundary terms* of the form (3.149) which, according to the general expression (2.10) can be identified with the central charges of the superalgebra. The precise identification is given below. See [11] for an earlier computation in spontaneously broken supersymmetric gauge theories.

[24] In this relation we have redefined the central charges also from a dimensional point of view. The central charge matrix Z_{ij} in the right hand side of (2.10) has dimension of a *length*$^{-1}$, while $\mathscr{Z}_{ij}$ has dimension of a charge. The actual relation is:

$$Z_{ij} = i\,\frac{1}{\ell_P^2}\frac{\sqrt{8\pi G}}{c^2}\,\mathscr{Z}_{ij} = i\,\frac{c}{\hbar}\sqrt{\frac{8\pi}{G}}\,\mathscr{Z}_{ij}\,. \tag{3.153}$$

The fact that the elementary fields do not carry $\mathscr{Z}_{ij}$ follows from the fact that they are massless and thus supersymmetry requires the central charges to vanish on them (see Eq. (2.139)). Solitons in supegravity are black holes to be discussed in the next section. The composite fields $T_{\mu\nu\, ij}$ are also called *graviphoton* field strengths since they are the objects actually sitting in the supergravity multiplet. Let us notice that, if we skew-diagonalize the central charge matrix $\mathscr{Z}_{ij}$ (or, equivalently Z_{ij}) by means of G_R-transformations in H, it will have the general form (2.127), (2.128):

$$\mathscr{Z}_{ij} = \mathscr{Z}_{(x,u),(y,v)} = \mathscr{Z}_u\, \epsilon_{xy}\delta_{uv}\ , \quad x, y = 1, 2, \quad u, v = 1, \dots, \left[\frac{\mathcal{N}}{2}\right]\ , \tag{3.156}$$

where the skew-eigenvalues $\mathscr{Z}_u = -i\, z_u$ are G_R-invariant. From (3.152) it follows that $\mathscr{Z}_u$, *as functions of* ϕ *and* $\Gamma = (e, m)$*, are* G*-invariant*:

$$\mathscr{Z}_u(g \star \phi; g\,\Gamma) = \mathscr{Z}_u(\phi; \Gamma)\,. \tag{3.157}$$

Since the vector $\mathscr{Z}^{\underline{M}}$ is an object transforming under H, we can compute the H-covariant derivative of it (just as we did for P in (3.47)) using the H-connection on the manifold ω:

$$\mathcal{D}^{(H)}\mathscr{Z} \equiv d\mathscr{Z} + \omega^c\, \mathscr{Z}\,. \tag{3.158}$$

Exercise: Prove that this is a covariant derivative, namely that:

$$\mathcal{D}^{(H)}\mathscr{Z}(g\star\phi; g\,\Gamma) = d\mathscr{Z}(g\star\phi; g\,\Gamma) + \omega^c(g\star\phi)\mathscr{Z}(g\star\phi; g\,\Gamma) = h_c(\phi, g)\,\mathcal{D}^{(H)}\mathscr{Z}(\phi; \Gamma)\,, \tag{3.159}$$

using the transformation property (3.46) of ω^c*:*

$$\omega^c(g \star \phi) = h_c\, \omega^c(\phi)\, h_c^{-1} + h_c dh_c^{-1} \tag{3.160}$$

We can express $\mathcal{D}^{(H)}\mathscr{Z}$ in terms of $\mathscr{Z}$ and of the complexified vielbein matrix P^c. We start from the definition (3.42) of P and ω in the complexified basis:

$$\mathbb{L}_c^{-1}d\mathbb{L}_c = P^c + \omega^c \;\Rightarrow\; d\mathbb{L}_c = \mathbb{L}_c\, P^c + \mathbb{L}_c\, \omega^c \;\Rightarrow\; d\mathbb{L}_c^\dagger = P^c\, \mathbb{L}_c^\dagger - \omega^c\, \mathbb{L}_c^\dagger\,. \tag{3.161}$$

From this we derive:

$$\mathcal{D}^{(H)}\mathscr{Z} = -(d + \omega^c)\mathbb{L}_c^\dagger\Gamma = -P^c\, \mathbb{L}_c^\dagger\Gamma = P^c\, \mathscr{Z}\,. \tag{3.162}$$

In components, using the matrix form (3.131), the above relation reads:

$$\mathcal{D}^{(H)}\mathscr{Z}_{ij} = \frac{1}{2}\, P_{ij\,kl}\, \mathscr{Z}^{kl} + P_{ij\,A}\, \mathscr{Z}^A\,, \tag{3.163}$$

$$\mathcal{D}^{(H)}\mathscr{Z}_A = \frac{1}{2}\, P_{A\,ij}\, \mathscr{Z}^{ij} + P_{A\,B}\, \mathscr{Z}^B\,. \tag{3.164}$$

For notational convenience, however, we shall use relation (3.155), remembering later, when making contact with our previous discussion on the Bogomolny bound for super-Poincaré representations, to make the replacement:

$$Z_{ij} \to \frac{\hbar}{c}\sqrt{\frac{G}{8\pi}}\, Z_{ij}\ ;\quad \mathscr{Z}_{ij} \to -i\,\frac{\hbar}{c}\sqrt{\frac{G}{8\pi}}\, Z_{ij}\,. \tag{3.154}$$

To understand why the physical charges are the "dressed" ones $\mathscr{Z}_{ij}$, $\mathscr{Z}_A$ instead of the quantized ones e_Λ, m^Λ, it is useful to refer to the higher dimensional origin of the $D = 4$ supergravity model. As mentioned in the introduction, four-dimensional supergravities can be interpreted as the effective theories describing superstring-M/theories compactified on suitable internal manifolds. This means that we consider the dynamics of the "microscopic" objects described by superstring-M/theories (string and extended objects called branes) propagating on a space-time of the form:

$$\mathcal{M}_4 \times \mathcal{M}_{int}\,, \tag{3.165}$$

where $\mathcal{M}_{int}$ is a compact internal manifold (which is six-dimensional in the case of superstring theory or seven-dimensional in the case of M-theory), such as a sphere, a torus or a space with more involved geometry. The smaller the volume of $\mathcal{M}_{int}$, the larger the energy required for propagating inside of it[25]: The modes describing propagation of the fields along the internal directions of $\mathcal{M}_{int}$ become energetically suppressed as the internal volume tends to zero. This is the Kaluza-Klein mechanism of dimensional reduction according to which the effective low-energy theory is a four-dimensional one describing the propagation of the lowest-lying superstring-M/theory modes only in four-dimensional space-time $\mathcal{M}_4$. In this setting the vector fields of the $D = 4$ theory originate from higher-order forms in the higher-dimensional parent theory which minimally couple to the microscopic extended objects in the spectrum of superstring-M/theories[26] and a four-dimensional (point-like) solution like a black hole results from a system of higher dimensional extended objects whose spatial extension in concealed in the effective $D = 4$ description since they extend over directions of the compact internal space (in order for the configuration to be stable, the extended objects are *wrapped* on unshrinkable cycles of $\mathcal{M}_{int}$). Such objects have quantized charges with respect to the fields they minimally couple to in ten or eleven dimensions, just as electric charge is quantized in four-dimensions. These are the e, m charges of the four-dimensional solution (typically a black hole). However, the extended objects are wrapped along cycles of the internal space and thus interact non-trivially with its geometry (besides interacting among themselves). The charges one would measure in $D = 4$ also depend on this interaction and thus depend not just on the intrinsic charges e, m, but also on those scalar fields which describe the *shape* and *size* of the internal cycle on which the microscopic objects are wrapped. In the definition of the dressed charges $\mathscr{Z}_{ij}$, $\mathscr{Z}_A$, this interaction of the extended objects with the geometry of the internal manifold and among themselves is taken into account. It is important to stress that the supergravity effective action is derived from superstring theory in the limit in which higher-order curvature terms are negligible and at order zero in the string coupling constant. The first condition requires the curvature to be small compared to $1/\ell_P^2$, $\ell_P \equiv \sqrt{\frac{G\hbar}{c^3}}$ being the Planck length, i.e. supergravity description fails in the vicinity of the black hole singularity.

[25]This is similar to the dependence of the energy of the normal modes of a vibrating string on the length of the string: the smaller the length the higher the energy of a same mode.

[26]Just as particles (i.e. objects with no spatial extension) minimally couple to 1-form potentials A_μ through their quantized electric charge, a string (i.e. an object with one spatial dimension in dimensions $D > 4$) minimally couples to a two-form field $B_{\mu\nu}$; a p-brane (i.e. an object with p-spatial dimensions) to a (p+1)-form field.

Therefore solutions which are well described within supergravity are those with large horizon area, outside of which the curvature can be small enough. These are called *large* black holes. On the other hand the superstring description favors a different limit, namely that in which (ten-dimensional) space-time is mainly flat and the extended (non-perturbative) building blocks of these microscopic constructions (the *D-branes*) are space-time defects. In this regime, very close to the branes the curvature explodes.

3.3 Black Holes in Supergravity

As a theory of gravity, supergravity has black hole solutions [13]. Seen as the effective low energy theory of superstring/M-theories suitably compactified on some internal manifold, supergravity provides a *macroscopic* (i.e. large scale) description of the solution, analogous to the thermodynamic description of gases, the microscopic description of the solution being provided by the higher dimensional superstring/M-theories. As mentioned in the previous subsection, a supergravity black hole can be realized in terms of a system of extended objects (which only extend over the internal space), belonging to the spectrum of superstring/M-theories, wrapping cycles of the internal manifold and intersecting among themselves. Just as the laws of thermodynamics can be derived from a molecular (i.e. microscopic) description of gases (kinetic theory of gasses), which also allows to interpret the entropy of the system in terms of the number of microscopic states realizing a same macroscopic one, the microscopic description of black holes provided by superstring/M-theories should explain, in principle, the laws of black hole thermodynamics and in particular account for the peculiar "area law" for the black hole entropy (see below), through a microscopic state-counting.

Let us briefly recall the main facts about black hole thermodynamics. The first exact solution to Einstein's field equations in the vacuum ($\mathcal{R}_{\mu\nu} = 0$) was found by in 1915 by Karl Schwarzschild. It describes space-time metric around a point particle of mass M, which has the form:

$$ds^2 = (1 - \frac{r_s}{r})\, dt^2 - \frac{dr^2}{(1 - \frac{r_s}{r})} - r^2\, (d\theta^2 + \sin^2(\theta)\, d\varphi^2)\,, \tag{3.166}$$

where $r_s \equiv 2\, G\, M/c^2$ is the *Schwarzschild radius.* Light from inside the sphere $r = r_s$ cannot escape from it to radial infinity. For this reason, this sphere is named *event horizon.* and the region inside it *black hole.* The horizon represents a *coordinate singularity*[27], it can be removed by a change in coordinates, while the point $r = 0$ is a true singularity (i.e. $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges). As long as the singularity is "hidden" by a event horizon, it does not pose a problem of predictability of events outside the black hole and the solution is perfectly acceptable. The Schwarzschild's solution is the most general spherically symmetric solution to the Einstein equations in the vacuum (Birkhoff, 1923).

Between 1916-1918, Reissner and Nordström found the spherically symmetric solution describing particle of mass M and charge Q' (here we express the charge in the rationalized

[27]This was proven by David Finkelstein in 1958.

Heaviside-Lorentz units):

$$ds^2 = \left(1 - \frac{2\,r_M}{r} + \frac{r_Q^2}{r^2}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2\,r_M}{r} + \frac{r_Q^2}{r^2}\right)} - r^2\,(d\theta^2 + \sin^2(\theta)\,d\varphi^2)\,, \tag{3.167}$$

where

$$r_M = \frac{G\,M}{c^2}\;;\;\; r_Q^2 = \frac{G}{4\pi c^4}\,Q'^2\,. \tag{3.168}$$

This solution has two horizons at $r_\pm = r_M \pm \sqrt{r_M^2 - r_Q^2}$ if $r_M > r_Q$ while is singular (the curvature singularity is not hidden inside a horizon) if $r_M < r_Q$.

In 1963 R. Kerr generalized Schwarzschild's solution to describe a spinning particle, further generalized by E. Newman in 1965 to describe a charged spinning particle (the Kerr-Newman solution). This represents the most general asymptotically flat, axisymmetric solution to Einstein's theory of gravity coupled to an electromagnetic field (Einstein-Maxwell theory)[28].

From a purely classical analysis of black holes in general relativity the following general properties were found [39]:

i) The surface gravity κ is uniform over the horizon;

ii) If a black hole absorbs a spinning, charged particle, its rest energy varies in the following relation to the variation of its horizon area A, angular momentum at the horizon J_H and charge Q:[29]

$$\delta M = \frac{\kappa\,\delta A}{8\pi G} + \frac{1}{c^2}\,\Omega_H\,\delta J_H + \Phi\,\delta Q\,, \tag{3.169}$$

Ω_H being the angular velocity at the horizon, Φ the electric potential and Q the electric charge;

iii) The total area of the black hole horizons in the universe can not decrease: $\delta A \geq 0$

iv) The solution with $\kappa = 0$ (*extremal solution*) can not be reached through a finite process.

There is a formal analogy between these properties and the zeroth, first, second and third laws of thermodynamics, provided we identify κ with the temperature and A with the entropy of the solution. That this is not just a formal analogy and that these are the actual laws of thermodynamics applied to a black hole solution was proven when Hawking discovered in 1974 [40] that black holes radiate and thus can be in thermal equilibrium with the surrounding radiation. Hawking's quantum analysis showed that black holes emit black-body radiation at a temperature:

$$T = \frac{\kappa\,\hbar}{2\pi\,k_B\,c}\,, \tag{3.170}$$

[28]For references see below in Subsect. 3.3.1 when we comment on the no-hair theorem.

[29]In the presence of scalar fields coupled to the solution, which is typical of supergravity black holes, a further term should be added, which depends on the *scalar charges* defined in terms of the radial derivatives of the scalar fields at spatial infinity.

where k_B is the Boltzmann constant. Property *ii*) is then the first law of thermodynamics and *iii*) the second law, provided we identify the entropy of the solution with:

$$S = \frac{k_B}{4\,\ell_P^2}\,A\,, \tag{3.171}$$

where $\ell_P \equiv \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length. This is the so called "area law" or Bekenstein-Hawking formula for the entropy [41].[30]

Explaining this formula from a microscopic point of view requires a microscopic description of black holes, i.e. a quantum theory of gravity, and is one of the most challenging problem in theoretical physics, besides being a testing ground for candidates for the quantum theory of gravity, as superstring theory is. One of the main successes of superstring theory has been indeed the derivation of the Bekenstein-Hawking formula (3.171) from a microstate counting. The first computation of this type was performed in Type IIB string theory by Strominger and Vafa [42]. They considered five-dimensional black holes originating from a system of a 1- and 5-branes (with a momentum along the overlapping dimension) suitably wrapped on an internal manifold. A considerable number of other computations generalizing this result followed.

The AdS/CFT duality conjecture put forward by Maldacena il 1998 [43], provided a new understanding of the "area law" (3.171). This duality, in its strongest version, is a statement that superstring theory realized on an anti-de -Sitter space-time solution is equivalent to a conformal field theory on the boundary of this space.[31] In other words the degrees of freedom of the theory on this background are localized on its boundary, namely on a space-time with one spatial dimension less. This *holographic principle* for gravity explains why, according to (3.171), the entropy, instead of scaling with a volume (as an extensive quantity should), actually scales as an area.

The microscopic description of a same supergravity solution is however not unique because the microscopic theory is not unique. The idea behind string/M-theory duality (not to be confused with the AdS/CFT mentioned above) is that these different constructions of a same supergravity solution are different descriptions of the *same* microscopic degrees of freedom. This correspondence between microscopic descriptions, which in general is non-perturbative in the string coupling constant, is realized at the level of low-energy effective supergravity in terms of global symmetries. They were conjectured in [33] to be described by the discrete group $G(\mathbb{Z})$ of the global symmetry group G of the classical theory (see discussion below Eq. (3.109)). If the black hole entropy S "counts" the number of microscopic degrees of freedom of a solution, it is reasonable to expect it not to depend on their description, namely to be $G(\mathbb{Z})$-invariant. In fact it is found in the known supergravity solutions to be even G-invariant as a function of the electric and magnetic charges and of the values of the scalar fields at infinity. In extremal black holes (i.e. solutions with vanishing Bekenstein-Hawking

[30] Here we restored all the $c, \hbar$ and G factors. In the sequel we set, as usual, $c = \hbar = 1 = 8\pi\, G$ and $k_B = 1/8\pi$.

