Lectures on Group Theory for Supergravity and Brane Theorists

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Overview

- We illustrate Group Theory from the stand-point of a Supergravity Theorist.
- These lectures deal only with mathematics and no supergravity is actually discussed, yet the chosen topics are motivated by Supergravity/Brane Theory.
- We deal with both finite and Lie groups in the perspective of their role in differential and complex algebraic geometry.
- We emphasize geometrical/group theoretical structures and conceptions that have been motivated and uncovered by supergravity.
- We aim at conveying the following message: *geometry and groups are fundamental items in brane theories, AdS/CFT and supergravity. Not only: these physical theories have introduced new visions and conceptions in Geometry and this last aspect might turn out to be the most important and durable contribution of Supersymmetry to Science in general*.

Finite Group Theory

Some elements of a theory which is quite old, usually not too much studied by high energy physicists, yet of growing relevance in the supergraivty/brane world

Recalling some fundamental notions

Cayley's theorem

Theorem *Any group G of* finite *order $|G|$ is* isomorphic to a subgroup of the permutation group *on $|G|$ objects, $S_{|G|}$.*

Lagrange's theorem

Theorem *The order of a subgroup H of a finite group G is a divisor of the order of G:*

$$\exists l \in \mathbb{N} \text{ such that } |G| = l\,|H| \ .$$

The integer l is called the index *of H in G.*

Lemma *The only finite group of order p, where p is a prime, is the cyclic group $\mathbb{Z}_p$.*

Left and right cosets

$$a_1 H = \{a_1, a_1 h_2, \ldots a_1 h_m\}$$

$$G = H \cup a_1 H \cup a_2 H \cup \ldots \cup a_l H$$

Order of elements and conjugacy classes

$$a \in G \quad a^h = e$$ ⟹ h = order of a

Corollary 4.2.1 *The* order *of any element of a finite group G is a* divisor *of the order of G.*

Conjugacy classes

$$[g] \equiv \{g' \in G \text{ such that } g' \sim g\} = \{h^{-1} g h\,,\ \text{for } h \in G\}$$

Conjugate subgroups

$$H_g \equiv \left\{h_g \in G :\ h_g = g^{-1} h g\,,\ \text{for } h \in H\right\}|$$

Invariant subgroups

A subgroup H of a group G is called an *invariant* (or *normal*) subgroup if it coincides with all its conjugate subgroups: $\forall g \in G,\ H_g = H$.

Still a few more general concepts

Factor groups

If H is an invariant subgroup of G, then G/H $(= H\backslash G)$ is a group, with respect to the product of classes defined as follows:

$$(g_1 H)(g_2 H) = g_1 g_2 H.$$

The centre of a group

The centre $Z(G)$ of a group G is the set of all those elements of G that commute (in the group sense) with all the elements of G:

$$Z(G) = \left\{ f \in G : g^{-1} f g = f \, , \, \forall g \in G \right\} \, .$$

$Z(G)$ is an abelian subgroup of G

The derived group

The group of commutators, or the *derived group* $\mathscr{D}(G)$ of a group G is the group *generated* by the set of all group commutators in G (that is it contains all group commutators and products thereof).

Solvable Groups, Simple Groups

In general a group G admits a chain of invariant subgroups, called its *subnormal series*[3]:

$$G = G_r \triangleright G_{r-1} \triangleright G_{r-2} \triangleright \ldots \triangleright G_1 \triangleright \{e\} \ ,$$

Definition *G is a* simple *group if it has* no proper invariant subgroup. *For simple groups, the subnormal series is minimal:*

$$G \triangleright \{e\} \ .$$

Definition *A group G is* solvable *if it admits a subnormal series such that all the factor groups G/G_1, G_1/G_2, ..., G_{k-1}/G_k, ... are* abelian.

Semi-direct products

$$G \ltimes K = \{(g,k) : g \in G\,,\ k \in K\}$$

$$(g_1,k_1)(g_2,k_2) = (g_1 \cdot g_2, k_1 \circ g_1(k_2)) \ .$$

The definition requires that the group G should act on the group K as a transformation group.

Linear Representations

Definition 4.3.1 *Let G be a group and let V be a vector space of dimension n. Any homomorphism:*

$$D : \quad G \rightarrow \mathrm{Hom}(V,V)$$

is named a linear representation of dimension n of the group G.

If the vectors $\{\mathbf{e}_i\}$ form a basis of the vector space V, then each group element $\gamma \in G$ is mapped into a $n \times n$ matrix $D_{ij}(\gamma)$ such that:

$$D(\gamma).\mathbf{e}_i = D_{ij}(\gamma)\mathbf{e}_j$$

In other words $D : \quad G \rightarrow \mathrm{GL}(n,\mathbb{F}) \qquad \mathbb{F} = \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$

Definition *Let* $D : G \rightarrow \mathrm{Hom}(V,V)$ *be a linear representation of a group G. A vector subspace* $W \subset V$ *is said to be* **invariant** *iff:*

$$\forall \gamma \in G, \forall \mathbf{w} \in W : \quad D(\gamma).\mathbf{w} \in W$$

Definition *A linear representation* $D : G \rightarrow \mathrm{Hom}(V,V)$ *of a group G is named* **irreducible** *iff the only invariant subspaces of V are* **0** *and V itself.*

Reducible Representations

$$V = \bigoplus_{i=1}^{r} W_i \qquad D = \bigoplus_{i=1}^{r} D_i \quad ; \quad D_i : G \to \mathrm{Hom}(W_i, W_i)$$

$$\forall \gamma \in G \quad ; \quad D(\gamma) = \left(\begin{array}{c|c|c|c|c|c} D_1(\gamma) & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & D_1(\gamma) & 0 & \dots & \dots & 0 \\ \hline \vdots & \dots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 & D_{r-2}(\gamma) & 0 \\ \hline 0 & \dots & \dots & \dots & 0 & D_r(\gamma) \end{array}\right)$$

Schurs's Lemmas

Lemma *Let V, W be two vector spaces of dimension n and m respectively, with $n > m$. Let G be a finite group and let $D_1 : G \to \mathrm{Hom}(V,V)$ and $D_2 : G \to \mathrm{Hom}(W,W)$ be two irreducible representations of dimension n and m respectively. Consider a linear map $\mathscr{A} : W \to V$ and impose the constraint*

$$\forall \gamma \in G \quad \forall \mathbf{w} \in W \qquad D_1(\gamma)\mathscr{A}.\mathbf{w} = \mathscr{A}.D_2(\gamma).\mathbf{w}$$

The only element $\mathscr{A} \in \mathrm{Hom}(W,V)$ that satisfies eq. (4.3.7) is $\mathscr{A} = 0$

Lemma *Let $D : G \to \mathrm{Hom}(V,V)$ be an n-dimensional irreducible representation of a finite group G. Let $C \in \mathrm{Hom}(V,V)$ be such that :*

$$\forall \gamma \in G \quad C\,D(\gamma) = D(\gamma)C$$

Then $C = \lambda\,\mathbf{1}$ where $\lambda \in \mathbb{C}$ and $\mathbf{1}$ is the identity map of the n-dimensional vector space V into itself.

Characters

Definition *Let $D : G \to Hom(V,V)$ a linear representation of a finite group G of dimension $n = dimV$. Let r be the number of conjugacy classes $\mathscr{C}_i$ into which the whole group is split:*

$$G = \bigcup_{i=1}^{r} \mathscr{C}_i \quad ; \quad \mathscr{C}_i \bigcap \mathscr{C}_j = \delta_{ij} \mathscr{C}_j$$
$$\forall \gamma, \tilde{\gamma} \in \mathscr{C}_i \quad \exists g \in G \quad / \quad \tilde{\gamma} = g \gamma g^{-1}$$

*We name **character** of the representation D the following r-dimensional vector:*

$$\chi[D] = \{\mathrm{Tr}\left[D(\gamma_1)\right], \mathrm{Tr}\left[D(\gamma_2)\right], \ldots, \mathrm{Tr}\left[D(\gamma_r)\right]\}$$

where:

$$\gamma_i \in \mathscr{C}_i$$

is any set of representatives of the r conjugacy classes.

Character orthogonality relations

$$
\begin{aligned}
g &= |\mathrm{G}| = \#\text{ of group elements} \\
g_i &= |\mathcal{C}_i| = \#\text{ of group elements in the conjugacy class } \mathcal{C}_i \quad i = i, \dots, r
\end{aligned}
$$

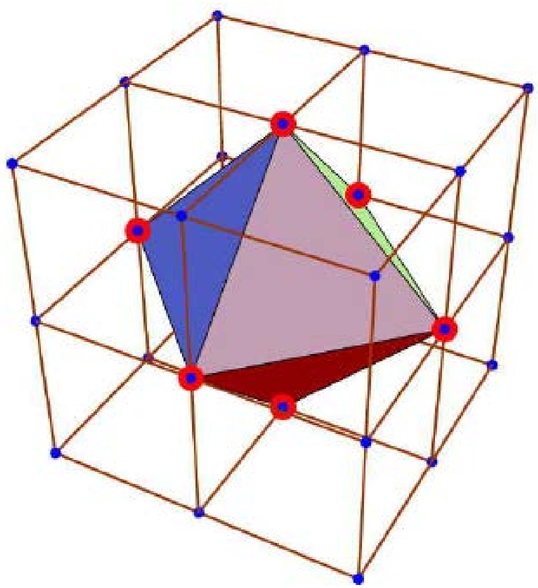
$$\sum_{\mu=1}^{r} n_\mu^2 = g \qquad \sum_{\mu=1}^{r} \chi_i^\mu \chi_j^\mu = \frac{g}{g_i} \delta_{ij} \qquad \sum_{i=1}^{r} g_i \chi_i^\mu \chi_i^\nu = g \, \delta^{\mu\nu}$$

$$\mathfrak{R} = \oplus_{\mu=1}^{r} a_\mu D^\mu$$

Where a_μ denotes the number of times the irrep D^μ is contained in the direct sum and it is named the *multiplicity*. Given the character vector of any considered representation $\mathfrak{R}$ the vector of its multiplicities is

$$a_\mu = \frac{1}{g} \sum_{i}^{r} g_i \chi_i^{\mathfrak{R}} \chi_i^\mu$$

Example: the octahedral Group



e	$1_1 = \{x,y,z\}$
C_3	$2_1 = \{-y,-z,x\}$
	$2_2 = \{-y,z,-x\}$
	$2_3 = \{-z,-x,y\}$
	$2_4 = \{-z,x,-y\}$
	$2_5 = \{z,-x,-y\}$
	$2_6 = \{z,x,y\}$
	$2_7 = \{y,-z,-x\}$
	$2_8 = \{y,z,x\}$
C_4^2	$3_1 = \{-x,-y,z\}$
	$3_2 = \{-x,y,-z\}$
	$3_3 = \{x,-y,-z\}$

C_2	$4_1 = \{-x,-z,-y\}$
	$4_2 = \{-x,z,y\}$
	$4_3 = \{-y,-x,-z\}$
	$4_4 = \{-z,-y,-x\}$
	$4_5 = \{z,-y,x\}$
	$4_6 = \{y,x,-z\}$
C_4	$5_1 = \{-y,x,z\}$
	$5_2 = \{-z,y,x\}$
	$5_3 = \{z,y,-x\}$
	$5_4 = \{y,-x,z\}$
	$5_5 = \{x,-z,y\}$
	$5_6 = \{x,z,-y\}$

There are 24 rotations of three dimensional space that map the octahedral into itself. They form a group O_{24} that consists of 5 conjugacy classes displayed above.