[31] In its original form it stated the duality between Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory on the $D = 4$ boundary of the AdS_5 space.

temperature, either non-rotating or under-rotating, i.e. rotating without ergosphere) the entropy only depends on the quantized charges e, m and not on the scalar fields at infinity. This reflects a general property of these solutions known as *attractor mechanism.* Let us review the main facts about static, spherically symmetric and asymptotically flat black hole solutions in extended supergravities.

3.3.1 Spherically Symmetric, Asymptotically Flat Black Hole Solutions

Ansaz and equations. We shall now restrict our discussion to static, spherically symmetric and asymptotically flat black hole solutions. The general ansatz for the metric and scalar fields has the following form:

$$\begin{aligned} ds^2 &= a(r)^2\,dt^2 - a(r)^{-2}\,dr^2 - b(r)^2\,(d\theta^2 + \sin^2(\theta)\,d\varphi^2)\,, &(3.172)\\ \phi^s &= \phi^s(r)\,,\\ \text{fermions} &= 0\,. &(3.173)\end{aligned}$$

where $a(r)$, $b(r)$ are functions of the radial variable to be determined by the equations of motion. If we consider dyonic solution with quantized electric and magnetic charges $\Gamma^M \equiv (m^\Lambda,\, e_\Lambda)$, the reader can verify that the following expression for $\mathbb{F}^M_{\mu\nu}$

$$\mathbb{F} = \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Lambda\,\mu\nu}\end{pmatrix} \frac{dx^\mu \wedge dx^\nu}{2} = \frac{1}{b^2}\mathbb{C}\cdot\mathcal{M}(\phi)\,\Gamma\,dt\wedge dr + \Gamma\,\sin(\theta)\,d\theta\wedge d\varphi\,, \tag{3.174}$$

satisfies the Maxwell equations (3.104).

Exercise: Verify this. The first of (3.104) directly follows from the fact that the scalar fields are taken to depend only on r*. As for the second, use the property that, in our notations*

$$\begin{aligned} {}^*(dx^\mu\wedge dx^\nu) &= \frac{1}{2e}\epsilon^{\mu\nu\rho\sigma}\,dx_\rho\wedge dx_\sigma \;\Rightarrow\\ &\Rightarrow\;\; {}^*(dt\wedge dr) = -b^2\,\sin(\theta)\,d\theta\wedge d\varphi\;\;;\;\; {}^*(d\theta\wedge d\varphi) = \frac{1}{b^2\,\sin(\theta)}\,dt\wedge dr\,,\end{aligned} \tag{3.175}$$

where $e = b^2\,\sin(\theta)$.

Let us write now the scalar field equations (3.102). Using the ansatz (3.174) we can rewrite the right hand side of this equation in a more compact way:

$$\begin{aligned}\mathbb{F}^T_{\mu\nu}\partial_s\mathcal{M}\mathbb{F}^{\mu\nu} &= 2\,\mathbb{F}^T_{tr}\partial_s\mathcal{M}\mathbb{F}_{tr}g^{tt}g^{rr} + 2\,\mathbb{F}^T_{\theta\varphi}\partial_s\mathcal{M}\mathbb{F}_{\theta\varphi}g^{\theta\theta}g^{\varphi\varphi} = -\frac{2}{b^4}\,\Gamma^T\mathcal{M}\mathbb{C}^T\partial_s\mathcal{M}\mathbb{C}\mathcal{M}\Gamma +\\ &+\frac{2}{b^4}\,\Gamma^T\partial_s\mathcal{M}\Gamma = \frac{4}{b^4}\,\Gamma^T\partial_s\mathcal{M}\Gamma = -\frac{8}{b^4}\,\partial_s V_{BH}\,,\end{aligned} \tag{3.176}$$

where we have introduced a new quantity $V_{BH}(\phi,\Gamma)$ called *black hole effective potential* defined as:

$$V_{BH}(\phi,\Gamma) \equiv -\frac{1}{2}\,\Gamma^T\mathcal{M}(\phi)\Gamma > 0\,. \tag{3.177}$$

The scalar field equation then reads:

$$\left(a^2\, b^2\, \phi^{s\prime}\right)' + \tilde{\Gamma}^s{}_{uv}\, \phi^{u\prime}\phi^{v\prime} = \frac{1}{b^2}\, G^{su}\, \partial_u V_{BH}\,, \tag{3.178}$$

where we have denoted by a prime the derivation with respect to r: $f'(r) \equiv \frac{df}{dr}(r)$. It is useful at this point to introduce a new radial variable $\tau = \tau(r)$ defined by the condition:

$$\frac{d\tau}{dr} = \frac{1}{a^2 b^2}\,. \tag{3.179}$$

Using the short-hand notation $\dot{f}(\tau) \equiv \frac{df}{d\tau}(\tau)$, equation (3.178) acquires the simpler form:

$$\ddot{\phi}^s + \tilde{\Gamma}^s{}_{uv}\, \dot{\phi}^u \dot{\phi}^v = a^2\, G^{su}\, \partial_u V_{BH}\,, \tag{3.180}$$

describing the motion of an imaginary "particle" in the manifold $\mathscr{M}_{scal}$, subject to a potential V_{BH} (if $V_{BH} = const.$, the motion would be geodesic,i.e. describing a free imaginary particle moving on the scalar manifold).

Let us consider now the Einstein equations (3.89).

Exercise: Using the ansatze (3.172) for the metric, prove that the vierbein and the spin-connection have the following form:

$$\begin{aligned}
&V^0 = a\, dt\ ;\ V^1 = \frac{1}{a}\, dr\ ;\ V^2 = b\, d\theta\ ;\ V^3 = b\sin(\theta)\, d\varphi\,,\\
&\omega^0{}_1 = a'\, V^0\ ;\ \omega^1{}_2 = -\frac{b'}{b} a\, V^2\ ;\ \omega^1{}_3 = -\frac{b'}{b} a\, V^3\ ;\ \omega^2{}_3 = -\frac{\cotan(\theta)}{b}\, V^3\,.
\end{aligned} \tag{3.181}$$

Exercise: Using the ansatze (3.172) for the metric, verify that the non-vanishing components of the Ricci tensor are:

$$\begin{aligned}
&\mathcal{R}^t{}_t = \frac{(aa'b^2)'}{b^2}\ ;\ \mathcal{R}^r{}_r = (aa')' + 2\,\frac{a}{b}\,(ab')'\,,\\
&\mathcal{R}^\theta{}_\theta = \mathcal{R}^\varphi{}_\varphi = -\frac{1}{b^2}\left(1 - (a^2bb')'\right)\,.
\end{aligned} \tag{3.182}$$

Exercise: Verify the following equations:

$$\begin{aligned}
&\mathbb{F}^T_{tr}\mathcal{M}\,\mathbb{F}_t{}^r = 2\,\frac{a^2}{b^4}\, V_{BH}\ ;\ \mathbb{F}^T_{rt}\mathcal{M}\,\mathbb{F}_r{}^t = -2\,\frac{1}{a^2 b^4}\, V_{BH}\,,\\
&\mathbb{F}^T_{\theta\varphi}\mathcal{M}\,\mathbb{F}_\theta{}^\varphi = \frac{2}{b^2}\, V_{BH}\ ;\ \mathbb{F}^T_{\varphi\theta}\mathcal{M}\,\mathbb{F}_\varphi{}^\theta = 2\,\frac{\sin^2(\theta)}{b^2}\, V_{BH}\,.
\end{aligned} \tag{3.183}$$

Using (3.183) the Einstein equations read:

$$\mathcal{R}_{rr} = G_{uv}\, \phi^{u\prime}\phi^{v\prime} - \frac{1}{a^2 b^4}\, V_{BH}\ ;\ \mathcal{R}_{tt} = \frac{a^2}{b^4}\, V_{BH}\,, \tag{3.184}$$

$$\mathcal{R}_{\varphi\varphi} = \frac{\sin^2(\theta)}{b^2}\, V_{BH}\ ;\ \mathcal{R}_{\theta\theta} = \frac{1}{b^2}\, V_{BH}\,. \tag{3.185}$$

From the above equations we find:

$$\mathcal{R}^t{}_t = \frac{1}{a^2}\,\mathcal{R}_{tt} = \frac{1}{b^4}\,V_{BH} = \frac{1}{b^2}\,\mathcal{R}_{\theta\theta} = -\mathcal{R}^\theta{}_\theta\,. \tag{3.186}$$

Now use the expression of the components of the Ricci tensor in terms of the metric (3.182) to find

$$\mathcal{R}^t{}_t = -\mathcal{R}^\theta{}_\theta \;\Rightarrow\; \frac{(aa'b^2)'}{b^2} = \frac{1}{b^2}\left(1-(a^2bb')'\right) \;\Rightarrow\; (a^2b^2)'' = 2\,. \tag{3.187}$$

Last condition, which is implied on the ansatz by the Einstein equation, is solved in general by setting[32]:

$$a^2b^2 = (r-r_0)^2 - \mathbf{c}^2 = (r-r_+)(r-r_-) \quad ; \quad r_\pm \equiv r_0 \pm \mathbf{c}\,. \tag{3.188}$$

Here we have assumed $\mathbf{c}^2 \geq 0$. If $\mathbf{c}^2 < 0$ the two roots $r_\pm$ are imaginary. As we shall see $r_\pm$ can be identified with an inner and outer horizon, just as in the non-extremal Reissner-Nordström solution [13], and thus if $\mathbf{c}^2 < 0$ the solution has no horizon to hide its singularity and thus it is not regular.

Equation (3.179) then defines the "affine parameter" τ:

$$\frac{d\tau}{dr} = \frac{1}{a^2b^2} = \frac{1}{(r-r_0)^2-\mathbf{c}^2} \;\Rightarrow\; r - r_0 = -\mathbf{c}\,\coth(\mathbf{c}\tau) \;\Leftrightarrow\; \tau = \frac{1}{2\mathbf{c}}\,\log\left(\frac{r-r_+}{r-r_-}\right)\,. \tag{3.189}$$

The coordinate τ is non-positive and runs from $-\infty$ at $r = r_+$ (corresponding, as we shall see, to the outer horizon of the black hole) to $\tau = 0$ at radial infinity. We also find:

$$\frac{d\tau}{dr} = \frac{1}{(r-r_0)^2-\mathbf{c}^2} = \frac{\sinh^2(\mathbf{c}\tau)}{\mathbf{c}^2}\,. \tag{3.190}$$

We can now change notation and write both functions $a(r)$, $b(r)$ in terms of a single function $U(r)$ as follows:

$$a(r) = e^{U(r)}\;;\; b(r)^2 = e^{-2U(r)}\,(r-r_+)(r-r_-) = e^{-2U(r)}\,\frac{\mathbf{c}^2}{\sinh^2(\mathbf{c}\tau)}\,. \tag{3.191}$$

The metric (3.172) now reads:

$$ds^2 = e^{2U}\,dt^2 - e^{-2U}\,\left[dr^2 + (r-r_+)(r-r_-)\,d\Omega^2\right]\,, \tag{3.192}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2(\theta)\,d\varphi^2$. In terms of the new radial variable τ the metric has the following form:

$$ds^2 \;=\; e^{2U}\,dt^2 - e^{-2U}\,\left(\frac{\mathbf{c}^4}{\sinh^4(\mathbf{c}\tau)}\,d\tau^2 + \frac{\mathbf{c}^2}{\sinh^2(\mathbf{c}\tau)}\,d\Omega^2\right)\,, \tag{3.193}$$

[32]Below we introduce the integration constant $\mathbf{c}$ (*extremality parameter*) not to be confused with the speed of light c.

where $U=U(\tau)$. Notice that in the new radial coordinate the metric has the property that the combination

$$e\, g^{\tau\tau} = \left(e^{-2U}\,\frac{\mathbf{c}^4}{\sinh^4(\mathbf{c}\tau)}\,\sin(\theta)\right)\left(e^{2U}\,\frac{\sinh^4(\mathbf{c}\tau)}{\mathbf{c}^4}\right) = \sin(\theta)\,, \tag{3.194}$$

is independent of τ.

Using the property:

$$aa'b^2 = \frac{\dot{a}}{a} = \dot{U}\,, \tag{3.195}$$

from (3.186) we find:

$$(aa'b^2)' = \frac{1}{b^2}\,V_{BH} \;\Leftrightarrow\; \ddot{U} = e^{2U}\,V_{BH}\,. \tag{3.196}$$

It is convenient to recompute the entries of the Ricci tensor in the coordinates t, τ, θ, φ.

Exercise: Verify that, in the new radial coordinate, the Ricci tensor corresponding to the metric (3.193) has the following non-vanishing entries:

$$\mathcal{R}_{tt} = \frac{1}{b^4}\,\ddot{U}\;;\;\; \mathcal{R}_{\tau\tau} = 2\mathbf{c}^2 - 2\dot{U}^2 + \ddot{U}\;;\;\; \mathcal{R}_{\theta\theta} = \frac{1}{\sin^2(\theta)}\,\mathcal{R}_{\varphi\varphi} = \frac{1}{a^2b^2}\,\ddot{U}\,. \tag{3.197}$$

From the first of Eq.s (3.184), using the second of (3.197), we find

$$2\mathbf{c}^2 - 2\dot{U}^2 + \ddot{U} = G_{uv}\dot{\phi}^u\dot{\phi}^v - e^{2U}\,V_{BH} \;\Leftrightarrow\; \dot{U}^2 + \frac{1}{2}\,G_{uv}\dot{\phi}^u\dot{\phi}^v - e^{2U}\,V_{BH} = \mathbf{c}^2\,, \tag{3.198}$$

where we have used (3.196). There is no further independent equation implied by the Einstein equations.

To summarize the results so far, we have found that the most general ansatz for the static solution depends on n_s+1 independent functions of the radial variable τ: $U(\tau)$, $\phi^s(\tau)$. These are subject to the equations:

$$\ddot{U} = e^{2U}\,V_{BH}\,, \tag{3.199}$$

$$\ddot{\phi}^s + \tilde{\Gamma}^s{}_{uv}\,\dot{\phi}^u\dot{\phi}^v = e^{2U}\,G^{su}\,\partial_u V_{BH}\,, \tag{3.200}$$

$$\dot{U}^2 + \frac{1}{2}\,G_{uv}\dot{\phi}^u\dot{\phi}^v - e^{2U}\,V_{BH} = c^2\,. \tag{3.201}$$

A distinctive feature of black hole solutions in supergravity theories is therefore the presence of the scalar fields which participate in the solution due to their non-minimal coupling to the vector fields, which determines the effective potential $V_{BH}(\phi;\, e, m)$. The scalar fields which do not couple to the electric-magnetic charges of the solution, do not enter the effective potential and thus do not exhibit a radial evolution.

The first two equations (3.199), (3.200) can be derived from an effective action:

$$S_{eff} \;=\; \int \mathcal{L}_{eff}\,d\tau = \int\left(\dot{U}^2 + \frac{1}{2}\,G_{su}(\phi)\,\dot{\phi}^s\,\dot{\phi}^u + e^{2\,U}\,V_{BH}(\phi;\,\Gamma)\right)\,d\tau\,. \tag{3.202}$$

This action describes an *autonomous Lagrangian system* in which the role of the time variable is played by the radial one τ. The corresponding Hamiltonian $\mathcal{H}$ is "conserved" on a solution, where by conserved we refer to the dependence on the radial variable τ and not on time (!): $\frac{d\mathcal{H}}{d\tau} = 0$, i.e. $\mathcal{H} = const.$. The Hamiltonian constraint, expressed in terms of $U(\tau)$, $\phi^s(\tau)$ and their derivatives, is nothing but (3.201):

$$\mathcal{H} = \dot{U}^2 + \frac{1}{2} G_{su}(\phi)\, \dot{\phi}^s\, \dot{\phi}^u - e^{2U}\, V_{BH}(\phi; \Gamma) = \mathbf{c}^2\,. \tag{3.203}$$

In this description the integration constant $\mathbf{c}^2$ plays the role that the energy would play in an ordinary Hamiltonian system.

Let us now study the physical properties of the solution. The solution has a time-like killing vector $\xi^\mu \partial_\mu = \frac{\partial}{\partial t}$ and the ADM mass is given by the Komar integral [13] over the sphere S^2_∞ spanned by θ, φ at radial infinity ($\tau = 0$):

$$M_{ADM} = \frac{c^2}{8\pi\, G} \int_{S^2_\infty} e\, \epsilon_{\theta\varphi\mu\nu}\, \nabla^\mu \xi^\nu\, d\theta\, d\varphi\,. \tag{3.204}$$

As a simple exercise the reader can prove that, on our general solution:

$$M_{ADM} = \frac{c^2}{G} \lim_{\tau \to 0^-} \dot{U}\,. \tag{3.205}$$

Exercise: Prove this by first proving that:

$$\nabla_t \xi^\tau = \Gamma^\tau_{tt} = e^{4U}\, \frac{\sinh^4(\mathbf{c}\tau)}{\mathbf{c}^4}\, \dot{U}\ ;\ \ \nabla_\tau \xi^t = \Gamma^t_{\tau t} = \dot{U}\,. \tag{3.206}$$

The solution is defined by the boundary conditions of the fields at radial infinity $\tau = 0$:

$$U(0) = 0\ ;\ \ \dot{U}(0) = \frac{G}{c^2}\, M_{ADM}\ ;\ \ \phi^s(0) = \phi^s_0\ ;\ \ \dot{\phi}^s(0) = \dot{\phi}^s_0\,, \tag{3.207}$$

the boundary conditions on the vector fields being already fixed by the electric and magnetic charges e, m. The first condition $U(0) = 0$ just expresses the requirement of asymptotic flatness of the metric.

We can write the constraint (3.203) at radial infinity, restoring the constants[33], in terms of the boundary data:

$$\frac{G^2}{c^4}\, M^2_{ADM} + \frac{1}{2} G_{su}(\phi_0)\, \dot{\phi}^s_0\, \dot{\phi}^u - \frac{8\pi\, G}{c^4}\, V_{BH}(\phi_0; \Gamma) = \mathbf{c}^2\,. \tag{3.208}$$

Regularity of the solution implies the existence of the two horizons $r_\pm$ (which may coincide) and this in turn requires $\mathbf{c}^2 \geq 0$ and a corresponding condition on the boundary data, according to (3.208).

[33] All terms in the constraint equation have dimension of a squared length. Since the scalar potential has dimension of a squared charge (in the Heaviside-Lorentz units), when restoring the constants we need to replace $V_{BH} \to \frac{8\pi\, G}{c^4}\, V_{BH}$.

No scalar hair. In all known black hole solutions the radial derivatives of the scalar fields $\dot{\phi}_0^s$ (which we shall also refer to as *scalar charges*) at infinity are not independent boundary data but are expressed in terms of the other quantities at infinity, namely the ADM mass, the electric and magnetic charges and the values ϕ_0^s of the scalar fields. A way to understand this dependence of $\dot{\phi}_0^s$ on the other boundary data is to recall that in this class of solutions the radial evolution of the scalar fields is only due to their non-minimal coupling to the electric-magnetic charges. In other words they are "dragged along" with the solution by the vector fields and have no independent dynamics.

Although there is no general proof of this feature in the context of supergravity theories, it seems to indicate that the most general static black hole solution is completely determined by its electric and magnetic charges, and its ADM mass (for stationary solutions we should also include the angular momentum)[34]. This would represent a generalization to supergravity black holes of the known "no-hair" theorem for ordinary black holes in general relativity [44]. This theorem stated that the most general, asymptotically flat, axisymmetric black hole in the Einstein-Maxwell theory is the Kerr-Newman solution [45], which is totally defined by its mass, electric, magnetic charges and angular momentum. This means that if a system of charged matter collapses into a black hole, any other physical property (hair) like multipole moments, baryon or lepton numbers etc. simply disappear. Let us stress, however, that a general proof of an analogous theorem for the scalar coupled supergravity black holes is still missing. Nevertheless, for black holes solutions in extended models with homogeneous-symmetric scalar manifold there is a general argument in favor of this conclusion, which makes use of an effective three dimensional description of the solution in which a larger global symmetry connecting stationary solutions of the $D = 4$ theory is manifest [37]. We shall not deal with it here.

The fact that on a black hole solution, once the electric-magnetic charges and the ADM mass are fixed, the radial evolution of the scalar fields is completely determined by their boundary values alone ϕ_0^s, suggests that for the scalar fields there exists an effective description in terms of a system of first order differential equations. This seems indeed to be a general feature and we shall explicitly work out this system for the BPS solutions (namely for the black holes preserving an amount of supersymmetry).

Near-horizon behavior. The two zeros $r_\pm = r_0 \pm c$ of the metric (3.192) are coordinate singularities representing an inner and outer horizons (just as in Reissner-Nordström solution (RN) [13]). To see this let us require the 2-sphere S^2 to have a finite area $A = 4\pi\, r_H^2$ as $r \to r_H = r_+$

$$A = \lim_{\tau \to -\infty} \int_{S_2} \sqrt{g_{\theta\theta} g_{\varphi\varphi}}\, d\theta\, d\varphi = \lim_{\tau \to -\infty} 4\pi\, e^{-2U}\, \frac{\mathbf{c}^2}{\sinh^2(\mathbf{c}\tau)}\,. \tag{3.209}$$

[34]Here we are just considering the physical quantities related to the radial derivatives of the fields at infinity. The boundary values of the scalar fields do not have a physical meaning in an ungauged supergravity.

Requiring this area to be finite, $A > 0$, implies for the warp factor e^U the following behavior for $r \to r_H = r_+$:

$$e^{-2U} \sim \frac{A_H}{4\pi} \frac{\sinh^2(\mathbf{c}\tau)}{\mathbf{c}^2} = \frac{r_H^2}{(r - r_+)(r - r_-)} \,. \tag{3.210}$$

Near $r = r_+$ the metric then reads:

$$ds^2 = \frac{(r - r_+)(r - r_-)}{r_H^2}\, dt^2 - \frac{r_H^2}{(r - r_+)(r - r_-)}\, dr^2 - r_H^2\, d\Omega^2 \,, \tag{3.211}$$

Notice that this is the near-horizon geometry of a non-extremal Reissner-Nordström solution. This justifies our identification of $r_\pm$ with the outer and inner horizons of the solution and the condition $\mathbf{c}^2 \geq 0$ as the regularity condition which implies the existence of these horizons.

We also require the scalar fields to have a regular behavior at the horizon. To this end we define the *physical distance* ρ from the horizon by the equation:

$$d\rho^2 = e^{-2U}\, dr^2 \,, \tag{3.212}$$

and require the scalar fields, as functions of ρ to run to finite values at the horizon (located at $\rho = \rho_H$):

$$\lim_{\rho \to \rho_H} \phi^s(\rho) = \phi^s_* \;\;,\;\; |\phi^s_*| < \infty \,, \tag{3.213}$$

we shall comment below on the implications of this condition.

From the behavior of the general solution in the near-horizon region, we can deduce the thermodynamic quantities like the temperature and the entropy. The temperature is given by (3.170) in terms of the surface gravity κ.

Exercise: Compute the surface gravity of the solution and verify that (restoring all constants):

$$\kappa = \frac{c^2\, \mathbf{c}}{r_H^2} \,. \tag{3.214}$$

Hint: Use the general formula [13]:

$$\kappa^2 = -\frac{c^4}{2} \nabla^\mu \xi^\nu \nabla_\mu \xi_\nu \,, \tag{3.215}$$

to prove, using Eq. (3.206), that

$$\kappa = c^2 \lim_{\tau \to -\infty} e^{2U} \frac{\sinh^2(\mathbf{c}\tau)}{\mathbf{c}^2} \dot{U} \,. \tag{3.216}$$

From the near-horizon behavior (3.210), (3.214) follows.

The temperature and the entropy then read:

$$T = \frac{\hbar}{2\pi k_B} \frac{c\,\mathbf{c}}{r_H^2} \;\;;\;\; S = \frac{k_B\, A\, c^3}{4\, G\, \hbar} \,, \tag{3.217}$$

so that we can identify:

$$\mathbf{c} = \frac{2\,G\,S\,T}{c^4}\,. \tag{3.218}$$

The constant $\mathbf{c}$ is the *extremality parameter*, it is zero if and only if the temperature is zero, namely when the solution is *extremal*. This is the case of the extremal RN solution in which the two horizons coincide: $r_+ = r_-$.

In order to have a better grip on the equations (3.199)-(3.201) and their solutions, let us look for a known solution: the Reissner-Nordström one. It can be shown that in a supergravity model, charges can be chosen ($e = Q$, $m = 0$, Q being the only non vanishing entry of e_Λ) so that the solution is electrically charged and at the origin of the scalar manifold ($\phi^s \equiv 0$) the derivatives of the potential $V_{BH}(\phi;\, e, m)$ vanish. It follows that $\phi^s(\tau) = 0$ all over space solves (3.200). Let us denote, in our units, by $Q^2/2$ the constant value of $V_{BH}(0;\, Q, 0)$, . The reader can verify that:

$$a^2 = e^{2U} = \left(1 - \frac{2\,r_M}{r} + \frac{r_Q^2}{r^2}\right) = \frac{(r - r_+)(r - r_-)}{r^2}\ ,\ \ b^2 = r^2\,, \tag{3.219}$$

where, restoring the constants[35]:

$$r_M = \frac{G\,M_{ADM}}{c^2}\ ;\ \ r_Q^2 = \frac{4\pi\,G}{c^4}\,Q^2 = \frac{8\pi\,G}{c^4}\,V_{BH}(0;\,Q,0)\,. \tag{3.220}$$

$$r_\pm = r_M \pm \sqrt{r_M^2 - r_Q^2}\,, \tag{3.221}$$

satisfies (3.199) (check it in the form of the first of Eq.s (3.196)).

Exercise: Check that this solution satisfies Eq. (3.201).

This is the RN solution with extremality parameter:

$$\mathbf{c} = \sqrt{r_M^2 - r_Q^2}\,, \tag{3.222}$$

and ADM mass (restoring the constants):

$$M_{ADM} = \frac{c^2 r_M}{G}\,. \tag{3.223}$$

The two lengths $r_\pm$ are the inner and outer horizons. Regularity requires

$$\mathbf{c}^2 \geq 0 \ \ \Leftrightarrow\ \ \ r_M \geq r_Q\ \ \Leftrightarrow\ \ M_{ADM} \geq \sqrt{\frac{4\pi}{G}}\,|Q| = \sqrt{\frac{8\pi}{G}\,V_{BH}(0)}\,. \tag{3.224}$$

This bound is saturated for the extremal solution whose temperature is zero.

[35] Recall that our charges are those in the rationalized-Heaviside-Lorentz units divided by 4π and those in the non-rationalized-Heaviside-Lorentz units divided by $\sqrt{4\pi}$.

Extremal solutions and the attractor mechanism. Consider now extremal solutions defined by the property $\mathbf{c}=0$. If we send $\mathbf{c}\to 0$, from (3.189) we find:

$$\tau = -1/r\,, \tag{3.225}$$

where we have redefined $r-r_0\to r$. The horizon is located at $r=0$, corresponding to $\tau\to-\infty$. The near-horizon behavior of the warp function U can be deduced from Eq. (3.210):

$$e^{-2U}\sim \lim_{\mathbf{c}\to 0} r_H^2\,\frac{\sinh^2(\mathbf{c}\tau)}{\mathbf{c}^2} = r_H^2\tau^2 \;\;\Rightarrow\;\; e^{-U}\sim -\tau r_H\,. \tag{3.226}$$

The physical distance ρ from the horizon is then defined by the condition (3.212):

$$d\rho = e^{-U}\,dr = \lim_{\mathbf{c}\to 0} e^{-U}\,\mathbf{c}^2\frac{d\tau}{\sinh^2(\mathbf{c}\tau)} = e^{-U}\frac{d\tau}{\tau^2}\sim -r_H\,\frac{d\tau}{\tau}\,, \tag{3.227}$$

from which we find:

$$\rho = -r_H\log(-\tau)\,. \tag{3.228}$$

The horizon is located at $\rho_H=-\infty$. Requiring regularity of the scalar fields at the horizon implies then:

$$\lim_{\rho\to-\infty}\phi^s(\rho) = \phi^s_*\;\;,\quad |\phi^s_*|<\infty\,. \tag{3.229}$$

This in turn implies that al the derivatives of the scalar fields with respect to ρ vanish in this limit:

$$\lim_{\rho\to-\infty}\frac{d^k}{d\rho^k}\phi(\rho) = 0\,. \tag{3.230}$$

This in particular implies, for $k=1$ and 2 that:

$$\lim_{\tau\to-\infty}\tau\,\dot\phi^s = \lim_{\tau\to-\infty}\tau^2\,\ddot\phi^s = 0\,. \tag{3.231}$$

Let us now consider the equations for the scalar fields (3.200) near the horizon:

$$\ddot\phi^s + \tilde\Gamma^s{}_{uv}\,\dot\phi^u\dot\phi^v = \frac{1}{r_H^2\tau^2}\,G^{su}\,\partial_u V_{BH} \;\Leftrightarrow\; \tau^2\ddot\phi^s + \tilde\Gamma^s{}_{uv}\,(\tau\dot\phi^u)(\tau\dot\phi^v) \;=\; \frac{1}{r_H^2}\,G^{su}\,\partial_u V_{BH}\,. \tag{3.232}$$

Taking the horizon limit of both sides and using (3.231), the left hand side vanishes, so that we have:

$$\lim_{\phi^s\to\phi^s_*}\partial_u V_{BH} = \partial_s V_{BH}(\phi_*;\,e,m) = 0\,. \tag{3.233}$$

We find that in going from radial infinity to the horizon of an extremal static black hole, the scalar fields flow toward values ϕ^s_* which define an extremum of the potential. In general V_{BH} may not depend on all the scalar fields, but have *flat directions*, which correspond to scalar fields which are not effectively coupled to the solution. Eq. (3.233) will then only fix those scalars along the non-flat directions as functions of the electric and magnetic charges only

$$\phi^s_* = \phi^s_*(e,m)\,. \tag{3.234}$$

As a consequence, the value of V_{BH} at the extremum ϕ^s_* will only depend on the electric and magnetic charges: $V_{ex} = V_{BH}(\phi_*;\, e, m) = V_{ex}(e, m)$.

If now we evaluate Eq. (3.199) near the horizon, we find:

$$\frac{1}{\tau^2} = \ddot{U} = e^{2U}\, V_{ex} = \frac{1}{r_H^2\, \tau^2}\, V_{ex} \;\Rightarrow\; V_{ex} = r_H^2\,. \tag{3.235}$$

In other words the area of the horizon can be expressed trough $V_{ex}(e, m)$ in terms of the electric and magnetic charges only:[36]

$$A = 4\pi\, V_{ex}(e, m) = A(e, m)\,. \tag{3.236}$$

The near horizon metric can be easily computed from (3.211) and reads:

$$ds^2 = \frac{r^2}{r_H^2}\, dt^2 - \frac{r_H^2}{r^2}\, dr^2 - r_H^2\, d\Omega^2\,. \tag{3.237}$$

It describes an $AdS_2 \times S^2$ space (Bertotti-Robinson solution) whose geometry only depends on the area A of the horizon S_2. It therefore *only depends on the quantized charges of the solution* and not on the boundary values $\phi_0 \equiv (\phi^r_0)$ of the scalar fields. This is the essence of the *attractor mechanism* [46]: The scalars along the non-flat directions of the potential V (namely which are non-trivially coupled to the black hole) flow from their values at radial infinity ϕ_0 towards fixed values at the horizon ϕ_*, solution to eq. (3.233) and only depending on the quantized charges. Notice that the extremal black holes interpolate between two vacua of the ungauged $\mathcal{N}$-extended supergravity: Minkowski space-time and $AdS_2 \times S^2$:

$$\text{Minkowski at radial infinity} \;\longleftrightarrow\; AdS_2 \times S^2 \text{ at the horizon}\,. \tag{3.238}$$

This is analogous to the general feature of solitonic solutions in field theory of interpolating between different vacua. In this sense extremal black hole solutions can be regarded as proper *solitons* of the ungauged supergravities.