Structure of the octahedral group

Abstractly the octahedral Group $O_{24} \sim S_{24}$ is isomorphic to the symmetric group of permutations of 4 objects. It is defined by the following generators and relations:

$$A, B \quad : \quad A^3 = \mathbf{e} \quad ; \quad B^2 = \mathbf{e} \quad ; \quad (B\,A)^4 = \mathbf{e}$$

$$A = 2_8 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad B = 4_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The group O_{24} is solvable

$$O_{24} \triangleright N_{12} \triangleright N_4$$

$$N_{12} \equiv \{1_1, 2_1, 2_2, \ldots, 2_8, 3_1, 3_2, 3_3\}$$
$$N_4 \equiv \{1_1, 3_1, 3_2, 3_3\}$$

$$N_4 \sim \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\frac{O_{24}}{N_{12}} \sim \mathbb{Z}_2 \quad ; \quad \frac{N_{12}}{N_4} \sim \mathbb{Z}_3$$

Irreps of O_{24}

$$\dim D_1 = 1 \; ; \; \dim D_2 = 1 \; ; \; \dim D_3 = 2 \; ; \; \dim D_4 = 3 \; ; \; \dim D_5 = 4$$

D_1 : the identity representation $\quad \forall \gamma \in O_{24} : \quad D_1(\gamma) = 1$

$$\chi_1 = \{1,1,1,1,1\}$$

D_2 : the quadratic Vandermonde representation $\quad \mathfrak{V}(x,y,z) = (x^2-y^2)(x^2-z^2)(y^2-z^2)$

$$\chi_2 = \{1,1,1,-1,-1\}$$

D_3 : the two-dimensional representation $\quad D_3 : \quad O_{24} \to SL(2,\mathbb{Z})$

$$D_3(A) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad ; \quad D_3(B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\chi_3 = \{2,-1,2,0,0\}$$

Irreps of O_{24} continued

D_4 **: the three-dimensional defining representation**

$$D_4(\mathrm{A}) = \mathrm{A} \quad ; \quad D_4(\mathrm{B}) = \mathrm{B}$$

$$\chi_3 = \{3, 0, -1, -1, 1\}$$

D_5 **: the three-dimensional unoriented representation**

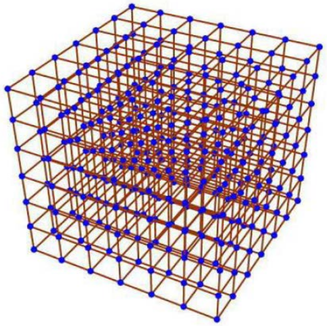
$$D_5(\mathrm{A}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad D_5(\mathrm{B}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\chi_5 = \{3, 0, -1, 1, -1\}$$

Character table of the octahedral group

Irrep \ Class	$\{\mathbf{e},1\}$	$\{C_3,8\}$	$\{C_4^2,3\}$	$\{C_2,6\}$	$\{C_4,6\}$
$D_1\,,\quad \chi_1 =$	1	1	1	1	1
$D_2\,,\quad \chi_2 =$	1	1	1	-1	-1
$D_3\,,\quad \chi_3 =$	2	-1	2	0	0
$D_4\,,\quad \chi_4 =$	3	0	-1	-1	1
$D_5\,,\quad \chi_5 =$	3	0	-1	1	-1

Crystallographic Groups



$$g_{\mu\nu} = \langle \mathbf{w}_\mu \,, \mathbf{w}_\nu \rangle$$

$$\mathrm{SO_g(n)} = \mathscr{S}\,\mathrm{SO(n)}\,\mathscr{S}^{-1}$$

$$\mathscr{S}^T\, g\, \mathscr{S} = \mathbf{1}$$

$$\mathbb{R}^n \ni \mathbf{q} \in \Lambda \Leftrightarrow \mathbf{q} = q^\mu\, \mathbf{w}_\mu \quad \text{where} \quad q^\mu \in \mathbb{Z}$$

Lattice

Definition *An abstract group Γ is named crystallographic in n-dimensions if there exists an n-dimensional lattice Λ_n with basis vectors $\mathbf{w}_\mu$ such that:*

1. *there is a isomorphism:*

$$\omega : \Gamma \to H \subset \mathrm{SO_g}(n)$$

   where $\mathrm{SO_g}(n)$ is the conjugate of the n-dimensional group rotation group respecting a metric g
2. *the metric g is that defined by the basis vectors of the lattice Λ_n*
3. *all elements of H are $n \times n$ matrices with integer valued entries.*

This is equivalent to the statement that Γ has an orthogonal action in $\mathbb{R}^n$ and preserves the lattice Λ_n.

The ADE classification of SU(2) finite subgroups

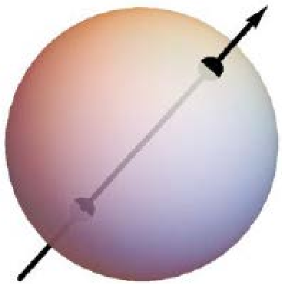
The problem of classifying finite rotation groups is Plato's problem of regular solids. This problem is isomorphic to the classification of simple Lie algebras, of modular invariant CFTs, of Arnold simple singularities and of ALE manifolds, namely of gravitational instantons.

It is one of the most profound items in the whole field of Mathematics and admits generalizations under the name of McKay correspondence.

It plays an important role in supergravirty and AdS/CFT theories, also in modern algebraic geometry.

SO(3) & SU(2) finite subgroups: preliminaries

Every rotation has an axis



$$\ell^2 + m^2 + n^2 = 1$$

$$A_{\ell,m,n} = \begin{pmatrix} 0 & -n & m \\ n & 0 & -\ell \\ -m & \ell & 0 \end{pmatrix} = -A^T_{\ell,m,n}$$

$$\mathcal{O}_{(\ell,m,n)} = \exp[\theta A_{\ell,m,n}] = \mathbf{1} + \sin\theta\, A_{\ell,m,n} + (1-\cos\theta)A^2_{\ell,m,n}$$

Homomorphism SU(2) into SO(3)

$$\forall\, \mathcal{U} \in \mathrm{SU}(2) : \omega[\mathcal{U}] = \mathcal{O} \in \mathrm{SO}(3) \quad / \quad \mathcal{O}_x{}^y = \tfrac{1}{2}\mathrm{Tr}\left[\mathcal{U}^\dagger \sigma_x \mathcal{U} \sigma^y\right]$$

We consider the preimage of the rotation in SU(2)

$$\omega\left[\mathcal{U}^{\pm}_{\ell,m,n}\right] = \mathcal{O}_{(\ell,m,n)} \qquad \mathcal{U}^{\pm}_{\ell,m,n} = \pm\begin{pmatrix} \rho + \mathrm{i}\nu & \mu - \mathrm{i}\lambda \\ -\mu - \mathrm{i}\lambda & \rho - \mathrm{i}\nu \end{pmatrix}$$

where

$$\lambda = \ell \sin\frac{\theta}{2} \quad ; \quad \mu = m \sin\frac{\theta}{2} \quad ; \quad \nu = n\sin\frac{\theta}{2} \quad ; \quad \rho = \cos\frac{\theta}{2}$$

Every SU(2) element has two poles

Each element $\mathscr{U} \in \mathrm{SU}(2)$ has two eigenvectors $\mathbf{z}_1$ and $\mathbf{z}_2$, such that

$$\mathscr{U}\,\mathbf{z}_1 = \exp[\mathrm{i}\theta]\mathbf{z}_1$$
$$\mathscr{U}\,\mathbf{z}_2 = \exp[-\mathrm{i}\theta]\,\mathbf{z}_2$$

$\exists$ orthogonal basis where $\mathscr{U}$ is diagonal and given by:

$$\mathscr{U} = \begin{pmatrix} \exp[\mathrm{i}\theta] & 0 \\ 0 & \exp[-\mathrm{i}\theta] \end{pmatrix}$$

Poles of $\mathscr{U} \in \mathrm{SU}(2)$

the rays $\{\lambda\,\mathbf{z}_1\}$ and $\{\mu\,\mathbf{z}_2\}$ where $\lambda, \mu \in \mathbb{C}$

$$p_i \equiv \{\lambda\,\mathbf{z}_i\}$$

Let $H \subset \mathrm{SO}(3)$ be a finite, discrete subgroup of the rotation group and let $\hat{H} \subset \mathrm{SU}(2)$ be its pre-image in $\mathrm{SU}(2)$ with respect to the homomorphism ω. Then the order of H is some positive integer number:

$$|H| = n \in \mathbb{N}$$

The total number of poles associated with H is:

$$\#\text{of poles} = 2n - 2$$

since $n-1$ is the number of elements in H that are different from the identity.

Formulating the argument

Poles are equivalent if they are mapped one into the other by the group H

$$p_i \sim p_j \quad \text{iff} \quad \exists \gamma \in H \,/\, \gamma p_i = p_j$$

Equivalence classes

$$\mathcal{Q}_\alpha = \{p_1^\alpha, \dots, p_{m_\alpha}^\alpha\} \quad ; \quad \alpha = 1, \dots, r$$

name m_α the cardinality of the orbit class $\mathcal{Q}_\alpha$

Stability subgroups of the classes

Each pole $p \in \mathcal{Q}_\alpha$ has a stability subgroup $K_p \subset H$:

$$\forall h \in K_p \quad : \quad h\, p = p$$

that is finite, abelian and cyclic of order k_α

$$H = K_p + v_1 K_p + \dots + v_{m_\alpha} K_p \quad m_\alpha \in \mathbb{N}$$

$$|K_{p_i}| = k_\alpha$$

$$\forall \mathcal{Q}_\alpha \quad ; \quad k_\alpha m_\alpha = n$$

(counting coincidences)

#of poles in $\mathcal{Q}_\alpha$ $= m_\alpha\,(k_\alpha - 1)$

$$2n - 2 = \sum_{\alpha=1}^{r} m_\alpha\,(k_\alpha - 1)$$

The Diophantine inequality

$$2\left(1-\frac{1}{n}\right)=\sum_{\alpha=1}^{r}\left(1-\frac{1}{k_\alpha}\right)$$

$$\forall \mathcal{Q}_\alpha \quad ; \quad k_\alpha m_\alpha = n$$

$$\Rightarrow \quad 2\left(1-\frac{1}{n}\right)=\sum_{\alpha=1}^{r}\left(1-\frac{1}{k_\alpha}\right)$$

$$r+\frac{2}{n}-2=\sum_{\alpha=1}^{r}\frac{1}{k_\alpha}$$

Since $k_\alpha \geq 2.$ $\quad r+\frac{2}{n}-2 \leq \frac{r}{2}$

Only two cases

$r = 2 \quad \text{or} \quad r = 3$


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
AVCTORE CLAVDIO GASPARE BACHETO
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobaea, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

The A solutions

Case $r = 2$: the infinite series of cyclic groups $\mathbb{A}_n$

$$\frac{2}{n} = \frac{1}{k_1} + \frac{1}{k_2} \qquad k_{1,2} \leq n \quad \Rightarrow \quad k_1 = k_2 = n$$

$$\mathscr{A} \in \mathrm{SU}(2) \quad : \quad \mathscr{Z} \equiv \mathscr{A}^n$$

$$\mathbb{Z}_{2n} \sim \Gamma_b[n,n,1] = \{\mathbf{1}, \mathscr{A}, \mathscr{A}^2, \ldots, \mathscr{A}^{n-1}, \mathscr{Z}, \mathscr{Z}\mathscr{A}, \mathscr{A}^2, \ldots, \mathscr{Z}\mathscr{A}^{n-1}\}$$

$$\omega[\Gamma_b[n,n,1]] = \Gamma[n,n,1] \sim \mathbb{Z}_n$$

$$\mathbb{A}_n \quad \Leftrightarrow \quad \begin{cases} \Gamma_b[n,n,1] = (\mathscr{A}, \mathscr{Z} \mid \mathscr{A}^n = \mathscr{Z} \quad ; \quad \mathscr{Z}^2 = \mathbf{1}) \\ \Gamma[n,n,1] = \qquad (\mathrm{A} \mid \mathrm{A}^n = \mathbf{1}) \end{cases}$$

Structure of the DE solutions

Case $r = 3$ ***and its solutions***

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1 + \frac{2}{n}$$

$$\mathscr{R} = 1 + \sum_{\alpha}^{r} k_\alpha$$

$$\Gamma_b\,[k_1,k_2,k_3] = \left(\mathscr{A},\mathscr{B},\mathscr{Z} \mid (\mathscr{A}\mathscr{B})^{k_1} = \mathscr{A}^{k_2} = \mathscr{B}^{k_3} = \mathscr{Z}\,;\,\mathscr{Z}^2 = \mathbf{1}\right)$$

$$\Gamma\,[k_1,k_2,k_3] = \left(\mathrm{A},\mathrm{B} \mid (\mathrm{AB})^{k_1} = \mathrm{A}^{k_2} = \mathrm{B}^{k_3} = \mathbf{1}\right)$$

There are in Γ elements of order at most of three different types

D-solutions

The solution $(k,2,2)$ **and the dihedral groups** Dih_k

$$\{k_1,k_2,k_3\} = \{k,2,2\} \quad ; \quad 2 < k \in \mathbb{Z}$$

The corresponding subgroups of SU(2) and SO(3) are:

$$\mathrm{Dih}_k \quad \Leftrightarrow \quad \begin{cases} \Gamma_b[k,2,2] = (\mathscr{A},\mathscr{B},\mathscr{Z} \mid (\mathscr{A}\mathscr{B})^k = \mathscr{A}^2 = \mathscr{B}^2 = \mathscr{Z}\,; \\ \qquad\qquad\qquad\qquad \mathscr{Z}^2 = \mathbf{1}) \\ \Gamma[k,2,2] = \qquad (\mathrm{A},\mathrm{B} \mid (\mathrm{AB})^k = \mathrm{A}^2 = \mathrm{B}^2 = \mathbf{1}) \end{cases}$$