If we consider extremal dyonic black holes, for a given set of charges e, m, we can always find boundary conditions on the scalar fields for which the scalar fields are constant all over space. It suffices to take:

$$\phi^s(\tau = 0) = \phi^s_*\,. \tag{3.239}$$

In this case, being

$$\partial_s V_{BH}(\phi_*;\, e, m) = 0\,, \tag{3.240}$$

the scalar field equations are solved by $\phi^s(\tau) \equiv \phi^s_*$. Such solutions are called *double extremal*. Being V_{BH} a constant $V_{BH}(\phi_*(e, m);\, e, m) = V_{ex}(e, m)$, the equation for U is easily integrated as in (3.219) and we find an *extremal Reissner Nordström solution* with

$$r_M = \frac{G\, M_{ADM}}{c^2}\;\;;\;\; r_Q^2 = \frac{8\pi\, G}{c^4}\, V_{ex}(e, m)\,, \tag{3.241}$$

[36] Restoring the constants and recalling that V has dimension of a charge squared, we would write:

$$A = 4\pi\left(\frac{8\pi G}{c^4}\, V_{ex}(e, m)\right)\,.$$

and

$$\mathbf{c}^2 = 0 \;\;\Leftrightarrow\;\; r_M = r_Q \;\;\Leftrightarrow\;\; M_{ADM} = \sqrt{\frac{8\pi}{G}\,V_{ex}(e,m)} = M_{ADM}(e,m)\,. \tag{3.242}$$

If we repeat this analysis for the non-extremal case, we find that ρ has the following form:

$$d\rho = -\frac{\mathbf{c}}{\sinh(\mathbf{c}\tau)}\,d\tau \sim 2\mathbf{c}\,e^{\mathbf{c}\tau}\,d\tau \;\;\Rightarrow\;\; \rho(\tau) = 2\,e^{\mathbf{c}\tau}\,. \tag{3.243}$$

Now the horizon is located at $\rho_H = 0$ and thus the regularity condition on the scalar fields:

$$\lim_{\rho\to 0}\phi^s(\rho) = \phi^s_* \;\;,\;\; |\phi^s_*| < \infty\,, \tag{3.244}$$

no longer implies the vanishing (3.230) of the derivatives of ϕ^s with respect to ρ. In particular equation (3.200) no longer implies that ϕ^s_* be an extremum, for the potential.

Exercise: Prove that near the horizon (3.200) becomes:

$$\frac{d^2\phi^s}{d\rho^2} + \frac{1}{\rho}\frac{d\phi^s}{d\rho} + \tilde{\Gamma}^s_{uv}\,\frac{d\phi^u}{d\rho}\frac{d\phi^v}{d\rho} = \frac{4\pi}{A}\,G^{su}\partial_u V_{BH}\,, \tag{3.245}$$

using the property

$$e^{2U} \sim \mathbf{c}^2\rho^2/r_H^2\,. \tag{3.246}$$

Expanding ϕ^s in Taylor series about $\rho = 0$ find that at the horizon

$$\lim_{\rho\to 0}\frac{d\phi^s}{d\rho} = 0\,, \tag{3.247}$$

while the second derivative is given in terms of the gradient of the potential at the origin, which therefore need not be zero.

From the the definition (3.243) of ρ, the property (3.247) and the near horizon behavior (3.246) of U we find in the non-extremal case:

$$\lim_{\rho\to 0} e^{-U}\dot{\phi}^s = \lim_{\rho\to 0} e^{-U}\,c\rho\frac{d\phi^s}{d\rho} = \lim_{\rho\to 0} r_H\frac{d\phi^s}{d\rho} = 0\,. \tag{3.248}$$

From this equation and from (3.231) we conclude that in both the extremal and non-extremal cases:

$$\lim_{\tau\to -\infty} e^{-U}\dot{\phi}^s = 0\,. \tag{3.249}$$

3.3.2 BPS-Solutions

In this subsection we shall focus on black hole solutions preserving a fraction of supersymmetries. Since black holes are bosonic backgrounds, this happens, see Eq.s (2.170) if the supersymmetry variations of the fermionic fields vanish on the solution along certain directions in the supersymmetry parameter space. As we did at the end of Sect. 2.3, we split the supersymmetry index i into the pair $i=(x,u)$ (do not mistake in this paragraph the index u with the one labeling the scalar fields), where $x=1,2$ and $u=1,\dots,[\mathcal{N}/2]$. Suppose the solution preserves one out of $\mathcal{N}$ supersymmetries. The corresponding (Killing spinor) parameter ϵ_i, once we transform the supersymmetry generators to the basis in which Z_{ij} is skew-diagonal, is defined by the condition (2.166):

$$\mathcal{S}^{(+)}_{(x,1),(y,v)}\,\epsilon_{(y,v)} = \epsilon_{(x,1)} + i\,\zeta_a\,\gamma^a\,\frac{\mathbb{Z}_1}{|z_1|}\,\epsilon_{xy}\epsilon_{(y,1)} = 0\ ,\quad u=1,\dots,q\,, \tag{3.250}$$

$$\epsilon_{(x,u)} = 0\ ,\quad u=2,\dots,\mathcal{N}/2\,, \tag{3.251}$$

where we have written:

$$\mathscr{Z}_{(x,u)(y,v)} = \mathscr{Z}_u\epsilon_{xy}\delta_{uv} = -i\,z_u\epsilon_{xy}\delta_{uv}\,.$$

Before evaluating the Killing spinor equations, let us compute the expression of the graviphoton field strength $T_{\mu\nu\,ij}$ on the solution in terms of the central charges. From (3.142) and (3.174) we find:

$$\begin{aligned}\mathbb{F}^{\pm} &= \mathbb{P}^{\pm}\,\mathbb{F} = \frac{1}{2}(\mathbf{1}\mp i\,\mathbb{C}\mathcal{M})\,\left[e^{2U}\,\mathbb{C}\,\mathcal{M}(\phi)\,\Gamma\,dt\wedge d\tau + \Gamma\,\sin(\theta)\,d\theta\wedge d\varphi\,,\right] = \\ &= \frac{1}{2}(\mathbf{1}\mp i\,\mathbb{C}\mathcal{M})\,\Gamma\,\left[\pm i\,e^{2U}\,dt\wedge d\tau + \sin(\theta)\,d\theta\wedge d\varphi\right] = \mathbb{P}^{\pm}\Gamma\,E^{\pm}\,,\end{aligned} \tag{3.252}$$

where:

$$E^{\pm} = \pm i\,e^{2U}\,dt\wedge d\tau + \sin(\theta)\,d\theta\wedge d\varphi\,. \tag{3.253}$$

Exercise: Verify that $^*E^{\pm} = \mp i\,E^{\pm}$.

Next we compute $\mathbb{T}^{\pm}$ from (3.144) and (3.145):

$$\mathbb{T}^{\pm} = -\mathbb{L}_c^{\dagger}\mathbb{C}\,\mathbb{F}^{\pm} = -\mathbb{L}_c^{\dagger}\mathbb{C}\,\mathbb{P}^{\pm}\Gamma\,E^{\pm} = \frac{1}{2}(\mathbf{1}\pm i\,\varpi)\,\mathscr{Z}\,E^{\pm}\,. \tag{3.254}$$

In particular we have:

$$T^{-}_{\mu\nu\,ij} = \mathscr{Z}_{ij}\,E^{-}_{\mu\nu}\ ;\quad T^{-}_{\mu\nu\,A} = \mathscr{Z}_A\,E^{-}_{\mu\nu}\,. \tag{3.255}$$

From the the gravitino (3.118) we derive one of the Killing spinor equations:

$$\delta\Psi_{\mu\,i} = \mathcal{D}_\mu\epsilon_i - \frac{1}{8}\,T^{-}_{\rho\sigma\,ij}\,\gamma^{\rho\sigma}\gamma_\mu\epsilon^j = \mathcal{D}_\mu\epsilon_i - \frac{1}{8}\,\mathscr{Z}_{ij}\,E_{\rho\sigma}\,\gamma^{\rho\sigma}\gamma_\mu\epsilon^j = 0\,, \tag{3.256}$$

where ϵ_i is subject to the conditions (3.251). Let us work out from (3.256) the corresponding conditions on the background fields.

We start evaluating $E_{\rho\sigma}\,\gamma^{\rho\sigma}$ on the solution:

$$E_{\rho\sigma}\,\gamma^{\rho\sigma} = 2\,(E_{t\tau}\,\gamma^{t\tau} + E_{\theta\varphi}\,\gamma^{\theta\varphi})\,. \tag{3.257}$$

Writing the metric in the τ radial variable:

$$ds^2 = a^2\,dt^2 - a^2\,b^4\,d\tau^2 - b^2\,d\Omega_2 \ \Rightarrow\ \ e = a^2\,b^4\,\sin(\theta)\,, \tag{3.258}$$

and using (A.47), we find:

$$\gamma_{\theta\varphi} = -i\, e\, \epsilon_{\theta\varphi t\tau}\, \gamma^{t\tau}\gamma^5 = -i\, a^2\, b^4\, \sin(\theta)\, \gamma^{t\tau}\gamma^5 \;\Rightarrow\; \gamma^{\theta\varphi} = -i\, \frac{a^2}{\sin(\theta)}\, \gamma^{t\tau}\gamma^5\,. \tag{3.259}$$

Substituting in (3.257) we have:

$$E_{\rho\sigma}\, \gamma^{\rho\sigma} = -2i\, a^2\, \gamma^{t\tau}(\mathbf{1}+\gamma^5) = -\frac{2i}{b^2}\, \gamma^{01}(\mathbf{1}+\gamma^5)\,, \tag{3.260}$$

where we have used: $\gamma^{01} = V_t{}^0 V_\tau{}^1\, \gamma^{t\tau} = a^2\, b^2\, \gamma^{t\tau}$. To evaluate the right hand side of (3.256) we also need to compute $\mathcal{D}_\mu \epsilon_i$. Let us use the following ansatz for the Killing spinor:

$$\epsilon_i = \epsilon_i(\tau) = f(\tau)\, \zeta_i\,, \tag{3.261}$$

where ζ_i are constant spinors subject to (3.251), so that:

$$\mathcal{D}_\mu \epsilon_i = \partial_\mu f\, \zeta_i + \frac{1}{4}\, \omega_{\mu\, ab}\gamma^{ab}\, f\, \zeta_i\,. \tag{3.262}$$

Let us evaluate the $\mu = t$ component of (3.256):

$$\begin{aligned}\frac{1}{f(\tau)}\, \delta\Psi_{t\, i} &= \frac{1}{4}\, \omega_{t\, ab}\gamma^{ab}\, \zeta_i - \frac{1}{8}\, \mathscr{Z}_{ij}\, E_{\rho\sigma}\, \gamma^{\rho\sigma}\gamma_\mu \zeta^j = \frac{1}{2}\, \omega_{t\, 01}\gamma^{01}\zeta_i + \frac{i\, a}{4b^2}\, \mathscr{Z}_{ij}\, \gamma^{01}\gamma_0(\mathbf{1}-\gamma^5)\zeta^j = \\ &= \frac{1}{2}\, \omega_{t\, 01}\gamma^{01}\zeta_i + \frac{i\, a}{2b^2}\, \mathscr{Z}_{ij}\, \gamma^{01}\gamma_0\zeta^j = \frac{1}{2}\, \frac{\dot{a}}{a\, b^2}\gamma^{01}\zeta_i + \frac{i\, a}{2b^2}\, \mathscr{Z}_{ij}\, \gamma^{01}\gamma_0\zeta^j\,. \end{aligned} \tag{3.263}$$

where we have used $\omega_{t\, 01} = a' a = \frac{\dot{a}}{a\, b^2}$. Now we write the first (3.251) for $u = 1$ in Weyl spinors:

$$\epsilon_{(x,1)} + i\, \gamma^0\, \frac{\mathbb{Z}_1}{|z_1|}\, \epsilon_{xy}\epsilon^{(y,1)} = 0\,, \tag{3.264}$$

Recall that $\mathbb{Z}_{ij} \equiv i\, \gamma^5 R_{ij} + I_{ij}$ and $Z_{ij} = R_{ij} + i\, I_{ij} = i\, \mathscr{Z}_{ij}$, so that:

$$\mathbb{Z}_{ij}\epsilon^i = (-i\, R_{ij} + I_{ij})\epsilon^j = -i\, Z_{ij}\epsilon^j = \mathscr{Z}_{ij}\, \epsilon^j\,. \tag{3.265}$$

Equation (3.264) then becomes:

$$0 = \epsilon_{(x,1)} + i\, \gamma^0\, \frac{\mathscr{Z}_1}{|z_1|}\, \epsilon_{xy}\epsilon^{(y,1)} = 0 \;\Rightarrow\; i\, \gamma^0\, \frac{\mathscr{Z}_1}{|z_1|}\, \epsilon_{xy}\epsilon^{(y,1)} = -\epsilon_{(x,1)}\,. \tag{3.266}$$

This condition now allows us to make the last two terms in the last line of (3.263) proportional to the same spinor. To see this write in (3.263) $i = (x, u)$, with $u = 1$:

$$\frac{1}{f(\tau)}\, \delta\Psi_{t\, (x,1)} = \frac{1}{2}\, \frac{\dot{a}}{a\, b^2}\gamma^{01}\zeta_{(x,1)} + \frac{i\, a}{2b^2}\, \mathscr{Z}_1\, \gamma^{01}\gamma_0\epsilon_{xy}\zeta^{(y,1)} = \frac{1}{2b^2}\, \left(\frac{\dot{a}}{a} - a|z_1|\right)\gamma^{01}\zeta_{(x,1)}\,, \tag{3.267}$$

which implies the following first order equation in the warp function $U(\tau)$:[37]

$$\dot{U} = e^U\,|z_1|\,, \tag{3.268}$$

recall that $|z_u|$ are field and charge-dependent:

$$z_u = z_u(\phi(\tau); e, m)\,. \tag{3.269}$$

The component $\mu = \tau$ of $\delta\Psi_\mu$ implies a differential equation for $f(\tau)$, while the other components imply no other condition.

Computing (3.268) at radial infinity, restoring the constants and recalling that $\dot{U}(\tau = 0) = \frac{G}{c^2}\,M_{ADM},\ U(\tau = 0) = 0$, one finds:[38]

$$M_{ADM} = \sqrt{\frac{8\pi}{G}}\,|z_1|_\infty\,, \tag{3.271}$$

which is nothing but the saturation of the bound (2.147) on the mass of the black hole. Here we have just required the preservation of one supersymmetry out of the $\mathcal{N}$ and found (3.268) as a necessary condition. If more supersymmetries were preserved then (3.268) would still hold, but $|z_1| = \cdots = |z_q| > |z_{q+1}| > \ldots$.

Let us prove form our previous analysis that BPS solutions are extremal, i.e. that $\mathbf{c}^2 = 0$. To this end it is useful to rewrite the potential V_{BH} in terms of central and matter charges:

$$V_{BH}(\phi^s; e, m) = -\frac{1}{2}\Gamma^T\mathcal{M}(\phi)\Gamma = \frac{1}{2}\Gamma^T\mathbb{C}^T\mathbb{L}_c\,\mathbb{L}_c^\dagger\mathbb{C}\Gamma = \frac{1}{2}\mathscr{Z}^\dagger\mathscr{Z} = \frac{1}{2}\,\mathscr{Z}_{ij}\mathscr{Z}^{ij} + \mathscr{Z}_A\mathscr{Z}^A\,, \tag{3.272}$$

where we have used (3.138) and the definition of $\mathscr{Z}$. In the basis in which $\mathscr{Z}_{ij}$ is skew-diagonal ($\mathscr{Z}_{(x,u)(y,v)} = \mathscr{Z}_u\epsilon_{xy}\delta_{uv} = -i\,z_u\epsilon_{xy}\delta_{uv}$) we can write:

$$V_{BH} = \sum_{u=1}^{[\frac{\mathcal{N}}{2}]}|z_u|^2 + \mathscr{Z}_A\mathscr{Z}^A\,. \tag{3.273}$$

Consider now the constraint (3.201) and use (3.268) together with the above expression of

[37]Restoring the constants we would write: $\dot{U} = e^U\,\frac{\sqrt{8\pi}\,G}{c^2}|z_1|$.