$\Gamma_b[k,2,2] \simeq \mathrm{Dih}_k^b$ is the binary dihedral subgroup. Its order is

$$|\,\mathrm{Dih}_k^b\,| = 4k$$

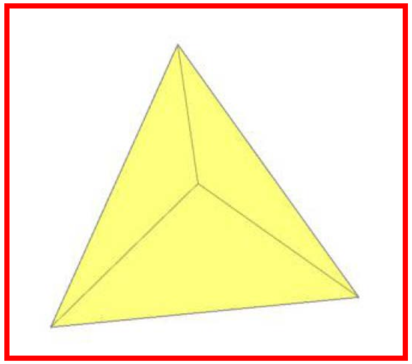
The E-solutions

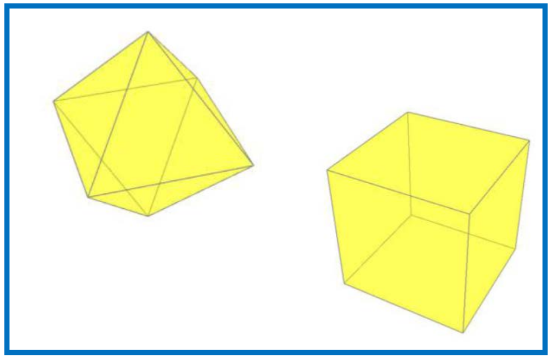
$$\{k_1,k_2,k_3\} = \{3,3,2\} \qquad \Gamma[3,3,2] \simeq \mathrm{T}_{12}$$

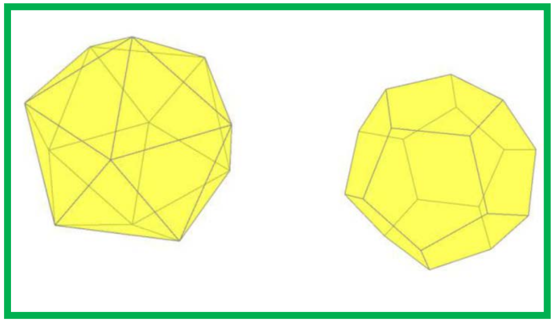
$$\{k_1,k_2,k_3\} = \{4,3,2\} \qquad \Gamma[4,3,2] \simeq \mathrm{O}_{24}$$

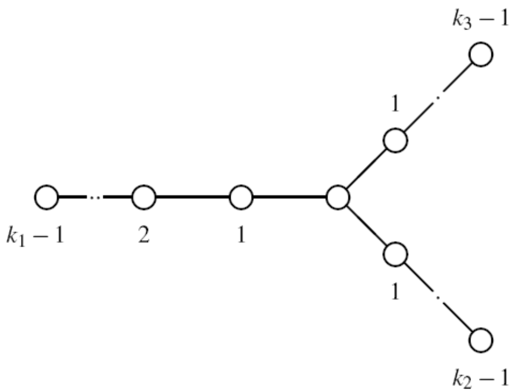
$$\{k_1,k_2,k_3\} = \{5,3,2\} \qquad \Gamma[5,3,2] \simeq \mathrm{I}_{60}$$

1. Tetrahedral
2. Octahedral
3. Icosahedral









The ADE classification

	Simple Lie Algebras	Finite subgroups of $\Gamma_b \subset \mathrm{SU}(2)$
r	number of simple chains in the Dynkin diagram	# of different types of group–element orders present in $\Gamma \equiv \omega[\Gamma_b]$
k_α	$k_\alpha - 1$ = lengths of the simple chains in the Dynkin diagram	group-element orders in $\Gamma \equiv \left(\mathrm{A},\mathrm{B} \mid (\mathrm{AB})^{k_1} = \mathrm{A}^{k_2} = \mathrm{B}^{k_3} = \mathbf{1}\right)$
$\mathscr{R} - 1 \equiv \sum_{\alpha=1}^{r}(k_\alpha - 1)$	$\mathscr{R}$ = rank of the Lie algebra	$\mathscr{R} + 1$ = # of conjugacy classes in Γ_b

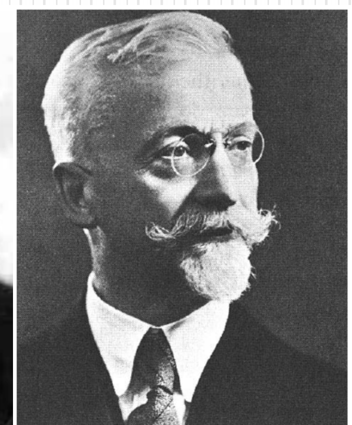
The classification of simple Lie algebras

The classification of semisimple Lie algebras is based on root spaces and Dynkin diagrams. Dynkin-Cartan theory, in all of its aspects, is of crucial relevance in many directions of Mathematical Physics and it is absolutely essential in Supergravity and Brane Theories.

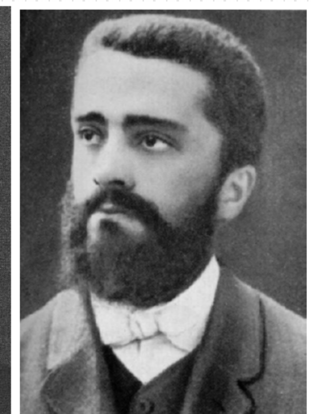
Sophus Lie

Felix Klein

Wilhelm Killing



Elie Cartan



Eugenio Levi

Cartan-Weyl basis of a complex Lie algebra

$$\kappa(H_i, H_j) = \delta_{ij} \quad \Rightarrow \quad H_\alpha = \alpha_i H_i$$
$$\kappa\left(E^\alpha, E^{-\alpha}\right) = 1$$
$$\kappa(H_i, E^\alpha) = 0$$

Cartan Killing metric

$$[H_i, H_j] = 0$$
$$[H_i, E^\alpha] = \alpha_i E^\alpha$$
$$\left[E^\alpha, E^{-\alpha}\right] = \alpha^i H_i$$
$$\left[E^\alpha, E^\beta\right] = N(\alpha,\beta)\, E^{\alpha+\beta} \qquad \text{if} \qquad \alpha+\beta \in \Delta$$
$$\left[E^\alpha, E^\beta\right] = 0 \qquad \text{if} \qquad \alpha+\beta \notin \Delta$$

The generators H_i span the Cartan subalgebra CSA (maximal abelian whose adjoint action is diagonalizable). The dimension **r** of the CSA is named the *rank*

Hermann Weyl

Δ is a finite collection of vectors in a r-dimensional Euclidian space

Theorem *If $\alpha, \beta \in \Delta$ are two roots, then the following two statements are true:*

1. $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$
2. $\sigma_\alpha(\beta) \equiv \beta - 2\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \Delta$ *is also a root.*

The elements $\alpha \in \Delta$ are named the ***roots***

Axiomatization of Root Systems

Definition *Let $\mathbb{E}$ be an euclidian space of dimension ℓ. A subset $\Delta \subset \mathbb{E}$ is named a* ***root system*** *if:*

1. *Δ is finite, spans $\mathbb{E}$ and does not contain $\mathbf{0}$.*
2. *If $\alpha \in \Delta$ the only multiples of α contained in Δ are $\pm\alpha$*
3. *$\forall \alpha, \beta \in \Delta$ we have $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$*
4. *$\forall \alpha, \beta \in \Delta$ we have $\sigma_\alpha(\beta) \equiv \beta - 2\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \Delta$*

Angles between the roots

$$\langle \beta , \alpha \rangle \equiv 2\alpha \frac{(\beta , \alpha)}{(\alpha , \alpha)} \quad \Rightarrow \quad \langle \beta , \alpha \rangle \langle \alpha , \beta \rangle = 4\cos^2\theta_{\alpha\beta} \in \mathbb{Z}$$

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\frac{\Vert\beta\Vert^2}{\Vert\alpha\Vert^2}$
0	0	$\frac{\pi}{2}$	undetermined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Lemma 9.1. *Let $\alpha,\beta\in\Delta$ be two non proportional roots. If $(\alpha,\beta)>0$ then $\alpha-\beta\in\Delta$ is a root. If $(\alpha,\beta)<0$ then $\alpha+\beta\in\Delta$ is a root.*

Simply laced Lie algebras

Non simply laced Lie algebras

Simple roots

Lemma *Let $\alpha, \beta \in \Delta$ be two non proportional roots. If $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Delta$ is a root. If $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Delta$ is a root.*

Definition *Given a root system $\Delta \subset \mathbb{E}^\ell$ in an euclidian space of dimension ℓ, a set Δ of exactly ℓ roots is named a simple root basis if:*

1. *Δ is a basis for the entire $\mathbb{E}^\ell$.*
2. *Every root $\alpha \in \Delta$ can be written as a linear combination of the elements α_i whose coefficients are either all positive or all negative integers*

$$\alpha = \sum_{i=1}^{\ell} k^i \alpha_i \quad ; \quad k^i \in \begin{cases} \mathbb{Z}_+ \\ \text{or } \mathbb{Z}_- \end{cases}$$

The vectors α_i comprised in Δ are named the simple roots of Δ.

Theorem *Every root system Δ admits a basis of simple roots.*

Weyl Group and Cartan matrix

Definition *Let Δ be a root system in dimension ℓ. The Weyl group of Δ, denoted $\mathscr{W}(\Delta)$ is the finite group generated by the reflections σ_α, $\forall \alpha \in \Delta$.*

Since for any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{E}$ we have:

$$(\sigma_\alpha(\mathbf{v}), \sigma_\alpha(\mathbf{w})) = (\mathbf{v}, \mathbf{w})$$

it follows that the Weyl group, which is finite, is always a finite subgroup of the rotation group in ℓ dimensions:

$$\mathscr{W}(\Delta) \subset \mathrm{SO}(\ell)$$

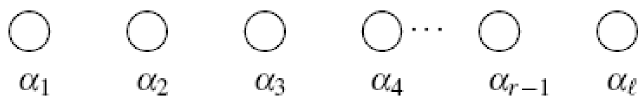
Having established that all possible irreducible root systems Δ are uniquely determined (up to isomorphisms) by the Cartan matrix:

$$C_{ij} = < \alpha_i , \alpha_j > \equiv 2 \frac{(\alpha_i , \alpha_j)}{(\alpha_j , \alpha_j)}$$

we can classify *all the complex simple Lie algebras* by classifying all possible Cartan matrices.

Coxeter-Dynkin diagrams

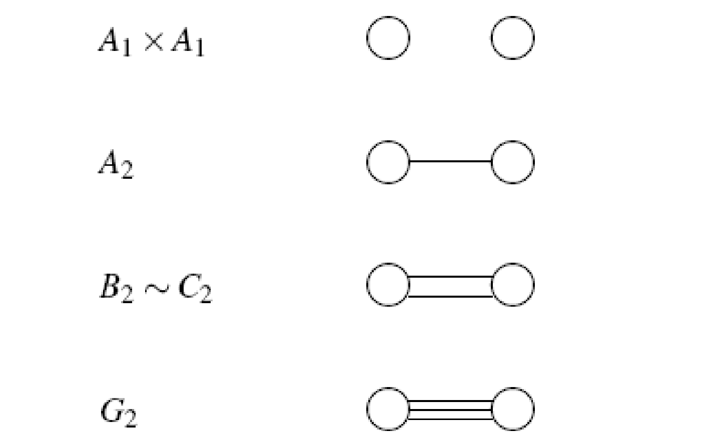
Simple roots



# of lines joining α_i with α_j

$$< \alpha_i \, , \alpha_j > < \alpha_j \, , \alpha_i > = 4 \cos^2 \theta_{ij} = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$$

Coxeter graphs



The four possible Coxeter graphs with two vertices

Eugene Dynkin Harold Coxeter

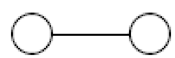
Dynkin diagrams

Add an arrow

By convention we decide that this *arrow points* in the direction of the *short root*.

A Coxeter graph equipped with the necessary arrows is named a **Dynkin diagram**.

$A_1 \times A_1$  $= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

A_2  $= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

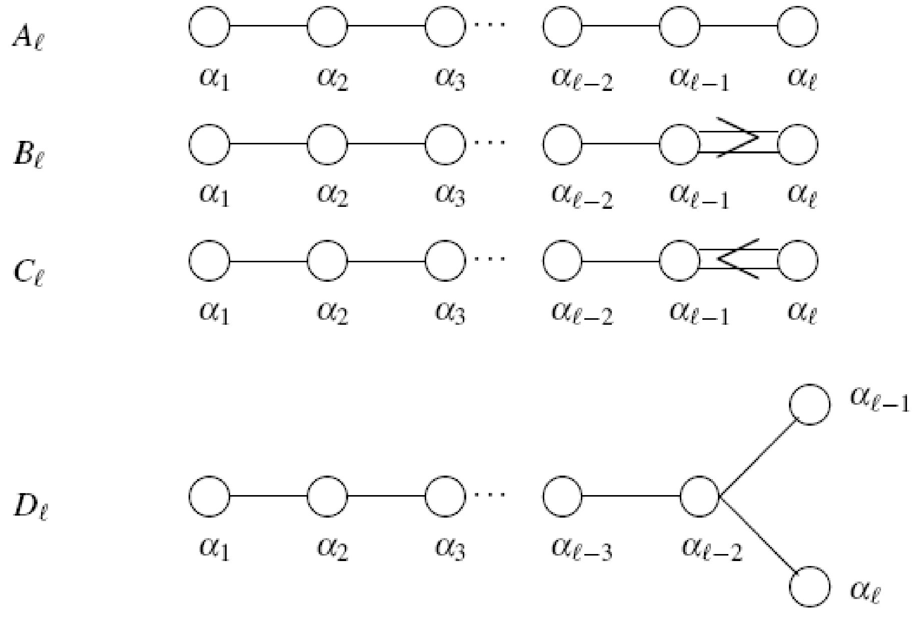
B_2  $= \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

C_2  $= \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

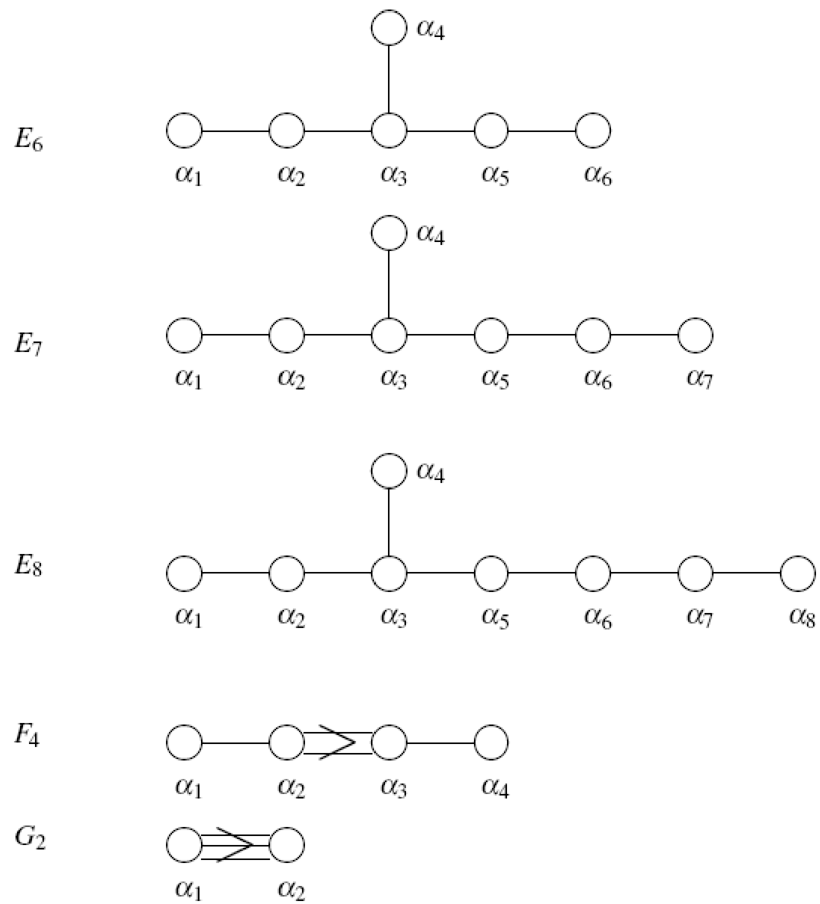
G_2  $= \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Classification Theorem

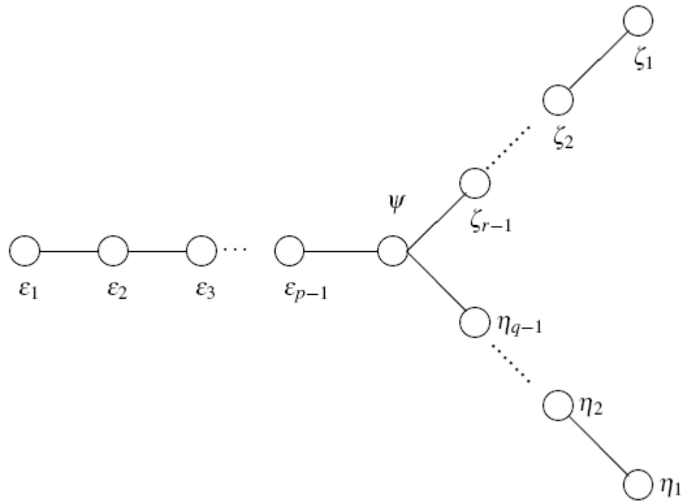
The possible Dynkin diagram and hence the simple complex Lie algebras are given by the following infinite series



or by these 5 exceptional cases



The ADE classification of simply laced Lie algebras



Elaborating the consequences of the axioms one reduces possible simply laced diagrams to the form on the right and finds the diophantine inequality

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$$

$$(p,q,r)=\begin{cases}(\ell,1,1) & \Rightarrow A_\ell \text{ Dynkin diagrams} \quad \ell\in\mathbb{N}\\ (\ell-2,2,2) & \Rightarrow D_\ell \text{ Dynkin diagrams } 4\le\ell\in\mathbb{N}\\ (3,3,2) & \Rightarrow E_6 \text{ Dynkin diagram}\\ (4,3,2) & \Rightarrow E_7 \text{ Dynkin diagram}\\ (5,3,2) & \Rightarrow E_8 \text{ Dynkin diagram}\end{cases}$$

Identification of the Classical Lie Algebras

$$A_r = SL(r+1,C)$$
$$B_r = SO(2r+1,C)$$
$$C_r = Sp(2r,C)$$
$$D_r = SO(2r,C)$$

The weight lattice and linear representations

$$\Lambda_{\text{root}} \subset \mathbb{R}^{\ell} \quad / \quad \mathbf{v} \in \Lambda_{\text{root}} \Leftrightarrow \mathbf{v} = n^i \alpha_i \quad , \quad n^i \in \mathbb{Z}$$

$$\Lambda_{\text{weight}} \equiv \Lambda^{\star}_{\text{root}}$$

Fundamental weights $\langle \lambda^i , \alpha_j \rangle \equiv 2 \frac{(\lambda^i , \alpha_j)}{(\alpha_j , \alpha_j)} = \delta^i_j$

For any representation Γ of the Lie Algebra $\mathbb{G}_c$ $\quad \Gamma : \mathbb{G}_c \Longrightarrow \mathrm{Hom}(V,V)$

$\Gamma(H_i) = -\mathrm{i}\,\Gamma(\mathrm{i}H_i) \quad ; \quad (i = 1, \ldots, \ell)$ are a set of ℓ *commuting hermitian matrices.*

$$V = \bigoplus_{\lambda \in \Pi_{\Gamma}} V_{\lambda} \quad ; \quad (\mathrm{dim} V_{\lambda} \geq 1)$$

where $\Pi_{\Gamma} \subset \mathscr{H}^{\star}$ is a subset of linear functionals on the Cartan subalgebra $\mathscr{H}$

$$\lambda : \mathscr{H} \to \mathbb{C}$$

Weigths of a representations

$$\forall \mathbf{v} \in V_\lambda \qquad \Gamma(h)\,\mathbf{v} = \lambda(h)\,\mathbf{v} \quad ; \quad \forall h \in \mathscr{H}$$

$$m(\lambda) = \dim V_\lambda \qquad n = \dim V = \sum_{\lambda \in \Pi_\Gamma} m(\lambda)$$

$$\lambda_i = \lambda(H_i)$$

Dirac notation (bra and kets)

$$|\lambda, p > \; ; \; p = 1, \dots, m(\lambda)$$

$$< \mu, r | \lambda, p > = \delta_{\lambda\mu}\, \delta_{rp}$$

$$\Gamma(h) | \lambda, p > = \lambda(h) | \lambda, p >$$

$$H_\alpha |\lambda, i > = \lambda(H_\alpha) |\lambda, i >$$

$$\lambda(H_\alpha) = 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \equiv < \lambda, \alpha > \in \mathbb{Z}$$

$$\Downarrow$$

$$\lambda \in \Lambda_{\text{weight}}$$

One fundamental result of Weyl. The weights of a representation belong to the weight lattice.

Another fundamental result of Weyl

Theorem *Given any finite dimensional representation Γ of a simple Lie algebra $\mathbb{G}$, the set of its weights Π_Γ is invariant with respect to the Weyl group*
Explicitly we have:

$$\forall \alpha \in \Delta \quad : \quad \lambda \in \Pi_\Gamma \Rightarrow \sigma_\alpha(\lambda) \in \Pi_\Gamma$$
$$m(\sigma_\alpha(\lambda)) = m(\lambda)$$

Theorem *Let α be a non vanishing root and let $\lambda \in \Pi_\Gamma$ be a weight of a representation Γ of a semisiple Lie algebra $\mathbb{G}$. Then there exist two integers p,q, such that $\lambda + k\alpha \in \Pi_\Gamma$ is a weight of the representation for all the integers $-p \leq k \leq q$. Furthermore we have:*

$$p - q = < \lambda \, , \, \alpha > \equiv 2 \frac{(\lambda, \alpha)}{(\alpha, \alpha)}$$

Utilizing these two theorems the construction of all the weights of a representation can be easily encoded into an iterative computer algorithm starting from a highest weight

Highest weight representations

Definition *Given two weights λ, μ of an irreducible representation Γ of a simple Lie algebra $\mathbb{G}$ we say that λ is larger than μ, if and only if their difference is a sum of positive roots*

$$\lambda \succ \mu \quad \Leftrightarrow \quad \lambda - \mu = \sum_{\alpha_k \in \Delta_+} \alpha_k$$

Theorem *If the irreducible representation Γ of a simple Lie algebra $\mathbb{G}$ is finite-dimensional than the set of weights Π_Γ admits a maximal weight Λ such that any other weight $\lambda \in \Pi_\Gamma$ is $\lambda \prec \Lambda$. Consequently we have:*

$$\Gamma(E^\alpha)|\Lambda >= 0$$

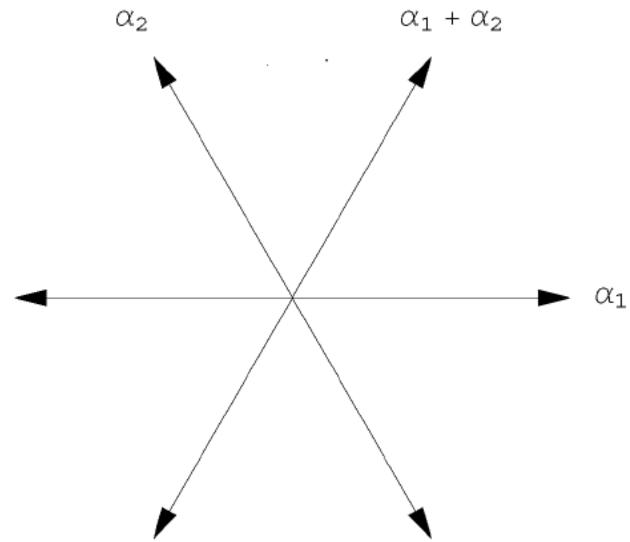
Corollary *The highest weights of all existing irreducible representations belong to the* ***Weyl chamber*** *$\mathfrak{W}$ which is the convex hull defined by the following conditions:*

$$\mathbf{v} \in \mathfrak{W} \quad \Leftrightarrow \quad 2\frac{(\mathbf{v}, \alpha_i)}{(\alpha_i, \alpha_i)} > 0 \quad i = 1, \ldots, \ell$$

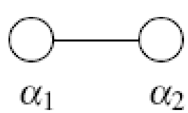
where α_i are the simple roots.