[38]Actually, see footnote 22, to make contact with our discussion about representations and Bogomolny bound we should further make the replacement (3.154), so as to finally find:

$$M_{ADM} = \frac{\hbar}{c}\,|z_1|_\infty\,, \tag{3.270}$$

which is the correct relation between the mass and the central charge of the algebra (which has dimension of a $length^{-1}$).

the potential:

$$\mathbf{c}^2 = \frac{1}{2}\,G_{uv}\dot{\phi}^u\dot{\phi}^v + e^{2U}|z_1|^2 - e^{2U}\,\left(\sum_{u=1}^{[\frac{\mathcal{N}}{2}]}|z_u|^2 + \mathscr{Z}_A\mathscr{Z}^A\right) =$$

$$= \frac{1}{2}\,G_{uv}\dot{\phi}^u\dot{\phi}^v - e^{2U}\,\left(\sum_{u=2}^{[\frac{\mathcal{N}}{2}]}|z_u|^2 + \mathscr{Z}_A\mathscr{Z}^A\right)\,. \tag{3.274}$$

Recall that at the horizon $\dot{\phi}^s$ and e^{2U} always vanish, while the central and matter charges tend to a finite value. Taking the near horizon limit of the above equation we find:

$$\mathbf{c}^2 = 0\,, \tag{3.275}$$

namely *the BPS solution is extremal.* This allows to rewrite the constraint in the following form:

$$\frac{1}{2}\,G_{uv}\dot{\phi}^u\dot{\phi}^v = e^{2U}\,\left(\sum_{u=2}^{[\frac{\mathcal{N}}{2}]}|z_u|^2 + \mathscr{Z}_A\mathscr{Z}^A\right)\,. \tag{3.276}$$

which has to hold for any τ. This condition is implied by the other Killing spinor equations, in particular from

$$\delta\chi_{ijk} = \delta\lambda_{iA} = 0\,, \tag{3.277}$$

which yield a system of first order differential equations in the scalar fields of the form:

$$\dot{\phi}^s = 2\,e^U\,G^{ss'}\partial_{s'}|z_1|\,. \tag{3.278}$$

This is best seen in the $\mathcal{N}=2$ theory. Since $i,j=1,2$ the components $P^{ijkl}=P^{[ijkl]}$ and $P_{ijkl}=P_{[ijkl]}$ of the scalar manifold are not present. Moreover the central charge matrix has only one skew-eigenvalue $\mathscr{Z}_{ij}=\epsilon_{ij}\,\mathscr{Z}$. From (3.163) we have:

$$\mathcal{D}^{(H)}\mathscr{Z} = P_A\,\mathscr{Z}^A\,, \tag{3.279}$$

where we have written $P_{ij,A}=\epsilon_{ij}\,P_A$. In the $\mathcal{N}=2$ theory there are no dilatinos χ_{ijk}, but just gauginos λ_{iA} and hyperinos $\zeta^{\underline{mi}}$. The scalar fields in the vector multiplets are complex z^α and $P_A=P_{\alpha,A}\,dz^\alpha$, $P^A=P^A_{\bar{\alpha}}\,d\bar{z}^{\bar{\alpha}}$ represent the complex viebein 1-forms of the corresponding special Kähler manifold:

$$G_{\alpha\bar{\beta}}(z,\bar{z}) = P_{\alpha,A}P_{\bar{\beta}}{}^A\ ;\quad P_{\alpha,A}P_{\bar{\beta}}{}^B\,G^{\alpha\bar{\beta}} = \delta^B_A\,. \tag{3.280}$$

The gaugino variation along the Killing spinor ϵ^i reads:

$$\delta\lambda_{A\,i} = i\,P_{\alpha,A}\partial_\mu z^\alpha\,\epsilon_{ij}\gamma^\mu\epsilon^j - \frac{i}{4}\,T^-_{\rho\sigma\,A}\gamma^{\rho\sigma}\epsilon_i = 0\,. \tag{3.281}$$

Now use the second of (3.255) and (3.260) to find:

$$0 = i\, P_{\alpha,A}\dot{z}^\alpha\, \epsilon_{ij}\gamma^\tau\epsilon^j - \frac{1}{b^2}\mathscr{Z}_A\gamma^{01}\epsilon_i = \frac{i}{ab^2}\, P_{\alpha,A}\dot{z}^\alpha\, \epsilon_{ij}\gamma^1\epsilon^j + \frac{i}{b^2}\mathscr{Z}_A\,\frac{\mathscr{Z}}{|\mathscr{Z}|}\gamma^{01}\gamma^0\epsilon_{ij}\epsilon^j =$$
$$= \frac{i}{ab^2}\left(P_{\alpha,A}\dot{z}^\alpha - a\,\frac{\mathscr{Z}_A\mathscr{Z}}{|\mathscr{Z}|}\right)\epsilon_{ij}\gamma^1\epsilon^j\,, \tag{3.282}$$

where we have used (3.266). The above condition yields the following first order differential equations on the scalar fields in the vector multiplets:

$$P_{\alpha,A}\dot{z}^\alpha = e^U\,\frac{\mathscr{Z}_A\mathscr{Z}}{|\mathscr{Z}|}\,. \tag{3.283}$$

Consider now the complex conjugate of Eq. (3.279) in components

$$\mathcal{D}^{(H)}\overline{\mathscr{Z}} = P^A\,\mathscr{Z}_A = P^A_{\bar{\alpha}}\,d\bar{z}^{\bar{\alpha}}\,\mathscr{Z}_A = \mathcal{D}^{(H)}_{\bar{\alpha}}\overline{\mathscr{Z}}\,d\bar{z}^{\bar{\alpha}}\,, \tag{3.284}$$

which in particular implies that:

$$\mathcal{D}^{(H)}_{\alpha}\overline{\mathscr{Z}} = \mathcal{D}^{(H)}_{\bar{\alpha}}\mathscr{Z} = 0\,. \tag{3.285}$$

This allows to rewrite (3.283) in the form:

$$\dot{z}^\alpha = e^U\,G^{\alpha\bar{\alpha}}P^A_{\bar{\alpha}}\frac{\mathscr{Z}_A\mathscr{Z}}{|\mathscr{Z}|} = e^U\,G^{\alpha\bar{\alpha}}\frac{\mathcal{D}^{(H)}_{\bar{\alpha}}\overline{\mathscr{Z}}\,\mathscr{Z}}{|\mathscr{Z}|} = 2\,e^U\,G^{\alpha\bar{\alpha}}\mathcal{D}^{(H)}_{\bar{\alpha}}|\mathscr{Z}| = 2\,e^U\,G^{\alpha\bar{\alpha}}\partial_{\bar{\alpha}}|\mathscr{Z}|\,, \tag{3.286}$$

where we have used the property that the norm $|\mathscr{Z}|$ is H-invariant, so that:

$$\mathcal{D}^{(H)}_{\bar{\alpha}}|\mathscr{Z}| = \partial_{\bar{\alpha}}|\mathscr{Z}|\,. \tag{3.287}$$

Finally the Killing spinor condition on the hyperini variation implies:

$$\dot{q}^m = 0\,. \tag{3.288}$$

From the requirement that a fraction of supersymmetries be preserved by the black hole in the $\mathcal{N}=2$ theory we have therefore found a set of first order equations

$$\dot{U} = e^U\,|\mathscr{Z}(\phi;e,m)|\ ;\quad \dot{z}^\alpha = e^U\,G^{\alpha\bar{\alpha}}P^A_{\bar{\alpha}}\frac{\mathscr{Z}_A\mathscr{Z}}{|\mathscr{Z}|} = 2\,e^U\,G^{\alpha\bar{\alpha}}\partial_{\bar{\alpha}}|\mathscr{Z}|\ ;\ \ \dot{q}^m = 0\,, \tag{3.289}$$

which in turn imply

$$\mathbf{c}^2 = \dot{U}^2 + G_{\alpha\bar{\beta}}(z,\bar{z})\dot{z}^\alpha\dot{\bar{z}}^{\bar{\beta}} + \frac{1}{2}\,G_{mn}(q)\dot{q}^m\dot{q}^n - e^{2U}\,\left(|\mathscr{Z}|^2 + \mathscr{Z}_A\mathscr{Z}^A\right) = 0\,, \tag{3.290}$$

in line with our general conclusion, namely (3.276). Notice that $\dot{q}^m = 0$ means that in a BPS solution the hyperscalars do not participate. This is due to the fact that they are not coupled to the vector fields, i.e they do not enter the matrix $\mathcal{M}(\phi)$, which in turn follows from the fact that they are not connected to vector fields by supersymmetry, i.e. there is no vector field-strength in the hyperini variation (3.121). The hyperscalars q^m are *flat directions* of the potential $V_{BH} = V_{BH}(z, \bar{z})$.

Let us now come back to the general case $\mathcal{N} > 2$ and make some other general considerations. Multiply both sides of (3.276) and consider the near horizon limit, using the general property (3.249):

$$\lim_{\tau \to 0} \left(\sum_{u=2}^{[\frac{\mathcal{N}}{2}]} |z_u|^2 + \mathscr{Z}_A \mathscr{Z}^A \right) = \lim_{\tau \to 0} \frac{e^{-2U}}{2} G_{uv} \dot{\phi}^u \dot{\phi}^v = 0 \,, \tag{3.291}$$

which in turn implies that all the skew-eigenvalues of the cental different from $\mathscr{Z}_1$ and all matter charges vanish at the horizon on a BPS solution:

$$\mathscr{Z}_u(\phi_*; e, m) = 0 \,, \; (u = 2, \dots, \left[\frac{\mathcal{N}}{2}\right]) \; ; \; \; \mathscr{Z}_A(\phi_*; e, m) = 0 \,. \tag{3.292}$$

Therefore the value of the potential at the horizon is (resuming constants):

$$V_{ex} = V_{BH}(\phi_*; \, e, m) = |\mathscr{Z}_1(\phi_*; e, m)|^2 = \frac{c^4}{8\pi \, G} \, r_H(e, m)^2 \,, \tag{3.293}$$

where we have used (3.236). The behavior of the warp function

$$e^{-U} \sim -\tau \, r_H = \frac{r_H}{r} = \frac{|\mathscr{Z}_1(\phi_*; e, m)|}{r} \,, \tag{3.294}$$

and the near horizon metric is a Bertotti-Robinson metric of the form (3.237) describing an $AdS_2 \times S^2$ space.

As a consequence of property (3.292), if the solutions preserves a fraction $q/\mathcal{N}$ of supersymmetries, with $q > 0$, we will have:

$$\mathscr{Z}_1(\phi_*; e, m) = \mathscr{Z}_2(\phi_*; e, m) = 0 \,, \tag{3.295}$$

that is *all the central charges vanish at the horizon.* This in turn implies, from Eq. (3.293), that the horizon area vanishes. The solution is therefore not regular: The horizon coincides with the singularity. Such solutions, named *small black holes*, having vanishing horizon area in the supergravity description, also have vanishing entropy, according to the Bekenstein-Hawking area law. We must however recall that the supergravity description of the solution, if we interpret supergravity as an effective low-energy theory, can be trusted if curvatures are sufficiently small, namely if there is a horizon which keeps the predictable region of space-time far enough from the singularity, where the curvature explodes. This is not the case for small black holes and we would expect corrections which are of higher-order in the curvatures, to play an important role near their singularity. This is indeed the case: higher order curvature string corrections do regularize the solution giving it a finite horizon area and finite entropy (see for instance [47] and references therein).

If we consider the double extremal BPS solution by choosing $\phi^s(\tau = 0) = \phi^s_*$ so that $\phi^s(\tau) \equiv \phi^s_*$ for any τ, we end up with an extremal Reissner-Nordström solution which, by construction, is BPS as well, with (restoring the constants)

$$M_{ADM} = \sqrt{\frac{4\pi}{G}}\,|Q| = \sqrt{\frac{8\pi}{G}}\,|\mathscr{Z}_1(\phi_*; e, m)|\,. \tag{3.296}$$

The BPS orbit of charges. We have learned that BPS solutions are characterized by the property that, at the extremum $\phi^s_*(e, m)$ of the potential all matter charges $\mathscr{Z}_A$ and all central charges $\mathscr{Z}_u$ but one must vanish, according to (3.292). Recalling that the quantities in (3.292) are to be computed at $\phi^s_*(e, m)$, this amounts to non-linear conditions the quantized charges. By virtue of (3.152) and (3.157), such conditions are invariant if we transform the charges under the duality group G: $\Gamma \to g\,\Gamma$ and thus define an orbit of the charge vector $\Gamma = (m, e)$ under the action of G. Such orbit is called *the BPS-orbit*. If the charges are not in the BPS-orbit, a solution to the Killing spinor equations is not a physical black hole.

Let us now make some more general comments. If a black hole, as a solitonic massive object, is to be described in the Hilbert space of the quantum theory, our general discussion in Sect. 2.3 implies that its mass must be greater than the norm of all the skew-eigenvalues of the central charge:

$$M_{ADM} \geq \sqrt{\frac{8\pi}{G}}\,|\mathscr{Z}_u|\;,\quad u = 1, \dots, \left[\frac{\mathcal{N}}{2}\right]\,. \tag{3.297}$$

Stability of BPS solutions. Suppose the quantized charges are in the BPS-orbit, let us give an argument for the stability of the BPS solution, following [38]. If $|\mathscr{Z}_1|$ is the largest of the eigenvalues of the central charge, the extremal solution is the BPS one in which the ADM mass coincides with $\sqrt{\frac{8\pi}{G}}\,|\mathscr{Z}_1|$. In a non-extremal solution therefore $M \neq |\mathscr{Z}_1|$ but condition (3.297) only allows for the possibility

$$M_{ADM} > \sqrt{\frac{8\pi}{G}}\,|\mathscr{Z}_1|\,, \tag{3.298}$$

corresponding to a solution with the same charges as the extremal one but a greater mass. From the constraint at infinity (3.208) we see then that $\mathbf{c}^2 > 0$, namely the solution is non-extremal, that is it has a non-vanishing temperature. Since we have proven that BPS solutions are extremal, this solution breaks all supersymmetries. Being $T \neq 0$, the black hole will radiate according to Hawking's quantum process. It can only emit elementary particles in the theory. Our model is an ungauged extended supergravity which describes massless fields (graviton, gravitino, fermions and gauge fields) which are *neutral*, since there is no minimal coupling of the gauge fields to any other field. Charges and mass are only carried by the solitonic black hole solution. Emitting neutral, massless particles, the mass M_{ADM} of the solution will decrease, while its charge $|\mathscr{Z}_1|$ will remain constant. This evaporation process will last until the solution becomes (after infinite time) extremal BPS (i.e. $M_{ADM} =$

$\sqrt{\frac{8\pi}{G}}\,|\mathscr{Z}_1|$) and the temperature drops to zero. In this limiting state the solution will no longer radiate. BPS solutions can then be regarded as (quantum mechanically) stable solutions of supergravity/superstring theory. Their mass has the minimum value allowed by the supersymmetry of the theory (not of the solution), according to (3.297). Since lower masses, for the same charges, would correspond to singular solutions ($\mathbf{c}^2 < 0$), the supersymmetry of the theory seems to provide a first principle for excluding solutions with naked singularities (i.e. acts as a *cosmic censor* [13, 14]).

BPS solutions have played an important role in the early nineties in the study of string dualities [48]. Let us recall that dualities are correspondences between superstring theories (and M-theory) realized on different backgrounds which allow to identify these effective theories as different descriptions of the same microscopic degrees of freedom. These correspondences may be non-perturbative, namely relate the strong-coupling regime (referred to the superstring coupling constant) of a superstring theory on some background to the weak-coupling regime of a different theory on some other background. The action of these dualities on the background fields are described, at the level of low-energy supergravity, by transformations in G (or in a suitable extension of G) [33]. Two dual theories must have the same spectrum and interactions. In particular the spectrum of BPS states should coincide. Verifying this coincidence on BPS states is definitely more affordable a task than on generic massive states because the BPS mass formula (3.296) is duality invariant (being G-invariant) and, most importantly, supersymmetry protects, to some extent, the masses from quantum corrections, by virtue of *non-renormalization* theorems. Quantum corrections may affect both sides of (3.296), the equality however must still hold at the quantum level. If this were not the case, a state belonging classically to a short (i.e. BPS) multiplet, at the quantum level would be described by a long one. In other words quantum corrections would introduce new degrees of freedom, which is unlikely [11, 12].