Example with A_2

$$\Delta_{A_2} = \begin{cases} \alpha_1 = \left(\sqrt{2}, 0\right) \ , \\ \alpha_2 = \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right) \ , \\ \alpha_1 + \alpha_2 = \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right) \ , \\ -\alpha_1 = \left(-\sqrt{2}, 0\right) \ , \\ -\alpha_2 = \left(\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}\right) \ , \\ -\alpha_1 - \alpha_2 = \left(-\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}\right) \ . \end{cases}$$



A_2



The Weyl group

$$\forall w \in \mathscr{W} \subset \mathrm{SO}(3) \quad : \quad w^T \mathscr{C}(\{\lambda_1,\lambda_2\})\, w = \mathscr{C}(w\{\lambda_1,\lambda_2\}) \in \mathrm{CSA}$$

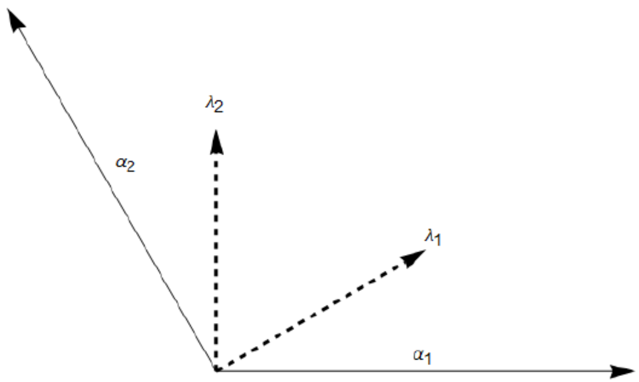
$$w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; (\lambda_1,\lambda_2,\lambda_3) \mapsto (\lambda_1,\lambda_2,\lambda_3) ,$$
$$w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; (\lambda_1,\lambda_2,\lambda_3) \mapsto (\lambda_2,\lambda_1,\lambda_3) ,$$
$$w_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} ; (\lambda_1,\lambda_2,\lambda_3) \mapsto (\lambda_3,\lambda_2,\lambda_1) ,$$
$$w_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; (\lambda_1,\lambda_2,\lambda_3) \mapsto (\lambda_1,\lambda_3,\lambda_2) ,$$
$$w_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} ; (\lambda_1,\lambda_2,\lambda_3) \mapsto (\lambda_2,\lambda_3,\lambda_1) ,$$
$$w_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ; (\lambda_1,\lambda_2,\lambda_3) \mapsto (\lambda_3,\lambda_1,\lambda_2) .$$

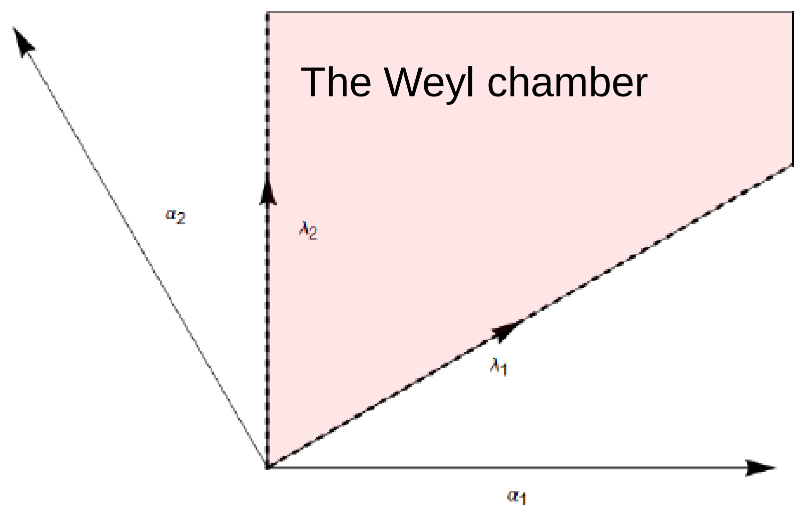
$$\mathrm{CSA} \ni \mathscr{C}(\{\lambda_1,\lambda_2\}) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix}$$

The complex Lie algebra SL(3,C) is provided by all 3×3 matrices that are traceless. The CSA is given by the diagonal set of traceless matrices.

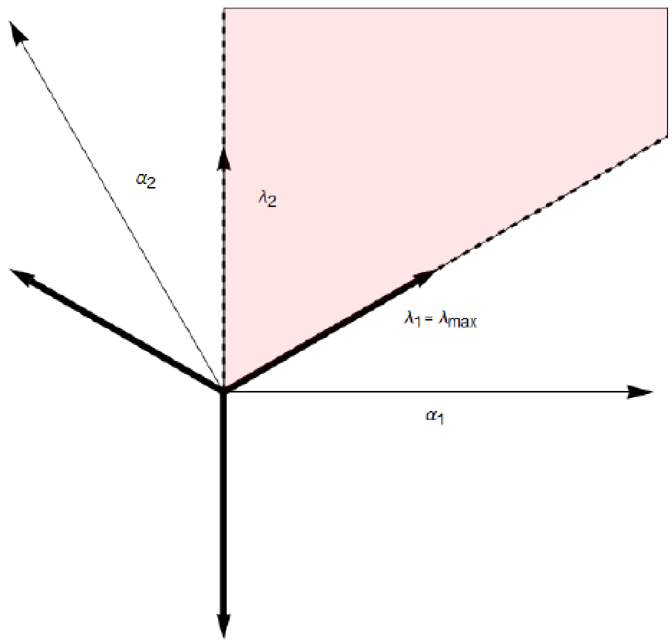
The Weyl group of A_2 is isomorphic to the permutation group of three objects S_3

The fundamental weights





Weights of the defining triplet representation



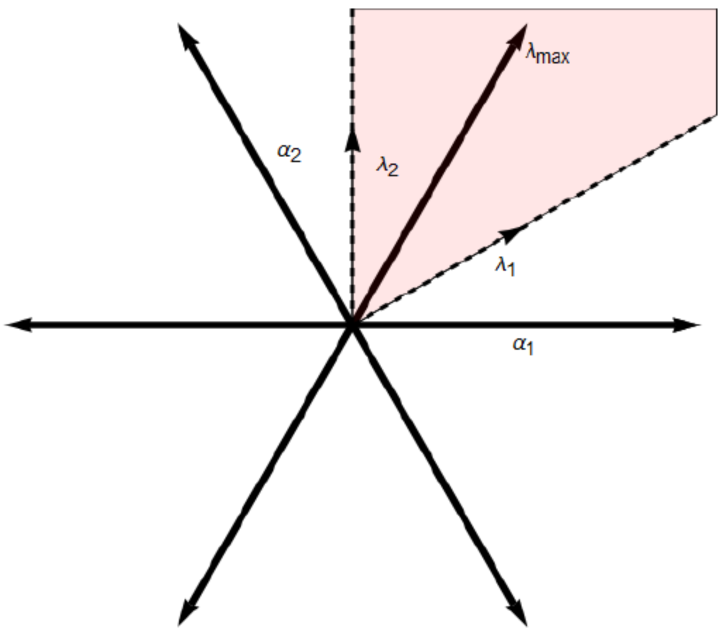
$$\Pi^{\text{triplet}} = \{\Lambda^1, \Lambda^2, \Lambda^3\}$$
$$\Lambda^1 = \lambda^1 \quad \Rightarrow (\mathbf{1},\mathbf{0})$$
$$\Lambda^2 = -\lambda^1 + \lambda^2 \quad \Rightarrow (-\mathbf{1},\mathbf{1})$$
$$\Lambda^3 = -\lambda^2 \quad \Rightarrow (\mathbf{0},-\mathbf{1})$$

Matrices of the representation

$$H_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; H_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix}$$
$$E^{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; E^{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} ,$$
$$E^{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$E^{-\alpha_1} = (E^{\alpha_1})^T \quad ; \quad E^{-\alpha_2} = (E^{\alpha_2})^T \quad ; \quad E^{-\alpha_1-\alpha_2} = \left(E^{\alpha_1+\alpha_2}\right)^T$$

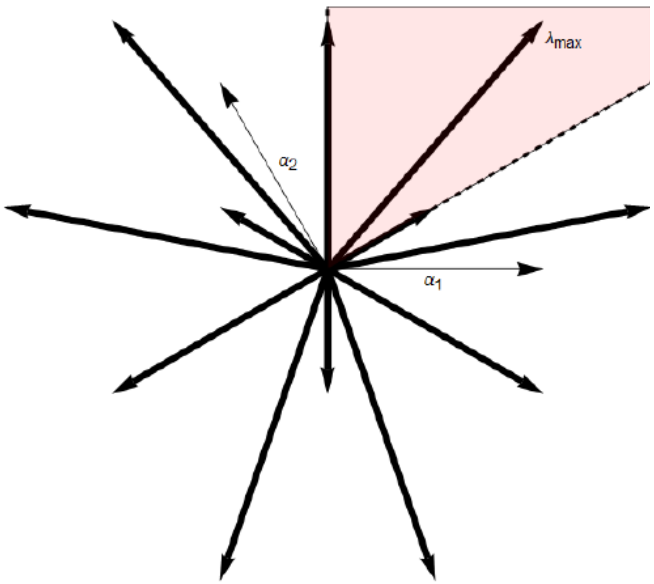
Weights of the adjoint representation, namely the roots



The root lattice is always sublattice of the weight lattice!

The highest root is the highest weight of the adjoint representation!

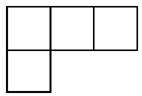
Weights of the 15-dim rep.



This provides a more complicated, less trivial example

Name		Orth. comp.	Dynk. lab.	mult
Λ_1	$= 2\lambda^1 + \lambda^2$	$= \left\{\sqrt{2}, 2\sqrt{\frac{2}{3}}\right\}$	$= (\mathbf{2},\mathbf{1})$	1
Λ_2	$= 3\lambda^1 - \lambda^2$	$= \left\{\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right\}$	$= (\mathbf{3},-\mathbf{1})$	1
Λ_3	$= 2\lambda^2$	$= \left\{0, 2\sqrt{\frac{2}{3}}\right\}$	$= (\mathbf{0},\mathbf{2})$	1
Λ_4	$= \lambda^1$	$= \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right\}$	$= (\mathbf{1},\mathbf{0})$	2
Λ_5	$= -2\lambda^1 + 3\lambda^2$	$= \left\{-\sqrt{2}, 2\sqrt{\frac{2}{3}}\right\}$	$= (-\mathbf{2},\mathbf{3})$	1
Λ_6	$= 2\lambda^1 - 2\lambda^2$	$= \left\{\sqrt{2}, -\sqrt{\frac{2}{3}}\right\}$	$= (\mathbf{2},-\mathbf{2})$	1
Λ_7	$= -\lambda^1 + \lambda^2$	$= \left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right\}$	$= (-\mathbf{1},\mathbf{1})$	2
Λ_8	$= -\lambda^2$	$= \left\{0, -\sqrt{\frac{2}{3}}\right\}$	$= (\mathbf{0},-\mathbf{1})$	2
Λ_9	$= -3\lambda^1 + 2\lambda^2$	$= \left\{-\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right\}$	$= (-\mathbf{3},\mathbf{2})$	1
Λ_{10}	$= -3\lambda^2 + \lambda^1$	$= \left\{\frac{1}{\sqrt{2}}, -\frac{5}{\sqrt{6}}\right\}$	$= (\mathbf{1},-\mathbf{3})$	1
Λ_{11}	$= -2\lambda^1$	$= \left\{-\sqrt{2}, -\sqrt{\frac{2}{3}}\right\}$	$= (-\mathbf{2},\mathbf{0})$	1
Λ_{12}	$= -\lambda^1 - 2\lambda^2$	$= \left\{-\frac{1}{\sqrt{2}}, -\frac{5}{\sqrt{6}}\right\}$	$= (-\mathbf{1},-\mathbf{2})$	1

Correspondence with Young tableaux and tensors

$\mathbf{15} \equiv$ 

The reason why certain weights have multiplicity >1 is that there is more than one inequivalent way to fill the boxes of the corresponding Young tableau

Weight	tensor comp.	weight mult.
Λ_1	$\begin{smallmatrix}1&1&1\\2&&\end{smallmatrix}$	1
Λ_2	$\begin{smallmatrix}1&1&1\\3&&\end{smallmatrix}$	1
Λ_3	$\begin{smallmatrix}1&1&2\\2&&\end{smallmatrix}$	1
Λ_4	$\left\{\begin{smallmatrix}1&1&2\\3&&\end{smallmatrix} \;\; \begin{smallmatrix}1&1&3\\2&&\end{smallmatrix}\right.$	2
Λ_5	$\begin{smallmatrix}1&2&2\\2&&\end{smallmatrix}$	1
Λ_6	$\begin{smallmatrix}1&1&3\\3&&\end{smallmatrix}$	1
Λ_7	$\left\{\begin{smallmatrix}1&2&2\\3&&\end{smallmatrix} \;\; \begin{smallmatrix}1&2&3\\2&&\end{smallmatrix}\right.$	2
Λ_8	$\left\{\begin{smallmatrix}1&2&3\\3&&\end{smallmatrix} \;\; \begin{smallmatrix}1&3&3\\2&&\end{smallmatrix}\right.$	2
Λ_9	$\begin{smallmatrix}2&2&2\\3&&\end{smallmatrix}$	1
Λ_{10}	$\begin{smallmatrix}1&3&3\\3&&\end{smallmatrix}$	1
Λ_{11}	$\begin{smallmatrix}2&2&3\\3&&\end{smallmatrix}$	1
Λ_{12}	$\begin{smallmatrix}2&3&3\\3&&\end{smallmatrix}$	1

The Golden Splitting

Supergravity has put into evidence some intrinsic properties of Lie Algebras that are purely mathematical yet obtain their proper interpretation only within the framework of those geometries that are well-adapted to supersymmetric field theories.

One of these hidden jewels is the golden splitting that is the Lie algebra seed of the c-map

$$\mathrm{adj}(\mathbb{U}_{\mathscr{Q}}) = \mathrm{adj}(\mathbb{U}_{\mathscr{S}\mathscr{K}}) \oplus \mathrm{adj}(\mathrm{SL}(2,\mathbb{R})_{\mathrm{E}}) \oplus \mathbf{W}_{(2,\mathbf{W})}$$

$$[T^a, T^b] = f^{ab}{}_c T^c$$

$$[L^x_E, L^y_E] = f^{xy}{}_z L^z,$$

$$[T^a, \mathbf{W}^{i\alpha}] = (\Lambda^a)^\alpha_\beta \mathbf{W}^{i\beta},$$

$$[L^x_E, \mathbf{W}^{i\alpha}] = (\lambda^x)^i{}_j \mathbf{W}^{j\alpha},$$

$$[\mathbf{W}^{i\alpha}, \mathbf{W}^{j\beta}] = \varepsilon^{ij} (K_a)^{\alpha\beta} T^a + \mathbb{C}^{\alpha\beta} k^{ij}_x L^x_E$$

$$\lambda^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad ; \quad \lambda^1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad ; \quad \lambda^2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

The golden splitting of the quaternionic algebra $\mathfrak{g}_2$

$$\text{adj}\left[\mathfrak{g}\right] = \left(\text{adj}\left[\mathfrak{sl}(2,\mathbb{C})_E\right], \mathbf{1}\right) \oplus \left(\mathbf{1}, \text{adj}\left[\mathfrak{sl}(2,\mathbb{C})\right]\right) \oplus \left(\mathbf{2}, \mathbf{4}\right)$$

$$\mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathscr{D}_3\left(\mathfrak{A}\right) = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & da^2+2bca & cb^2+2adb & b^2d \\ ac^2 & bc^2+2adc & ad^2+2bcd & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}$$

Theory of Coset Manifolds

A fundamental item in Supergravity and in many other branches of Physics.

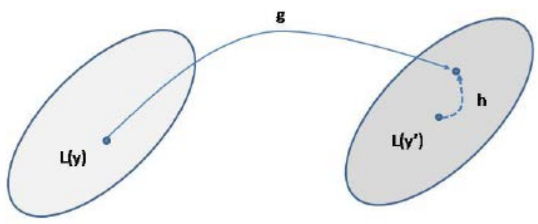
The classification of symmetric coset manifolds was the monumental work of Elie Cartan. Such a classification is also a classification of Real-Forms of the complex Lie algebras and contains the seeds of the Tits Satake projection, a very fundamental item in the applications of coset manifolds to Supergravity.

Coset Manifolds

$$\forall g, g' \in \mathrm{G} \quad : \quad g \sim g' \quad \text{iff} \quad \exists h \in \mathrm{H} \quad \backslash \quad g\,h = g'$$

$$d = \dim \frac{\mathrm{G}}{\mathrm{H}} \equiv \dim \mathrm{G} - \dim \mathrm{H}$$

$$\forall g \in G \quad : \quad g\,\mathbb{L}(y) = \mathbb{L}(y')\,h(g,y) \quad ; \quad h(g,y) \in \mathrm{H}$$



The geometry of coset manifolds

Decomposition of the Lie algebra

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \qquad [\mathbb{H}, \mathbb{H}] \subset \mathbb{H} \qquad [\mathbb{H}, \mathbb{K}] \subset \mathbb{H} \oplus \mathbb{K}$$

If $[\mathbb{H}, \mathbb{K}] \subset \mathbb{K}$ ⟺ The coset is reductive

$[\mathbb{K}, \mathbb{K}] \subset \mathbb{H}$ ⟺ Symmetric space

All the geometric properties of the manifold are essentially encoded in the structure constants

Basis of generators of the Lie algebra

$$T_A \; (A = 1, \ldots, n) \qquad [T_A\,, T_B] = C^C{}_{AB} T_C$$

→

$$[T_j\,, T_k] = C^i{}_{jk} T_i$$
$$[T_i\,, T_b] = C^a{}_{ib} T_a$$
$$[T_b\,, T_c] = C^i{}_{bc} T_i + C^a{}_{bc} T_a$$

The structure of isometries

Infinitesimal transformations and Killing vectors

$$g \simeq 1 + \varepsilon^A T_A$$
$$h(y,g) \simeq 1 - \varepsilon_A W^i_A(y) T_i$$
$$y'^\alpha \simeq y^\alpha + \varepsilon^A k^\alpha_A$$

Killing vector fields on G/H

$$\mathbf{k}_A = k^\alpha_A(y)\frac{\partial}{\partial y^\alpha}$$

$$g_2^{-1} g_1^{-1} g_2 g_1 \mathbb{L}(y) \simeq \left(1 - \varepsilon_1^A \varepsilon_2^B [T_A , T_B]\right) \mathbb{L}(y)$$

$$[T_A , T_B] \mathbb{L}(y) = T_A T_B \mathbb{L}(y) - T_B T_A \mathbb{L}(y)$$
$$= [\mathbf{k}_A , \mathbf{k}_B] \mathbb{L}(y) - \left(\mathbf{k}_A W^i_B - \mathbf{k}_B W^i_A + 2C^i{}_{jk} W^j_A W^k_B\right) \mathbb{L}(y) T_i$$

$$[T_A , T_B] \mathbb{L}(y) = C^C{}_{AB} T_C \mathbb{L}(y) = C^C{}_{AB} \left(\mathbf{k}_C \mathbb{L}(y) - W^i_C \mathbb{L}(y) T_i\right)$$

$$[\mathbf{k}_A , \mathbf{k}_B] = C^C{}_{AB} \mathbf{k}_C$$
$$\mathbf{k}_A W^i_B - \mathbf{k}_B W^i_A + 2C^i{}_{jk} W^j_A W^k_B = C^C{}_{AB} W^i_C$$

An example: the hyperbolic hyperplane

$$\mathbb{H}_-^{(n+1,1)} = \frac{\mathrm{SO}(n+1,1)}{\mathrm{SO}(n+1)} \qquad \eta_{IJ} = \mathrm{diag}\left(\underbrace{+,+\ldots,+}_{n+1-m},\underbrace{-,-,\ldots,-}_{m}\right)$$

$$\mathbf{x} \in \mathbb{H}_\pm^{(n+1-m,m)} \quad \Leftrightarrow \quad \langle \mathbf{x},\mathbf{x}\rangle_\eta \equiv \pm 1 \qquad\qquad \forall g \in \mathrm{G} \quad : \quad \langle \mathbf{x},\mathbf{x}\rangle_\eta = \pm 1 \quad \Leftrightarrow \quad \langle g\,\mathbf{x}, g\,\mathbf{x}\rangle_\eta = \pm 1$$

For $\mathbb{H}_-^{(n,1)}$

$$\mathrm{SO}(n,1) \supset \mathrm{SO}(n) \ni h = \left(\begin{array}{c|c} \mathcal{O} & 0 \\ \hline 0 & 1 \end{array}\right) \quad ; \quad \mathcal{O}^T \mathcal{O} = \mathbf{1}_{n\times n}$$

$$\mathbb{L}(\mathbf{y})^T \, \eta \, \mathbb{L}(\mathbf{y}) = \eta$$

$$\eta = \mathrm{diag}\,(+,+,\ldots,+,-)$$

Standard parameterization of the coset

$$\mathbb{L}(\mathbf{y}) = \left(\begin{array}{c|c} \mathbf{1}_{n\times n} + 2\,\frac{\mathbf{y}\,\mathbf{y}^T}{1-\mathbf{y}^2} & -2\,\frac{\mathbf{y}}{1-\mathbf{y}^2} \\ \hline -\quad 2\,\frac{\mathbf{y}^T}{1-\mathbf{y}^2} & \frac{1+\mathbf{y}^2}{1-\mathbf{y}^2} \end{array}\right)$$

$$\mathbf{x}(\mathbf{y}) \equiv \mathbb{L}(\mathbf{y})\,\mathbf{x}_0 = \mathbb{L}(\mathbf{y}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \end{pmatrix} = \frac{1}{1-\mathbf{y}^2} \begin{pmatrix} 2\,y^1 \\ \vdots \\ 2y^n \\ \hline \frac{1+\mathbf{y}^2}{1-\mathbf{y}^2} \end{pmatrix}$$

$$\mathbf{x}(\mathbf{y})^T \, \eta \, \mathbf{x}(\mathbf{y}) = -1$$

Spheres or Hyperplanes

$$\mathbb{L}_\kappa(\mathbf{y}) = \left(\begin{array}{c|c} \mathbf{1}_{n\times n} + 2\,\mathbf{y}\,\mathbf{y}^T\,\frac{\kappa}{1+\kappa\mathbf{y}^2} & -2\,\frac{\mathbf{y}}{1+\kappa\mathbf{y}^2} \\ \hline 2\,\kappa\,\frac{\mathbf{y}^T}{1+\kappa\,\mathbf{y}^2} & \frac{1-\kappa\,\mathbf{y}^2}{1+\kappa\mathbf{y}^2} \end{array}\right)$$

$$\mathbf{x}_\kappa(\mathbf{y}) \equiv \mathbb{L}_\kappa(\mathbf{y}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \end{pmatrix}$$

$$\mathbb{L}_\kappa(\mathbf{y})^T \underbrace{\left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ \hline 0 & \dots & 0 & 0 & \kappa \end{array}\right)}_{\eta_\kappa} \mathbb{L}_\kappa(\mathbf{y}) = \underbrace{\left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ \hline 0 & \dots & 0 & 0 & \kappa \end{array}\right)}_{\eta_\kappa}$$

$$\mathbf{x}_\kappa(\mathbf{y})^T\,\eta_\kappa\,\mathbf{x}_\kappa(\mathbf{y}) = \kappa$$

The decomposition of the Lie algebra and the Killing vector field

$$J_{ij} = \mathscr{I}_{ij} - \mathscr{I}_{ji} \quad ; \quad i,j = 1,\dots,n$$

$$\mathscr{I}_{ij} = \left(\begin{array}{cccc|c} 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 \\ \hline 0 & \dots & \underbrace{\dots}_{\text{j-th column}} & 0 & 0 \end{array}\right) \begin{array}{l} \\ \}\text{i-th row} \\ \\ \\ \end{array}$$

The commutation relations of the $\mathfrak{so}(n)$ generators are very simple.

$$[J_{ij}\,,J_{k\ell}] = -\delta_{ik}J_{j\ell} + \delta_{jk}J_{i\ell} - \delta_{j\ell}J_{ik} + \delta_{i\ell}J_{jk}$$

The coset generators can instead be chosen as the following matrices:

$$P_i = \left(\begin{array}{cccc|c} 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \\ \hline 0 & \dots & \underbrace{-\kappa}_{\text{i-th column}} & 0 & 0 \end{array}\right) \begin{array}{l} \\ \}\text{i-th row} \\ \\ \\ \end{array}$$

$$[J_{ij}\,,P_k] = -\delta_{ik}P_j + \delta_{jk}P_i$$
$$[P_i\,,P_j] = -\kappa J_{ij}$$

$$\mathbf{k}_{ij} = y_i\partial_j - y_j\partial_i$$
$$\mathbf{k}_i = \tfrac{1}{2}\left(1-\kappa\,\mathbf{y}^2\right)\partial_i + \kappa\,y_i\,\mathbf{y}\cdot\partial$$

$$J_{ij}\,\mathbb{L}_\kappa(\mathbf{y}) = \mathbf{k}_{ij}\,\mathbb{L}_\kappa(\mathbf{y}) + \mathbb{L}_\kappa(\mathbf{y})\,J_{pq}\,W^{pq}_{ij}(y)$$

$$P_i\,\mathbb{L}_\kappa(\mathbf{y}) = \mathbf{k}_i\,\mathbb{L}_\kappa(\mathbf{y}) + \mathbb{L}_\kappa(\mathbf{y})\,J_{pq}\,W^{pq}_{i}(y)$$