Horizon as an stable attractor point. Summarizing, we have seen that, if we choose certain quantized electric and magnetic charges (belonging to the $1/\mathcal{N}$-BPS orbit), at the extremum of the potential all matter and central charges vanish except one central charge skew-eigenvalue ($\mathscr{Z}_1$). The regular BPS solution is solution to a system of first order differential equations of the form:[39]

$$\dot{U} = e^U\, W(\phi;\, e, m)\,, \tag{3.299}$$

$$\dot{\phi}^s = 2\, e^U\, G^{ss'}\partial_{s'}W(\phi;\, e, m)\,, \tag{3.300}$$

where we have defined $W(\phi;\, e, m) = |\mathscr{Z}_1(\phi;\, e, m)|$. At the horizon, since $e^{-U}\dot{\phi}^s$ vanishes, not only V_{BH}, but also $|\mathscr{Z}_1(\phi;\, e, m)|$ has an extremum:

$$\partial_{s'}W(\phi_*(e, m);\, e, m) = \partial_{s'}|\mathscr{Z}_1(\phi_*(e, m);\, e, m)| = 0\,. \tag{3.301}$$

The horizon point $\phi^s = \phi^s_*(e, m)$ is an equilibrium point for the dynamical system (3.300), since the right-hand-side vanishes. Let us notice that, by definition, W is always positive

[39]Restoring the constants we would write: $\dot{U} = e^U\, \frac{\sqrt{8\pi\, G}}{c^2}\, W(\phi;\, e, m)$.

definite. Moreover its derivative along the solution $\phi^r(\tau)$ is positive definite as well (except in ϕ_* where it vanishes):

$$\frac{dW}{d\tau} = \dot{\phi}^r \partial_r W = \frac{1}{2}\, e^{-U}\, G_{rs}(\phi)\, \dot{\phi}^r\, \dot{\phi}^s > 0\,. \tag{3.302}$$

We see that, if ϕ_* is isolated[40], W has the properties of a *Liapunov's function* and thus, by virtue of Liapunov's theorem, ϕ_* is a *stable attractor point* (we refer the reader to Appendix C for a brief review of the notion of asymptotic stability in the sense of Liapunov and of Liapunov's theorem, see also standard books like [49]). This conclusion extends to models based on a generic (not necessarily homogeneous) scalar manifold: The very existence of a W-function even just in a neighborhood of an isolated critical point ϕ_*, in terms of which the evolution of the scalar filed is described by a dynamical system of the form (3.300), is enough to guarantee asymptotic stability of ϕ_*, and thus that *the horizon is a stable attractor* (see the second of [54]). Let us emphasize that in this case we need not evaluate the Hessian of the potential on ϕ_*. In other words the (local) existence of W can be taken as an alternative and more powerful characterization of the attractiveness and stability properties of the horizon point ϕ_*.

Our analysis also extends to multicenter BPS solutions, see for instance [50] and references therein. We shall not deal with them here.

3.3.3 Non-BPS Extremal Solutions

BPS extremal solutions were the first to be studied. Eventually new extremal non-BPS (i.e. breaking all supersymmetries) solutions were found [51]. These are defined by quantized electric and magnetic charges belonging to orbits with respect to the duality action of G which are different from the BPS one, which we shall refer to also as "orbit-I". One of these orbits, to be dubbed "orbit-II", has the distinctive feature that at the extremum of the potential all central and matter charges vanish, except one matter charge $\mathscr{Z}_{A_0}$. The corresponding regular solution is described by a system of first order differential equations of the form (3.299),(3.300), where now $W(\phi;\, e, m) = |\mathscr{Z}_{A_0}(\phi;\, e, m)|$. A class of $\mathcal{N} < 8$ models can be obtained as consistent truncations of the maximal $\mathcal{N} = 8$ theory. This means that they are obtained by setting fields of the maximal theory to zero so that all solutions of these models are also solutions to the $\mathcal{N} = 8$ theory. Since in the $\mathcal{N} = 8$ model there are no matter multiplets, all charges at infinity are central charges $\mathscr{Z}_{ij}$. In particular the matter charge $\mathscr{Z}_{A_0}$ of the $\mathcal{N} < 8$ model defining the non-BPS extremal solution, in the context of the $\mathcal{N} = 8$ theory is one of the four central charge skew-eigenvalues $\mathscr{Z}_u$. The non-BPS extremal black hole becomes then BPS if viewed as solution to the maximal theory. It preserves one

[40]In $\mathcal{N} = 2$ supergravity, the BPS fixed point is isolated in the special-Kähler manifold spanned by the scalar fields $z^\alpha, \bar{z}^{\bar{\alpha}}$ in the vector multiplets. For $\mathcal{N} > 2$, the scalar potential V_{BH} has flat directions. Just as the hyperscalars q^m in the $\mathcal{N} = 2$ models, the scalar fields corresponding to the flat directions of V_{BH} do not have an independent evolution. Once we fix them at radial infinity, all the other scalar fields evolve towards a single fixed point at the horizon, defining in their evolution an hypersurface inside $\mathscr{M}_{scal}$ on which the fixed point isolated and the theorem applies. In the whole $\mathscr{M}_{scal}$ the fixed points ϕ^s_* define a hypersurface parametrized by the flat directions which is a *stable attractor hypersurface*.

of its eight supersymmetries, which is among the $8-\mathcal{N}$ supersymmetries which are lost in the truncation to the $\mathcal{N}<8$ model.

There is a further orbit of charges, to be referred to as "orbit-III", for which the moduli of all the central charge skew-eigenvalues $\mathscr{Z}_u$ and matter charges $\mathscr{Z}_A$ become equal at the extremum of V_{BH}. The corresponding solutions cannot be BPS, since, as we have seen, if in a BPS solution all of the $\mathscr{Z}_u$ coincide at the horizon, the whole central charge matrix has to vanish at that point. They were first studied in [52]. The corresponding regular solution is described by a system of first order differential equations of the form (3.299),(3.300). The explicit form of the corresponding W-function is not known. Only parametric or integral expressions of W were found [53],[54]. In the maximal theory, if we denote by $\sigma(e,m)$ the common value of the four central charge skew-eigenvalues $\mathscr{Z}_u$ at the horizon, the W function at that point is given by:

$$W(\phi_*(e,m);\, e,m) = 2\,\sigma(e,m)\,. \tag{3.303}$$

This is also the ADM mass of the corresponding double-extremal solution. It is greater than the value $\sigma(e,m)$ of the central charge skew-eigenvalues $\mathscr{Z}_u$ at the horizon, consistently with the general condition (3.297). Also for regular (non-BPS) extremal black holes of type II and III the function W defining the system of first order equations is a Liapunov function whose existence implies that the corresponding horizons are *stable attractor points* (or better *stable attractor hypersurfaces*, see footnote 32, if we take the flat directions into account) of the dynamical system of the scalar fields.

Extremal (i.e. $T=0$), asymptotically flat black holes are expected to belong to irreducible representations of the super-Poincaré algebra.[41] BPS solutions are described by massive *short* multiplets, while non-BPS ones by massive *long* multiplets.

In each of these orbits, as we have seen in the previous subsection, we can find a *double extremal* solution in which all scalar fields are constant and fixed all along the radial direction to their horizon values. These solutions are defined by the boundary condition:

$$\phi^s(\tau=0) = \phi^s_*(e,m)\,. \tag{3.304}$$

and are of extremal Reissner-Nordström type. This means that there is more than one embedding of the extremal Reissner-Nordström solution an extended supergravity, of which only one, modulo transformations of the global symmetry group, preserves supersymmetries, the other being non-BPS.[42]

3.3.4 The Black Holes and Duality

We have learned that the on-shell global symmetries of an extended supergravity, at the classical level, are encoded in the isometry group G of the scalar manifold (if non-empty),

[41] Supersymmetry is well defined only at zero-temperature. Non-extremal solutions have non-vanishing temperature and therefore are not described by pure states, but rather by *non-supersymmetric statistical ensembles of states* [38]. Their description within a Hilbert space generated by (pure) states in representations of the super-Poincaré algebra implies however the inequality (3.297).

[42] Modulo transformations of the global symmetry group, since the orbits of regular extremal solutions are at most three, see below, there are at most three inequivalent embeddings of the extremal Reissner-Nordström solution an extended supergravity.

whose action on the fields of the model is described in (3.116): Its non-linear action on the scalar fields ϕ^s is combined with a simultaneous linear symplectic action on the field strengths F^Λ and their duals G_Λ. This duality action of G is defined by a symplectic representation $\mathbf{S}$ of G. The fermion fields transform under the compensating transformation $h(g,\phi)$ in H. Under this action static black hole solutions, defined by the general ansatz (3.172), are mapped into solutions of the same kind. More precisely a duality transformation $g \in G$ maps a black hole solution $U(\tau), \phi^r(\tau)$, with charges $\Gamma^M = (m^\Lambda,\, e_\Lambda)$ and ADM mass M_{ADM}, into a new solution $U'(\tau) = U(\tau)$, $\phi'^r(\tau) = g \star \phi^r(\tau)$ with charges $\Gamma' = \mathbf{S}[g]\,\Gamma$ and the same ADM mass (the ADM mass, being a property of the metric of the solution, is not affected by duality transformations which leave the metric unaltered). In particular if, for given charges and ADM mass, the solution $U(\tau), \phi^s(\tau)$ is uniquely defined by the boundary condition ϕ^s_0 for the scalar fields, $U'(\tau) = U(\tau)$, $\phi'^s(\tau)$ is the *unique solution* with charges Γ' defined by the boundary condition $\phi'_0 = g \star \phi_0$

$$g \in G \quad : \quad \begin{cases} U(\tau) \\ \phi(\tau) \\ \Gamma \end{cases} \quad \longrightarrow \quad \begin{cases} U'(\tau) = U(\tau) \\ \phi'(\tau) = g \star \phi(\tau) \\ \Gamma' = \mathbf{S}[g]\,\Gamma \end{cases} \quad . \tag{3.305}$$

Using eq.s (3.90) and (3.305), we see that the effective potential $V_{BH}(\phi;\,\Gamma)$, as a function of the scalar fields and of the quantized charges, is invariant under the simultaneous action (3.305:

$$V_{BH}(\phi, \Gamma) \quad = \quad V(g \star \phi, \mathbf{S}[g]\,\Gamma)\,. \tag{3.306}$$

This implies that V_{BH}, as a function of the scalar fields and quantized charges, is G-invariant. From this property of V_{BH} it follows that the effective action (3.202) and the extremality constraint (3.203) are manifestly duality invariant. A consequence of this is that *black holes in extended supergravities can be classified in orbits with repsect to the action (3.305) of the global symmetry gorup G*.

We have denoted by $\phi^s_*(\Gamma) = \phi^s_*(e,m)$ the extremum of $V_{BH}(\phi;\,\Gamma)$:

$$\partial_s V_{BH}(\phi_*(\Gamma);\,\Gamma) = 0\,. \tag{3.307}$$

From (3.306) we find:

$$\partial_s V_{BH}(\phi_*(\Gamma);\,\Gamma) = 0 \;\Leftrightarrow\; \partial_s V_{BH}(g \star \phi_*(\Gamma);\,\mathbf{S}[g]\,\Gamma) = 0\,. \tag{3.308}$$

This implies that the point $g \star \phi_*(\Gamma)$ extremizes the potential $V(\phi', \mathbf{S}[g]\,\Gamma)$. But such extremum was denoted by $\phi_*(\mathbf{S}[g]\,\Gamma)$, so we can write:

$$g \star \phi^s_*(\Gamma) = \phi^s_*(\mathbf{S}[g]\,\Gamma)\,. \tag{3.309}$$

This has an important implication for extremal solutions:

$$V_{ex}(\Gamma) = V_{BH}(\phi_*(\Gamma);\,\Gamma) = V_{BH}(g \star \phi_*(\Gamma);\,\mathbf{S}[g]\,\Gamma) = V_{BH}(\phi_*(\mathbf{S}[g]\,\Gamma);\,\mathbf{S}[g]\,\Gamma) = V_{ex}(\mathbf{S}[g]\,\Gamma)\,. \tag{3.310}$$

In other words *the value of scalar potential at the extremum, which defines the horizon area A and thus the entropy of the solution, is described by a G-invariant function of the quantized charges only.* Therefore the entropy of the extremal solution is a G-invariant function of Γ. In all the models with homogeneous-symmetric scalar manifold in Table 3.1.1, except the $\mathcal{N}=2$ ones with $G=\mathrm{U}(1,n)$ and the $\mathcal{N}=3$ supergravity, the representation $\mathbf{S}$ of G in which the electric and magnetic charges transform has a single invariant function $I_4(\Gamma)=I_4(e,m)$ of the electric-magnetic charge vector Γ, which has degree four in the charges. Denoting by $(T_A)_M{}^N$ the matrices $\mathbf{S}[T_A]$ representing the generators T_A of G in the symplectic duality representation $\mathbf{S}$, the quartic invariant in these models can be written inn the general form:

$$I_4(\Gamma) = -\frac{n_V(2n_V+1)}{6d}\,(T_A)_{MN}\,(T^A)_{PQ}\,\Gamma^M\,\Gamma^N\,\Gamma^P\,\Gamma^Q\,, \tag{3.311}$$

where the symplectic indices are raised and lowered by $\mathbb{C}^{MN}$ and $\mathbb{C}_{MN}$, the index A is raised by the inverse of $\eta_{AB}\equiv(T_A)_M{}^N(T_B)_N{}^M$ and d is the dimension of G. In terms of I_4 the potential at the extremum reads:

$$V_{ex}(e,m) = \sqrt{|I_4(e,m)|}\,, \tag{3.312}$$

and the horizon area reads (resuming the constants):

$$A(e,m) = 4\pi\left(\frac{8\pi G}{c^4}\sqrt{|I_4(e,m)|}\right)\,, \tag{3.313}$$

and the entropy of the extremal solution therefore reads:

$$S(e,m) = \frac{k_B}{\ell_P^2}\,\pi\left(\frac{8\pi G}{c^4}\sqrt{|I_4(e,m)|}\right)\,. \tag{3.314}$$

The orbits I (BPS), II and III of Γ with respect to the action of G, discussed in the previous subsection, have the following features:

$$\begin{aligned} \text{Orbit I (BPS)}\quad & I_4>0\,,\\ \text{Orbit II (non-BPS)}\quad & I_4>0\,,\\ \text{Orbit III (non-BPS)}\quad & I_4<0\,. \end{aligned} \tag{3.315}$$

Orbits of the electric and magnetic charges with vanishing quartic invariant $I_4(e,m)=0$ define *small black holes.* It was shown that orbits I, II and III exhaust all possible orbits of regular black holes in extended supergravities [55]. Within each of these orbits the extremal solutions are defined by the condition that, at radial infinity:

$$M_{ADM} = \sqrt{\frac{8\pi}{G}}\,W(\phi_0;\,e,m)\,. \tag{3.316}$$

Only for the BPS solution (orbit I), the value of the ADM mass corresponds to one of the skew-eigenvalues (the one with largest modulus) of the central charge matrix. In the other orbits its value is strictly greater than the moduli of any of the central charge skew-eigenvalues. Non-extremal solutions ($\mathbf{c}^2 > 0$) in each orbit are characterized, for the same charges, by a larger ADM mass than the corresponding extremal solution:

$$M_{ADM} > \sqrt{\frac{8\pi}{G}}\, W(\phi_0;\, e, m)\,. \tag{3.317}$$

Hawking-evaporation will then reduce their mass keeping the charges and therefore $W(\phi_0;\, e, m)$ constant. Just as for the BPS orbit, ADM mass of the black hole will tend to its lower bound $W(\phi_0;\, e, m)$ defined by the corresponding extremal solution. This lower bound for the non-BPS orbits of electric and magnetic charges is strictly larger than any of the $|\mathscr{Z}_u|$.