G is an H-bundle

Vielbeins, connections and metrics on G/H

$$\Sigma(y) = \mathbb{L}^{-1}(y)\,\mathrm{d}\mathbb{L}(y) \qquad 0 = \mathrm{d}\Sigma + \Sigma \wedge \Sigma$$

Cartan Maurer forms

$$\Sigma = V^a T_a + \omega^i T_i$$

$$\mathscr{P}\left(\frac{\mathrm{G}}{\mathrm{H}},\mathrm{H}\right) \quad : \quad \mathrm{G} \xrightarrow{\pi} \frac{\mathrm{G}}{\mathrm{H}}$$

G is an H-bundle on G/H

$$\begin{aligned} \mathrm{d}V^a + C^a{}_{ib}\,\omega^i \wedge V^b &= -\tfrac{1}{2} C^a{}_{bc} V^b \wedge V^c \\ \mathrm{d}\omega^i + \tfrac{1}{2} C^i{}_{jk}\omega^j \wedge \omega^k &= -\tfrac{1}{2} C^i{}_{bc} V^b \wedge V^c \end{aligned}$$

$$\mathrm{adj}\,\mathbb{G} = \underbrace{\mathrm{adj}\mathbb{H}}_{=\mathfrak{A}_{\mathbb{H}}} \oplus \mathfrak{D}_{\mathbb{H}}$$

$$\mathscr{A} = \exp\left[\mathfrak{A}_{\mathbb{H}}\right] \qquad \mathscr{D} = \exp\left[\mathfrak{D}_{\mathbb{H}}\right]$$

$$V^a(y') = V^b(y)\,\mathscr{D}_b{}^a\left(h(y,g)\right)$$

$$\omega(y') = \mathscr{A}\left[h(y,g)\right]\,\omega(y)\,\mathscr{A}^{-1}\left[h(y,g)\right] + \mathscr{A}\left[h(y,g)\right]\,\mathrm{d}\mathscr{A}^{-1}\left[h(y,g)\right]$$

G/H as (pseudo)-Riemannian manifolds

Invariant metrics on coset manifolds

$$ds^2 = \tau_{ab} V^a \otimes V^b = \underbrace{\tau_{ab} V^a_\alpha(y) V^b_\beta(y)}_{g_{\alpha\beta}(y)} dy^\alpha \otimes dy^\beta$$

$$\begin{aligned} \ell_{\mathbf{k}_A} ds^2 &= \tau_{ab} \left(\ell_{\mathbf{k}_A} V^a \otimes V^b + V^a \otimes \ell_{\mathbf{k}_A} V^b \right) \\ &= \underbrace{\tau_{ab} \left([\mathfrak{D}_{\mathbb{H}}(W_A)]^a{}_c \, \delta^b_d + [\mathfrak{D}_{\mathbb{H}}(W_A)]^b{}_c \, \delta^a_d \right)}_{=0 \quad \text{by invariance}} V^c \otimes V^d \\ &= 0 \end{aligned}$$

rank of the coset manifold

$$\mathfrak{D}_{\mathbb{H}} = \left(\begin{array}{c|c|c|c|c} \mathfrak{D}_1 & 0 & \dots & 0 & 0 \\ \hline 0 & \mathfrak{D}_2 & 0 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \dots & 0 & \mathfrak{D}_{\mathfrak{r}-1} & 0 \\ \hline 0 & 0 & \dots & 0 & \mathfrak{D}_{\mathfrak{r}} \end{array}\right)$$

we have $\mathfrak{r}$ irreducible invariant tensors $\tau^{(i)}_{a_i b_i}$ in correspondence of such irreducible blocks and we can introduce $\mathfrak{r}$ independent scale factors:

$$\tau = \left(\begin{array}{c|c|c|c|c} \lambda_1 \tau^{(1)} & 0 & \dots & 0 & 0 \\ \hline 0 & \lambda_2 \tau^{(2)} & 0 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & \dots & 0 & \lambda_{p-1} \tau^{(p-1)} & 0 \\ \hline 0 & 0 & \dots & 0 & \lambda_p \tau^{(p)} \end{array}\right)$$

If rank=1 $\quad \eta_{ab} = \delta_{ab}$

$$ds^2 = \lambda^2 \eta_{ab} V^a \otimes V^b$$

$$E^a = \lambda V^a$$

In any case

The spin connection ω^{ab} uniquely determined ⟵ $0 = \mathrm{d}E^a - \omega^{ab} \wedge E^c \eta_{bc}$

The Riemann tensor

$$\omega^{ab}\,\eta_{bc} \equiv \omega^a{}_c = \frac{1}{2\lambda} C^a{}_{cd} E^d + C^a{}_{ci}\,\omega^i$$

$$\mathfrak{R}^a{}_b = \mathrm{d}\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b$$

$$\mathfrak{R}^a{}_b = R^a{}_{bcd} E^c \wedge E^d$$

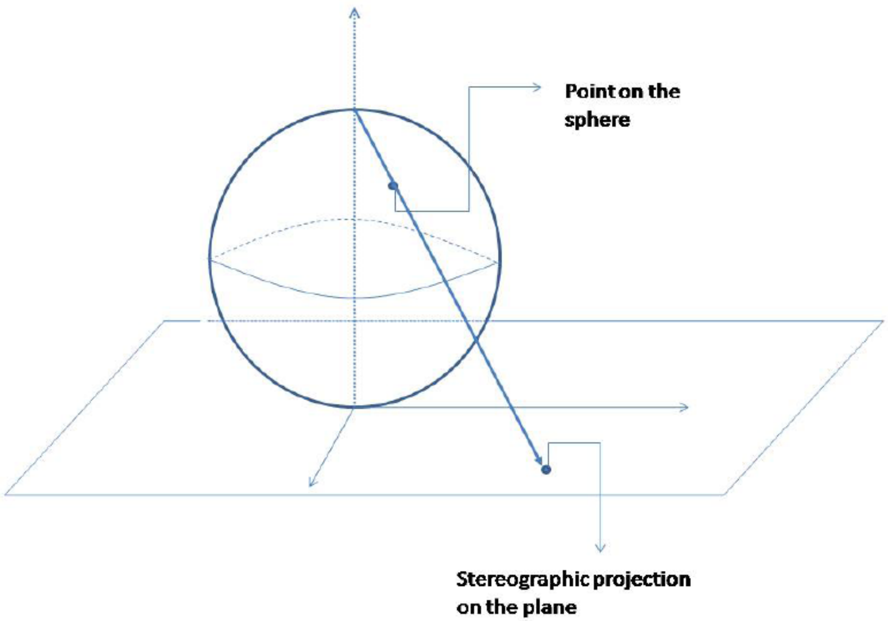
$$R^a{}_{bcd} = \frac{1}{\lambda^2}\Big(-\tfrac{1}{4}\,\frac{1}{2\lambda} C^a{}_{be} C^e{}_{cd} - \tfrac{1}{8} C^a{}_{ec} C^e{}_{bd} + \tfrac{1}{8} C^a{}_{ed} C^e{}_{bc} - \tfrac{1}{2} C^a{}_{bi} C^i{}_{cd}\Big)$$

In the case of symmetric spaces $C^a{}_{be} = 0$

$$R^a{}_{bcd} = -\frac{1}{2\,\lambda^2} C^a{}_{bi} C^i{}_{cd}$$

Example (spheres or pseudo-spheres)

$$E^a = -\frac{2}{\lambda}\,\frac{\mathrm{d}y^a}{1+\kappa\mathbf{y}^2} \qquad R^{ab}{}_{cd} = \frac{\kappa}{\lambda^2}\,\delta^{[a}_{[c}\,\delta^{b]}_{d]}$$



Real sections and non-compact cosets

In the context of coset manifolds a very interesting class that finds important applications in supergravity and superstring theories is the following one:

$$\mathscr{M}_{\mathbb{G}_R} = \frac{G_R}{H_c}$$

where G_R is some semi-simple Lie group and $H_c \subset G_R$ is its maximal compact subgroup. The Lie algebra $\mathbb{H}_c$ of the denominator H_c is the maximal compact subalgebra $\mathbb{H} \subset \mathbb{G}_R$ which has typically rank $r_{compact} > r$. Denoting, as usual, by $\mathbb{K}$ the orthogonal complement of $\mathbb{H}_c$ in $\mathbb{G}_R$:

$$\mathbb{G}_R = \mathbb{H}_c \oplus \mathbb{K}$$

$$r_{nc} = \text{rank}\,(G_R/H) \equiv \dim \mathscr{H}^{n.c.} \quad ; \quad \mathscr{H}^{n.c.} \equiv \text{CSA}_{\mathbb{G}(\mathbb{C})} \bigcap \mathbb{K}$$

we obtain that $r_{nc} < r$.

Two extremal real sections

a) **The maximally split real section** $\mathbb{G}_{\mathbf{max}}$. This is defined by assuming that the allowed coefficients c^A are all real. In any linear representation of $\mathbb{G}_{\mathbf{max}}$ the matrices representing

$$T_A \equiv \left\{H_i\,, E^{\alpha}\,, E^{-\alpha}\right\}$$

are all *real*. From the representations of $\mathbb{G}_{\mathbf{max}}$, by taking linear combinations of the generators with complex coefficients one obtains all the linear representations of the complex Lie algebra $\mathbb{G}(\mathbb{C})$.

b) **The maximally compact real section** $\mathbb{G}_c$. This real section, whose exponentiation produces a compact Lie group, is obtained by allowing linear combinations with real coefficients of the set of generators:

$$\mathfrak{T}_A \equiv \left\{\mathrm{i}H_i\,, \mathrm{i}\left(E^{\alpha}+E^{-\alpha}\right)\,, \left(E^{\alpha}-E^{-\alpha}\right)\right\}$$

In all linear representations of $\mathbb{G}_{\mathbf{c}}$ the matrices representing the generators $\mathfrak{T}_A$ are *anti-hermitian*.

Classification of real sections

$$\mathrm{Bor}\,[\mathbb{G}] = \mathrm{span}\,\{H_i, E^{\alpha}\} \quad ; \quad \alpha > 0 \qquad\qquad E^{-\alpha} = (E^{\alpha})^T$$

CLASSIFICATION OF ALL THE REAL SECTIONS

STEP ONE

Definition *Let:*

$$\theta \quad : \quad \mathfrak{g} \to \mathfrak{g}$$

be a linear automorphism of the compact Lie algebra $\mathfrak{g} = \mathbb{G}_c$*, where* $\mathbb{G}_c$ *is the maximal compact real section of a complex semi-simple Lie algebra* $\mathbb{G}(\mathbb{C})$*. By definition we have:*

$$\forall \alpha, \beta \in \mathbb{R} \quad , \quad \forall X, Y \in \mathfrak{g} \quad : \quad \begin{cases} \theta\,(\alpha X + \beta Y) = \alpha\,\theta(X) + \beta\,\theta(Y) \\ \theta\,([X\,,Y]) = [\theta(X)\,,\theta(Y)] \end{cases}$$

If $\theta^2 = \mathbf{Id}$ *then* θ *is named a Cartan involution of the Lie algebra* $\mathfrak{g}$.