The general relation of the scalar potential to the W-function defining the extremal solution is derived from the general constraint (3.201) and from the system of first-order equations (3.299) and (3.300):

$$V_{BH}(\phi; e, m) = W^2 + G^{ss'}(\phi)\partial_s W \partial_{s'} W\,. \tag{3.318}$$

This can be viewed as a partial differential equation defining W. In fact it corresponds to the Hamilton-Jacobi equations (for "zero energy solutions" $\mathbf{c} = 0$) [53] associated with the effective *autonomous Lagrangian system* (3.202) which describes the black hole solutions. At the horizon W is extremized as well as the potential and therefore:

$$V_{ex}(e, m) = V_{BH}(\phi_*(e, m); e, m) = W(\phi_*(e, m); e, m)^2 = r_H^2\,. \tag{3.319}$$

Recalling that the W function, if evaluated on the solution ($W(\phi^s(\tau);\, e, m)$), is monotonically increasing from the horizon to radial infinity (see Eq. (3.302)), it monotonically interpolates between the horizon area for $\tau \to -\infty$ and the ADM mass for $\tau = 0$, so that, restoring the constants:

$$\frac{c^2}{\sqrt{8\pi\, G}}\, r_H \;\overset{\tau\to-\infty}{\longleftarrow}\; W(\phi^s(\tau);\, e, m) \;\overset{\tau\to 0}{\longrightarrow}\; \sqrt{\frac{G}{8\pi}}\, M_{ADM}\,. \tag{3.320}$$

A Notations

Poincaré transformations. We use the "mostly minus" convention for the signature of the metric, so that the Lorentz metric in flat Minkowski space-time is $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. A generic Poincaré transformation $(\Lambda,\, x_0)$ is defined by a Lorentz transformation $\Lambda = (\Lambda_\mu{}^\nu)$ and a space-time translation by $x_0 = (x_0^\mu)$:

$$x^\mu \overset{(\Lambda,\, x_0)}{\longrightarrow} x'^\mu = \Lambda^\mu{}_\nu\, x^\nu - x_0^\mu\,, \tag{A.1}$$

where $\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu}$. We also use the convention: $\epsilon_{0123} = -\epsilon^{0123} = 1$, so that:

$$dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = d^4x = -\epsilon^{0123}\, d^4x \;\Rightarrow\; dx^\mu \wedge dx^\nu \wedge dx^\nu \wedge dx^\rho = -\epsilon^{\mu\nu\rho\sigma}\, d^4x\,. \tag{A.2}$$

The abstract generators of the Lorentz group and of space-time translations are denoted by $\mathcal{L}_{\mu\nu}$, $\mathcal{P}_\mu$ and satisfy the commutation relations:

$$[\mathcal{L}_{\mu\nu},\, \mathcal{L}_{\rho\sigma}] = \eta_{\nu\rho}\, \mathcal{L}_{\mu\sigma} + \eta_{\mu\sigma}\, \mathcal{L}_{\nu\rho} - \eta_{\nu\sigma}\, \mathcal{L}_{\mu\rho} - \eta_{\mu\rho}\, \mathcal{L}_{\nu\sigma}\,, \tag{A.3}$$

$$[\mathcal{L}_{\mu\nu},\, \mathcal{P}_\rho] = \mathcal{P}_\mu\, \eta_{\nu\rho} - \mathcal{P}_\nu\, \eta_{\mu\rho}\,. \tag{A.4}$$

On 4-vectors V^μ, in the $(\mathbf{\frac{1}{2}},\mathbf{\frac{1}{2}})$ of SL(2, $\mathbb{C}$), $\mathcal{L}_{\mu\nu}$ has the matrix form: $(\mathcal{L}_{\mu\nu})_\rho{}^\sigma = \delta^\sigma_\mu\,\eta_{\nu\rho} - \delta^\sigma_\nu\,\eta_{\mu\rho}$. An abstract Poincaré transformation is then given by:

$$T(\Lambda,\, x_0) = e^{x_0^\mu\,\mathcal{P}_\mu}\cdot\Lambda = \Lambda\cdot\, e^{x_0^{\prime\mu}\,\mathcal{P}_\mu} \;\;;\;\; \Lambda = e^{\frac{1}{2}\,\theta^{\mu\nu}\,\mathcal{L}_{\mu\nu}}\,, \tag{A.5}$$

where $x_0' = \Lambda^{-1}\,x_0$.

Let $\Phi^m(x^\mu)$ be a classical field transforming in a representation D of the Lorentz group. Under a generic Poincaré transformation it transforms as:

$$\Phi^m(x) \stackrel{(\Lambda,\,x_0)}{\longrightarrow} \Phi'^m(x') = D(\Lambda)^m{}_n\,\Phi^n(x) = \left[\mathcal{O}_{(\Lambda,\,x_0)}\cdot\Phi\right]^m(x')\,, \tag{A.6}$$

where $\mathcal{O}_{(\Lambda,\,x_0)}$ is a realization of the group on the fields in terms of differential operators:

$$\begin{aligned}\mathcal{O}_{(\Lambda,\,x_0)} &= \exp(x_0\cdot\mathcal{O}(\mathcal{P}))\cdot\exp(\frac{1}{2}\,\theta^{\mu\nu}\,\mathcal{O}(\mathcal{L}_{\mu\nu}))\,,\\ \mathcal{O}(\mathcal{P}_\mu) &= \partial_\mu\;;\;\; \mathcal{O}(\mathcal{L}_{\mu\nu}) = D(\mathcal{L}_{\mu\nu}) + x_\mu\partial_\nu - x_\nu\partial_\mu\,.\end{aligned} \tag{A.7}$$

In the quantum theory the Poincaré generators are represented by anti-hermitian operators $\hat{\mathcal{L}}_{\mu\nu}$, $\hat{\mathcal{P}}_\mu$ on the infinite dimensional Hilbert space of states. The four-momentum and the total angular momentum operators are

$$\hat{P}_\mu = i\hbar\,\hat{\mathcal{P}}_\mu\;;\;\; \hat{J}_i = \frac{i\,\hbar}{2}\epsilon_{ijk}\,\hat{\mathcal{L}}^{jk} = \hat{S}_i + \hat{M}_i\,, \tag{A.8}$$

$\hat{M}_i$ and $\hat{S}_i$ being the orbital and spin components, respectively. We shall set $\hbar = c = 1$. The action of a (unitary) Poincaré transformation on the hermitian operator $\hat{O}$, representing an observable O is:

$$\hat{O} \stackrel{(\Lambda,\,x_0)}{\longrightarrow} \hat{O}' = U(\Lambda,\,x_0)^\dagger\,\hat{O}\,U(\Lambda,\,x_0)\,, \tag{A.9}$$

and on a field operator $\hat{\Phi}^m(x)$, in the Heisenberg representation:

$$\hat{\Phi}^m(x) \stackrel{(\Lambda,\,x_0)}{\longrightarrow} \hat{\Phi}'^m(x') = U(\Lambda,\,x_0)^\dagger\,\hat{\Phi}^m(x')\,U(\Lambda,\,x_0) = D(\Lambda)^m{}_n\,\hat{\Phi}^n(x) = \left[\mathcal{O}_{(\Lambda,\,x_0)}\cdot\hat{\Phi}\right]^m(x')\,. \tag{A.10}$$

The infinitesimal variation of $\hat{\Phi}^m(x)$ reads:

$$\delta\hat{\Phi}^m(x) = \frac{\epsilon^{\mu\nu}}{2}\,[\hat{\Phi}^m(x),\,\hat{\mathcal{L}}_{\mu\nu}] + \epsilon^\mu\,[\hat{\Phi}^m(x),\,\hat{\mathcal{P}}_\mu]\,. \tag{A.11}$$

In the Heisenberg representation the effect of two consecutive transformations g_1 and g_2, which in the Schroedinger picture would transform a state $|s\rangle$ as follows:

$$|s\rangle \stackrel{g_1}{\longrightarrow} |g_1\,s\rangle = U(g_1)|s\rangle \stackrel{g_2}{\longrightarrow} |g_2 g_1\,s\rangle = U(g_2)|g_1 s\rangle = U(g_2 g_1)|s\rangle\,, \tag{A.12}$$

is implemented by keeping $|s\rangle$ unchanged and transforming the field operators as follows:

$$\begin{aligned}\hat{\Phi}^m(x) &\stackrel{g_1}{\longrightarrow} U(g_1)^\dagger\hat{\Phi}^m(x')U(g_1) \stackrel{g_2}{\longrightarrow} U(g_1)^\dagger U(g_2)^\dagger\hat{\Phi}^m(x'')U(g_2)U(g_1) =\\ &= U(g_2 g_1)^\dagger\hat{\Phi}^m(x'')U(g_2 g_1)\,.\end{aligned} \tag{A.13}$$

The one used above is the *active* description of transformations on $\hat{\Phi}$, if $g_1 = e^{T_1} \approx \mathbf{1} + T_1$ and $g_2 = e^{T_2} \approx \mathbf{1} + T_2$, T_1, T_2 infinitesimal, $\delta_2\delta_1\hat{\Phi} = \mathcal{O}(T_2)\delta_1\hat{\Phi} = \mathcal{O}(T_2)\mathcal{O}(T_1)\hat{\Phi}$, so that:

$$(\delta_2\delta_1)_{\text{active}}\hat{\Phi} = \mathcal{O}(T_2{\cdot}T_1)\hat{\Phi} = [[\hat{\Phi},\, T_2],\, T_1] \;\Rightarrow\; [\delta_2,\, \delta_1]_{\text{active}}\hat{\Phi} = \mathcal{O}([T_2,\, T_1])\hat{\Phi} = [\hat{\Phi},\, [T_2,\, T_1]]\,, \tag{A.14}$$

where we have used the Jacobi identity on commutators. Using the *passive* description instead, the original fields $\hat{\Phi}$ are expressed in terms of the transformed ones $\hat{\Phi}'$ so that $\delta\hat{\Phi} = f(\hat{\Phi}')$ and a subsequent transformation will be effected by expressing $\hat{\Phi}'$ inside f in terms of the new one $\hat{\Phi}''$. Clearly we have:

$$[\delta_2,\, \delta_1]_{\text{passive}}\hat{\Phi} = -[\delta_2,\, \delta_1]_{\text{active}}\hat{\Phi}\,. \tag{A.15}$$

From this it follows that, if $\{T_A\}$ are the infinitesimal generators of a Lie group which close a Lie algebra with structure constants ${f_{AB}}^C$:

$$[T_A,\, T_B] = {f_{AB}}^C\, T_C\,, \tag{A.16}$$

and if $\delta_A\hat{\Phi} = [\hat{\Phi},\, T_A]$ denote an infinitesimal variations generated by T_A, the commutators of these variations in the active description close an algebra with the same structure constants, while in the passive one they close with the opposite structure constants:

$$[\delta_A,\, \delta_B]_{\text{active}} = {f_{AB}}^C\, \delta_C \;,\quad [\delta_A,\, \delta_B]_{\text{passive}} = -{f_{AB}}^C\, \delta_C\,. \tag{A.17}$$

The invariants of the Poincaré group are constructed out of the momentum generator $\hat{P}_\mu$ and the Pauli-Lubanski 4-vector $\hat{W}_\mu \equiv \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\,\hat{\mathcal{L}}^{\nu\rho}\hat{P}^\sigma$. In particular, for massive single-particle representations, in the rest-frame:

$$\langle\hat{W}_\mu\hat{W}^\mu\rangle = -\sum_{I=1}^{3}\langle\hat{W}_I\hat{W}_I\rangle = -m^2\,c^2\,\langle|\hat{S}|^2\rangle = -m^2\,c^2\,\hbar^2\,s\,(s+1) \;\;;\;\; \langle\hat{P}_\mu\hat{P}^\mu\rangle = m^2\,c^2\,. \tag{A.18}$$

On a massless state of momentum p_μ

$$\langle\hat{W}_\mu\rangle = p_\mu\,\langle\hat{\Gamma}\rangle\,, \tag{A.19}$$

where $\hat{\Gamma}$ is the helicity operator.

Spinors. We mostly follow the notations of [2]. In flat space-time we choose the γ-matrices, satisfying the relation $\{\gamma^\mu,\, \gamma^\nu\} = 2\,\eta^{\mu\nu}$, to have the following form:

$$\gamma^\mu = \begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbf{0} \end{pmatrix} \;;\;\; \sigma^\mu = (\mathbf{1}, \sigma^I)\;;\;\; \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^I) \quad (I = 1,2,3)\,, \tag{A.20}$$

σ^I being the three Pauli matrices. The spinor representation of the Lorentz generators is given by:

$$D(\mathcal{L}_{\mu\nu}) = \frac{\gamma_{\mu\nu}}{2} \;\;;\;\; \gamma_{\mu\nu} \equiv \frac{1}{2}\,[\gamma^\mu,\, \gamma^\nu]\,. \tag{A.21}$$

One can verify that:

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger = \eta^{\mu\mu}\,\gamma^\mu\,. \tag{A.22}$$

A 4-component spinor is an array of four complex Grassmannian entries transforming in the $(\mathbf{\frac{1}{2}},\mathbf{0})+(\mathbf{0},\mathbf{\frac{1}{2}})$ of $\mathrm{SL}(2,\mathbb{C})$. We use for the complex conjugation of Grassmann numbers the following convention: $(\xi_1\,\xi_2)^* = \xi_2^*\,\xi_1^*$. Dirac and complex conjugations on a spinor ψ as defined, respectively, as follows:

$$\bar{\psi} \equiv \psi^\dagger\gamma^0\ ;\ \ \psi_c \equiv C\,\bar{\psi}^T\,, \tag{A.23}$$

where the charge conjugation matrix C is defined as $C=-i\gamma^2\,\gamma^0$ and satisfies the properties:

$$C^{-1}\gamma^\mu\,C = (\gamma^\mu)^T\ ;\ \ C = C^* = -C^T = -C^{-1}\,. \tag{A.24}$$

Defining

$$\gamma^{\mu_1\cdots\mu_k} \equiv \gamma^{[\mu_1}\dots\gamma^{\mu_k]}\,, \tag{A.25}$$

the following properties hold:

$$\begin{aligned}(C\gamma^{\mu_1\cdots\mu_k})^T &= -(-1)^{\frac{k(k+1)}{2}}\,C\gamma^{\mu_1\cdots\mu_k}\,,\\ \bar{\chi}_c\gamma^{\mu_1\cdots\mu_k}\lambda &= (-1)^{\frac{k(k+1)}{2}}\,\bar{\lambda}_c\gamma^{\mu_1\cdots\mu_k}\chi\,,\\ (\bar{\chi}_c\gamma^{\mu_1\cdots\mu_k}\lambda)^* &= (-1)^k\,\bar{\chi}\gamma^{\mu_1\cdots\mu_k}\lambda_c\,.\end{aligned} \tag{A.26}$$

The spinor representation can be reduced by imposing the Majorana condition on spinors:

$$\psi = \psi_c = C\,\bar{\psi}^T\,. \tag{A.27}$$

From the last of Eq.s (A.26) it follows that, if χ, λ are Majorana spinors, $\bar{\chi}\lambda$ and $i\bar{\chi}\gamma^\mu\lambda$ are real.

We define the matrix γ^5 as follows

$$\gamma^5 \equiv \frac{i}{4}\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}\,. \tag{A.28}$$

The following properties hold:

$$\begin{aligned}\gamma^5\gamma_\mu &= -\frac{i}{3!}\,\epsilon_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma}\ ;\ \ \gamma^5\gamma_{\mu\nu} = -\frac{i}{2}\,\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\,,\\ \gamma^5\gamma_{\mu\nu\rho} &= i\,\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\ ;\ \ \gamma^5\gamma_{\mu\nu\rho\sigma} = i\,\epsilon_{\mu\nu\rho\sigma}\,.\end{aligned} \tag{A.29}$$

Of particular use is the basic Fierz identity:

$$\lambda\,\bar{\chi} = -\frac{1}{4}\,(\bar{\chi}\lambda) - \frac{1}{4}\,(\bar{\chi}\gamma^5\lambda)\,\gamma^5 - \frac{1}{4}\,(\bar{\chi}\gamma^\mu\lambda)\,\gamma_\mu + \frac{1}{4}\,(\bar{\chi}\gamma^5\gamma^\mu\lambda)\,\gamma^5\gamma_\mu + \frac{1}{8}\,(\bar{\chi}\gamma^{\mu\nu}\lambda)\,\gamma_{\mu\nu}\,. \tag{A.30}$$

If we apply the above identity to a single spinor 1-form $\Psi = \Psi_\mu\, dx^\mu$, the only non-vanishing bilinears are $\bar{\Psi} \wedge \gamma^\mu \Psi$ and $\bar{\Psi} \wedge \gamma^{\mu\nu} \Psi$, so that:

$$\Psi_{[\rho}\,\bar{\Psi}_{\sigma]} = -\frac{1}{4}\,(\bar{\Psi}_{[\sigma}\gamma^\mu\Psi_{\rho]})\,\gamma_\mu + \frac{1}{8}\,(\bar{\Psi}_{[\sigma}\gamma^{\mu\nu}\Psi_{\rho]})\,\gamma_{\mu\nu} = \frac{1}{4}\,(\bar{\Psi}_{[\rho}\gamma^\mu\Psi_{\sigma]})\,\gamma_\mu - \\ - \frac{1}{8}\,(\bar{\Psi}_{[\rho}\gamma^{\mu\nu}\Psi_{\sigma]})\,\gamma_{\mu\nu}\,. \tag{A.31}$$

From the above identity one can verify that:

$$\gamma^\mu\,\Psi\,\wedge\bar{\Psi}\,\wedge\gamma_\mu\Psi = 0\,. \tag{A.32}$$

Below are other useful properties of the γ-matrices:

$$\begin{aligned}
\gamma_{\mu\nu}\gamma^\rho &= 2\,\gamma_{[\mu}\delta^\rho_{\nu]} + \gamma_{\mu\nu}{}^\rho = 2\,\gamma_{[\mu}\delta^\rho_{\nu]} + i\epsilon_{\mu\nu}{}^{\rho\sigma}\gamma^5\gamma_\sigma\,,\\
\gamma_{\mu\nu}\gamma^{\rho\sigma} &= \gamma_{\mu\nu}{}^{\rho\sigma} - 4\,\delta^{[\rho}_{[\mu}\gamma_{\nu]}{}^{\sigma]} - 2\,\delta^{\rho\sigma}_{\mu\nu}\,,\\
\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]} &= \gamma_{\mu\nu}{}^{\rho\sigma} + 2\,\delta^{\rho\sigma}_{\mu\nu} = 2\,(\delta^{\rho\sigma}_{\mu\nu} + \frac{i}{2}\,\epsilon_{\mu\nu}{}^{\rho\sigma}\,\gamma^5)\,,\\
\gamma_\rho\gamma^{\mu_1\dots\mu_k}\gamma^\rho &= 2(-1)^k(2-k)\,\gamma^{\mu_1\dots\mu_k}\,.
\end{aligned} \tag{A.33}$$

We shall also use the 2-component notation and write:

$$\chi = \begin{pmatrix}\xi_\alpha\\ \bar{\zeta}^{\dot{\alpha}}\end{pmatrix}\,, \tag{A.34}$$

where $\bar{\zeta}^{\dot{\alpha}} = (\zeta^\alpha)^*$, $\bar{\zeta}_{\dot{\alpha}} = (\zeta_\alpha)^*$ and the indices $\alpha,\,\dot{\alpha} = 1, 2$ are raised and lowered as follows:

$$\begin{aligned}
\lambda^\alpha &= \epsilon^{\alpha\beta}\lambda_\beta\ ;\ \ \lambda_\beta = \epsilon_{\alpha\beta}\,\lambda^\alpha\,,\\
\bar{\lambda}^{\dot{\alpha}} &= \epsilon^{\dot{\beta}\dot{\alpha}}\bar{\lambda}_{\dot{\beta}}\ ;\ \ \bar{\lambda}_{\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}}\,\bar{\lambda}^{\dot{\alpha}}\,,
\end{aligned}$$

where $\epsilon_{12} = \epsilon^{12} = -\epsilon_{\dot{1}\dot{2}} = -\epsilon^{\dot{1}\dot{2}} = 1$. Clearly we have $\zeta_1^\alpha\,\xi_{2\,\alpha} = -\zeta_{1\,\alpha}\,\xi_2^\alpha$ and the same for the dotted components. The action of the spin SU(2) on ξ_α and $\bar{\zeta}^{\dot{\alpha}}$ is the same:

$$S^I = \frac{i}{2}\epsilon_{IJK}\,D(\mathcal{L}^{JK}) = \frac{1}{2}\begin{pmatrix}\sigma^I & \mathbf{0}\\ \mathbf{0} & \sigma^I\end{pmatrix} \quad (I, J, K = 1, 2, 3)\,, \tag{A.35}$$

while the action of the Lorentz generators reads:

$$\begin{aligned}
D(\mathcal{L}^{\mu\nu}) &= -\frac{i}{2}\begin{pmatrix}(\sigma^{\mu\nu})_\alpha{}^\beta & \mathbf{0}\\ \mathbf{0} & (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\end{pmatrix}\,,\\
\sigma^{\mu\nu} &\equiv i\,\sigma^{[\mu}\bar{\sigma}^{\nu]}\ ;\ \ \bar{\sigma}^{\mu\nu} \equiv i\,\bar{\sigma}^{[\mu}\sigma^{\nu]}\,.
\end{aligned} \tag{A.36}$$

The Dirac conjugate of a spinor then, in the 2-component notation, reads:

$$\bar{\chi} = (\zeta^\alpha,\,\bar{\xi}_{\dot{\alpha}})\,. \tag{A.37}$$

One can also verify that:

$$(\sigma^\mu)_\alpha{}^{\dot\beta} = (\bar\sigma^\mu)^{\dot\beta}{}_\alpha\ ,\ \ (\sigma^{\mu\nu})_\beta{}^\alpha = (\sigma^{\mu\nu})^\alpha{}_\beta\ ;\ \ (\bar\sigma^{\mu\nu})_{\dot\beta}{}^{\dot\alpha} = (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,,$$
$$\bar\sigma^{\mu\nu} = (\sigma^{\mu\nu})^\dagger = \epsilon\,(\sigma^{\mu\nu})^*\,\epsilon\,, \tag{A.38}$$

where $\epsilon \equiv (\epsilon_{\alpha\beta})$. In these notations the charge conjugation matrix reads:

$$C = -\begin{pmatrix} \epsilon_{\alpha\beta} & \mathbf{0} \\ \mathbf{0} & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}\,, \tag{A.39}$$

so that:

$$\chi_c = \begin{pmatrix} \xi^\alpha \\ \bar\zeta_{\dot\alpha} \end{pmatrix}\,. \tag{A.40}$$

The Lorentz-invariant contraction between two spinors χ_1, χ_2 reads:

$$\bar\chi_1\,\chi_2 = (\zeta_1^\alpha,\,\bar\xi_{1\,\dot\alpha})\begin{pmatrix} \xi_{2\,\alpha} \\ \bar\zeta_2^{\dot\alpha} \end{pmatrix} = \zeta_1^\alpha\,\xi_{2\,\alpha} + \bar\xi_{1\,\dot\alpha}\,\bar\zeta_2^{\dot\alpha} \tag{A.41}$$

The general form of a Majorana spinor is:

$$\chi = \begin{pmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{pmatrix}\,, \tag{A.42}$$

that is the two 2-spinor components ξ and ζ coincide. In the case of two Majorana spinors spinors χ_1, χ_2, one can easily verify that

$$\bar\chi_1\,\chi_2 = \xi_1^\alpha\,\xi_{2\,\alpha} + \bar\xi_{1\,\dot\alpha}\,\bar\xi_2^{\dot\alpha} = \bar\chi_2\,\chi_1\,. \tag{A.43}$$

In curved space $\mathcal{M}_4$, we define *rigid indices* $a, b, c, \dots$, labeling the vierbein $V^a = V^a_\mu$ basis of $T^*\mathcal{M}_4$, or their duals $V_a = V_a^\mu\,\partial_\mu$ in $T\mathcal{M}_4$. The vierbein basis defines the *moving* or *free-falling* frame and satisfies the defining condition:

$$g_{\mu\nu}(x) = V^a_\mu(x)\,V^b_\nu(x)\,\eta_{ab}\,, \tag{A.44}$$

where $g_{\mu\nu}(x)$ is the metric tensor. We define constant γ-matrices γ^a by the condition $\{\gamma^a,\,\gamma^b\} = 2\,\eta^{ab}$. All the above definitions and properties given for flat space, apply to the matrices γ^a, and therefore we just need to replace the *curved indices* $\mu, \nu, \dots$ with rigid ones. We can also define point-dependent matrices $\gamma^\mu(x) \equiv V_a{}^\mu(x)\,\gamma^a$, satisfying

$$\{\gamma^\mu,\,\gamma^\nu\} = 2\,g^{\mu\nu}(x)\,. \tag{A.45}$$

The above properties still apply to these matrices provided we take the dependence of the vierbein in due care. The constant matrix γ^5 is now defined as:

$$\gamma^5 \equiv \frac{i}{4}\,\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = \frac{i\,e}{4}\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\,, \tag{A.46}$$

where $e = \det(V_\mu{}^a) = \sqrt{|\det(g_{\mu\nu})|}$. We will then write:

$$\gamma^5\gamma_\mu = -\frac{ie}{3!}\,\epsilon_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma}\ ;\ \gamma^5\gamma_{\mu\nu} = -\frac{ie}{2}\,\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\,,$$
$$\gamma^5\gamma_{\mu\nu\rho} = i\,e\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\ ;\ \gamma^5\gamma_{\mu\nu\rho\sigma} = ie\,\epsilon_{\mu\nu\rho\sigma}\,, \tag{A.47}$$

and

$$\gamma_{\mu\nu}\gamma^\rho = 2\,\gamma_{[\mu}\delta^\rho_{\nu]} + \gamma_{\mu\nu}{}^\rho = 2\,\gamma_{[\mu}\delta^\rho_{\nu]} + ie\,\epsilon_{\mu\nu}{}^{\rho\sigma}\gamma^5\gamma_\sigma\,,$$
$$\gamma_{\mu\nu}\gamma^{\rho\sigma} = \gamma_{\mu\nu}{}^{\rho\sigma} - 4\,\delta^{[\rho}_{[\mu}\gamma_{\nu]}{}^{\sigma]} - 2\,\delta^{\rho\sigma}_{\mu\nu}\,,$$
$$\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]} = \gamma_{\mu\nu}{}^{\rho\sigma} + 2\,\delta^{\rho\sigma}_{\mu\nu} = 2\,(\delta^{\rho\sigma}_{\mu\nu} + \frac{ie}{2}\,\epsilon_{\mu\nu}{}^{\rho\sigma}\,\gamma^5)\,. \tag{A.48}$$

B Massive Representations of the Supersymmetry Algebra

We have listed in Tables 2, 3 and 4 all possible massive representations with highest spin $s_{MAX} \le 3/2$ for $N \le 8$. We have denoted the spin states by (s) and the number in front of them is their multiplicity. In the fundamental multiplet, with spin $s_0 = 0$ vacuum, the multiplicity of the spin $(N-q-k)/2$ is the dimension of the k-fold antisymmetric Ω-traceless representation of $USp(2(N-q))$. For multiplets with $s_0 \neq 0$ one has to make the tensor product of the fundamental multiplet with the representation of spin s_0. We also indicate if the multiplet is long or short.

C Stability and Asymptotic Stability in the Sense of Liapunov

Let us briefly recall the notion of stability (in the sense of Liapunov) and of attractiveness of an equilibrium point. Given an autonomous dynamical system:

$$\dot{\phi}^r \;=\; f^r(\phi)\,, \tag{C.1}$$

an equilibrium point ϕ_* ($f^r(\phi_*) = 0$), is *attractive* (or an attractor), for $\tau \to -\infty$, if there exist a neighborhood $\mathcal{I}_{\phi_*}$ of ϕ_*, such that all trajectories $\phi^r(\tau, \phi_0)$ originating at $\tau = 0$ in $\phi_0 \in \mathcal{I}_{\phi_*}$ evolve towards ϕ_* as $\tau \to -\infty$:

$$\lim_{\tau\to-\infty}\phi^r(\tau,\phi_0) \;=\; \phi^r_*\ ,\ \ \forall\phi_0 \in \mathcal{I}_{\phi_*}\,. \tag{C.2}$$

An equilibrium point ϕ_* (not necessarily attractive) is *stable* (in the sense of Liapunov) if, for any $\epsilon > 0$, there exist a ball $\mathcal{B}_\delta(\phi_*)$ of radius $\delta > 0$ centered in ϕ_*, such that:

$$\forall\phi_0 \in \mathcal{B}_\delta(\phi_*)\ ,\ \ \forall\tau < 0\ \ :\ \ \phi(\tau,\phi_0) \in \mathcal{B}_\epsilon(\phi_*)\,, \tag{C.3}$$

N	massive spin 3/2 multiplet	long	short
8	none		
6	$2 \times \left[(\frac{3}{2}), 6(1), 14(\frac{1}{2}), 14'(0)\right]$	no	$q = 3$, ($\frac{1}{2}$BPS)
5	$2 \times \left[(\frac{3}{2}), 6(1), 14(\frac{1}{2}), 14'(0)\right]$	no	$q = 2$, ($\frac{2}{5}$BPS)
4	$2 \times \left[(\frac{3}{2}), 6(1), 14(\frac{1}{2}), 14'(0)\right]$	no	$q = 1$, ($\frac{1}{4}$BPS)
	$2 \times \left[(\frac{3}{2}), 4(1), 6(\frac{1}{2}), 4(0)\right]$	no	$q = 2$, ($\frac{1}{2}$BPS)
3	$\left[(\frac{3}{2}), 6(1), 14(\frac{1}{2}), 14'(0)\right]$	yes	no
	$2 \times \left[(\frac{3}{2}), 4(1), 6(\frac{1}{2}), 4(0)\right]$	no	$q = 1$, ($\frac{1}{3}$BPS)
2	$\left[(\frac{3}{2}), 4(1), 6(\frac{1}{2}), 4(0)\right]$	yes	no
	$2 \times \left[(\frac{3}{2}), 2(1), (\frac{1}{2})\right]$	no	$q = 1$, ($\frac{1}{2}$BPS)
1	$\left[(\frac{3}{2}), 2(1), (\frac{1}{2})\right]$	yes	no

Table 2: Massive spin 3/2 multiplets.

that is, provided we take the starting point ϕ_0 sufficiently close to ϕ_*, the entire solution will stay, for all $\tau < 0$, in any given, whatever small, neighborhood of ϕ_*. Finally an equilibrium point is *asymptotically stable* (in the sense of Liapunov) if it is attractive and stable.

Liapunov's Theorem: If there exist a function $v(\phi)$ which is positive definite in a neighborhood of ϕ_* (that is positive in a neighborhood of ϕ_* and $v(\phi_*) = 0$) and such that also the derivative of v along the solution, in the same neighborhood, is positive definite[43]: $\frac{dv}{d\tau} = \dot{\phi}^r \partial_r v > 0$, then ϕ_* is an asymptotically stable equilibrium point or, equivalently, a stable attractor.

For large extremal black holes such function is $v(\phi) = W(\phi) - W(\phi_*) = W(\phi) - \sqrt{|I_4|}$.

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[43] Here we require positive definiteness because our critical point is located at $\tau \to -\infty$ and not at $+\infty$ as in standard textbooks.

N	massive spin 1 multiplet	long	short
8,6,5	none		
4	$2\times\big[(1),4(\frac{1}{2}),5(0)\big]$	no	$q=2$, $(\frac{1}{2}\text{BPS})$
3	$2\times\big[(1),4(\frac{1}{2}),5(0)\big]$	no	$q=1$, $(\frac{1}{3}\text{BPS})$
2	$\big[(1),4(\frac{1}{2}),5(0)\big]$	yes	no
	$2\times\big[(1),2(\frac{1}{2}),(0)\big]$	no	$q=1$, $(\frac{1}{2}\text{BPS})$
1	$\big[(1),2(\frac{1}{2}),(0)\big]$	yes	no

Table 3: Massive spin 1 multiplets.

N	massive spin 1/2 multiplet	long	short
8,6,5,4,3	none		
2	$2\times\big[(\frac{1}{2}),2(0)\big]$	no	$q=1$, $(\frac{1}{2}\text{BPS})$
1	$\big[(\frac{1}{2}),2(0)\big]$	yes	no

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