From involution a new Lie algebra

$$\mathfrak{g} = \mathfrak{H}_\theta \oplus \mathfrak{p}_\theta \quad \Rightarrow \quad \begin{array}{l} [\mathfrak{H}_\theta \,, \mathfrak{H}_\theta] \subset \mathfrak{H}_\theta \\ [\mathfrak{H}_\theta \,, \mathfrak{p}_\theta] \subset \mathfrak{p}_\theta \\ [\mathfrak{p}_\theta \,, \mathfrak{p}_\theta] \subset \mathfrak{H}_\theta \end{array}$$

$$\mathscr{H}_c = \mathrm{span}\{\mathrm{i}H_{\alpha_i}\}$$

$$\theta : \mathscr{H}_c \to \mathscr{H}_c$$

The compact CSA is mapped into itself

$$\mathscr{M}_\theta = \frac{\mathrm{G_c}}{\mathrm{H}_\theta} \quad \text{where} \quad \mathrm{H}_\theta \equiv \exp[\mathfrak{H}_\theta] \quad ; \quad \mathrm{G_c} \equiv \exp[\mathfrak{g}]$$

$$\mathfrak{g}^\star_\theta = \mathfrak{H}_\theta \oplus \mathfrak{p}^\star_\theta \quad ; \quad \mathfrak{p}^\star_\theta \equiv \mathrm{i}\mathfrak{p}_\theta$$

$$\mathscr{M}^\star_\theta = \frac{\mathrm{G}^\star_\theta}{\mathrm{H}_\theta} \quad ; \quad \mathrm{H}_\theta \equiv \exp[\mathfrak{H}_\theta] \quad ; \quad \mathrm{G}^\star_\theta \equiv \exp[\mathfrak{g}^\star_\theta]$$

$$\mathrm{CSA}_{\mathbb{G}_R} = \mathrm{i}\mathscr{H}^{comp} \oplus \mathscr{H}^{n.c.}$$

$$\Updownarrow \qquad \Updownarrow$$

$$\mathrm{CSA}_{\mathbb{G}_{\mathbf{max}}} = \mathscr{H}^{comp} \oplus \mathscr{H}^{n.c.}$$

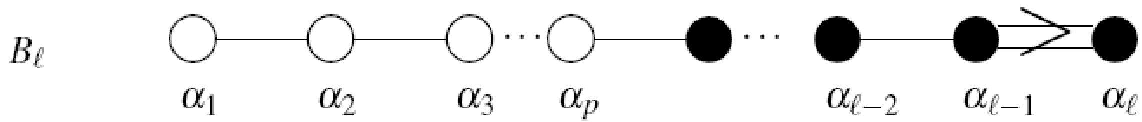
$$h \in \mathscr{H}^{comp} \Leftrightarrow \theta(h) = h$$

$$h \in \mathscr{H}^{n.c} \Leftrightarrow \theta(h) \neq h$$

real sections and simple roots

$$\alpha = \alpha_{||} \oplus \alpha_{\perp}$$

Tits Satake diagrams



Definition *A Riemannian space $(\mathscr{M}, g)$ is named* ***normal*** *if it admits a completely solvable*[5] *Lie group* $\exp[\mathrm{Solv}(\mathscr{M})]$ *of isometries that acts on the manifold in a simply transitive manner (i.e. for every 2 points in the manifold there is one and only one group element connecting them). The group* $\exp[\mathrm{Solv}(\mathscr{M})]$ *is generated by a so-called* ***normal metric Lie algebra****,* $\mathrm{Solv}(\mathscr{M})$ *that is a completely solvable Lie algebra endowed with a suitable, invariant Euclidean metric.*

$$\mathscr{M} \simeq \exp\left[\mathrm{Solv}(\mathscr{M})\right]$$
$$g\,|_{e\in\mathscr{M}} = <,>$$

All non compact cosets are normal

The key point is that non-compact coset manifolds of the form 1 are all normal. This is so because there is always, *for all real forms except the maximally compact one* a **solvable subalgebra** with the following features:

$$\mathrm{Solv}\left(\frac{\mathrm{G}_R}{\mathrm{H}_c}\right) \subset \mathrm{G}_R$$

$$\dim\left[\mathrm{Solv}\left(\frac{\mathrm{G}_R}{\mathrm{H}_c}\right)\right] = \dim\left(\frac{\mathrm{G}_R}{\mathrm{H}_c}\right)$$

$$\exp\left[\mathrm{Solv}\left(\frac{\mathrm{G}_R}{\mathrm{H}_c}\right)\right] = \text{transitive on } \frac{\mathrm{G}_R}{\mathrm{H}_c}$$

For maximally split real sections

$$\mathbb{H}_c = \mathrm{span}\left\{\left(E^{\alpha} - E^{-\alpha}\right)\right\} \quad ; \quad \forall \alpha \in \Delta_+ \qquad \mathbb{K} = \mathscr{H} \oplus \mathrm{span}\left[\left(E^{\alpha} + E^{-\alpha}\right)\right]$$

$$\mathrm{Bor}\left(\mathbb{G}_{\mathbf{max}}\right) \equiv \mathscr{H} \oplus \mathrm{span}\left(E^{\alpha}\right) \quad ; \quad \forall \alpha \in \Delta_+$$

$$\mathrm{Bor}(\mathrm{G}_{\mathbf{max}}) = \mathrm{Solv}\left(\frac{\mathrm{G}_{\mathbf{max}}}{\mathrm{H}_c}\right).$$

Kaehler Geometry and Hypergeometry

Lie Group Theory is inextricably entangled with Differential Geometry. One of the most important and durable contributions of Supergravity to Science is encoded in the new geometrical structures that it has introduced or better clarified and developed.

Kaehler, Hodge-Kaehler, Special Kaehler HyperKaehler and Quaternionic Kaehler manifolds are all advocated by and integrated into the fabrics of Supergravity

Complex Manifolds

Definition 3.2.1 *A 2n-dimensional manifold $\mathcal{M}$ is called almost complex if it has an almost complex structure. An almost complex structure is a linear operator J : $\Gamma(T\mathcal{M},\mathcal{M}) \to \Gamma(T\mathcal{M},\mathcal{M})$ which satisfies the following property:*

$$J^2 = -\mathbb{1}$$

$$\mathbf{e}_\alpha = \partial_\alpha = \frac{\partial}{\partial \phi_\alpha}$$

in a well-adapted frame we have

$$J\mathbf{e}_\alpha = -\mathbf{e}_{\alpha+n} \quad if \quad \alpha \le n$$
$$J\mathbf{e}_\alpha = \mathbf{e}_{\alpha-n} \quad if \quad \alpha > n$$

$$\mathbf{E}_i = \mathbf{e}_i - i\mathbf{e}_{i+n}$$
$$\mathbf{E}_{i^*} = \mathbf{e}_i + i\mathbf{e}_{i+n}$$

$$\Rightarrow \quad J\mathbf{E}_i = i\mathbf{E}_i \qquad J\mathbf{E}_{i^*} = -i\mathbf{E}_{i^*}$$

$$\Downarrow$$

$$z^i = \phi^i + i\phi^{i+n} \quad \Rightarrow \quad \mathbf{E}_i = \partial_i = \frac{\partial}{\partial z^i} \quad \mathbf{E}_{i^*} = \partial_{i^*} = \frac{\partial}{\partial z^{i*}}$$

Locally we can introduce complex coordinates

The question is whether we can do that globally

Holomorphic transition functions

Moreover every two well-adapted frames are related to each other by a coordinate transformation which is a holomorphic function of the corresponding complex coordinates. Indeed let

$$\phi^\alpha \to \phi^\alpha + \zeta^\alpha(\phi)$$

be an infinitesimal coordinate transformation connecting two well adapted frames. By definition this means

$$\partial_\alpha \zeta^\beta J^\gamma_\beta = J^\beta_\alpha \partial_\beta \zeta^\gamma$$

which is nothing but the Cauchy–Riemann equation for the real and imaginary parts of a holomorphic function. Hence eq. (3.2.20) can be replaced by

$$z^i \to z^i + \zeta^i(z)$$

We can establish a global complex structure if J is integrable. This requires

$$T^\alpha_{\beta\gamma} \partial_\alpha \phi \, dx^\beta \wedge dx^\gamma = 0$$

where the following is the Nienjuis tensor that should vanish

$$T^\alpha_{\beta\gamma} = \partial_{[\beta} J^\alpha_{\gamma]} - J^\mu_\beta J^\nu_\gamma \partial_{[\mu} J^\alpha_{\nu]}$$

Definition *Let $\mathscr{M}$ be a complex manifold and E be another complex manifold. A holomorphic vector bundle with total space E and base manifold $\mathscr{M}$ is given by a projection map:*

$$\pi \; : \; E \longrightarrow \mathscr{M}$$

such that

a) π is a holomorphic map of E onto $\mathscr{M}$

b) Let $p \in \mathscr{M}$, then the fibre over p

$$E_p \; = \; \pi^{-1}(p)$$

is a complex vector space of dimension r. (The number r is called the rank of the vector bundle.)

c) For each $p \in \mathscr{M}$ there is a neighbourhood U of p and a holomorphic homeomorphism

$$h \; : \; \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r$$

such that

$$h\left(\pi^{-1}(p)\right) \; = \; \{p\} \times \mathbb{C}^r$$

(The pair (U, h) is called a local trivialization.)

d) The transition functions between two local trivializations (U_α, h_α) and (U_β, h_β):

$$h_\alpha \circ h_\beta^{-1} \; : \; (U_\alpha \cap U_\beta) \otimes \mathbb{C}^r \longrightarrow (U_\alpha \cap U_\beta) \otimes \mathbb{C}^r$$

induce holomorphic maps

$$g_{\alpha\beta} \; : \; (U_\alpha \cap U_\beta) \longrightarrow \mathrm{GL}(r, \mathbb{C})$$

Connections on holomorphic bundles

Let $E \longrightarrow \mathscr{M}$ be a holomorphic vector bundle of rank r and $U \subset \mathscr{M}$ an open subset of the base manifold. A frame over U is a set of r holomorphic sections $\{s_1, \ldots, s_r\}$ such that $\{s_1(z), \ldots, s_r(z)\}$ is a basis for $\pi^{-1}(z)$ for any $z \in U$. Let $f \equiv \{e_I(z)\}$ be a frame of holomorphic sections. Any other holomorphic section ξ is described by

$$\xi = \xi^I(z)\, e_I$$

$$\overline{\partial}\,\xi^I = d\overline{z}^{j^\star}\,\overline{\partial}_{j^\star}\,\xi^I = 0$$

$$D\xi = d\xi + \theta\,\xi$$

$$\theta = \theta^{(1,0)} + \theta^{(0,1)}$$
$$\theta^{(1,0)} = dz^i\,\theta_i$$
$$\theta^{(0,1)} = d\overline{z}^{i^\star}\,\theta_{i^\star}$$

Let now a *fiber hermitian metric* h be defined on the holomorphic vector bundle.

$$\langle \xi , \eta \rangle_h \equiv \overline{\xi}^{I^\star}(\overline{z})\,\eta^J(z)\,h_{I^\star J}(z,\overline{z}) = \xi^\dagger\, h\,\eta$$

Connections are simpler on holomorphic bundles than in real geometry

Definition *A hermitian metric for a complex manifold $\mathcal{M}$ is a hermitian fibre metric on the canonical tangent bundle $T\mathcal{M}$. In this case the transition functions $g_{\alpha\beta}$ are given by the jacobians of the coordinate transformations.*

$$A) \quad d\langle \xi , \eta \rangle_h = \langle D\xi , \eta \rangle_h + \langle \xi , D\eta \rangle_h$$

$$B) \quad D^{(0,1)}\xi \equiv \left[\overline{\partial} + \theta^{(0,1)}\right] \xi = 0$$

$$\theta(f) = h(f)^{-1} \partial h(f) \qquad \theta^I{}_J = dz^i \, h^{IJ^\star} \partial_i h_{K^\star J}$$

$$dz^k \, \Gamma^i_{kj} = - g^{il^\star} \partial g_{l^\star j}$$ Levi Civita connection

$$\Theta(f) = \partial \theta + \overline{\partial} \theta + \theta \wedge \theta = \overline{\partial} \theta$$ Curvature 2-form

This identity follows from $\partial \theta + \theta \wedge \theta = 0$, which is identically true

For the Levi Civita case we have

$$
\begin{aligned}
\Gamma^i_j &= \Gamma^i_{kj} dz^k \\
\Gamma^i_{kj} &= -g^{i\ell^*}(\partial_j g_{k\ell^*}) \\
\Gamma^{i^*}_{j^*} &= \Gamma^{i^*}_{k^*j^*} d\overline{z}^{k^*} \\
\Gamma^{i^*}_{k^*j^*} &= -g^{i^*\ell}(\partial_{j^*} g_{k^*\ell})
\end{aligned}
$$

$$\mathcal{R} = \bar{\partial}\,\Gamma \quad \Longrightarrow$$

$$
\begin{aligned}
\mathscr{R}^i_j &= \mathscr{R}^i_{jk^*\ell} d\overline{z}^{k^*} \wedge dz^\ell \\
\mathscr{R}^i_{jk^*\ell} &= \partial_{k^*}\Gamma^i_{j\ell} \\
\mathscr{R}^{i^*}_{j^*} &= \mathscr{R}^{i^*}_{j^*k\ell^*} dz^k \wedge d\overline{z}^{\ell^*} \\
\mathscr{R}^{i^*}_{j^*k\ell^*} &= \partial_k \Gamma^{i^*}_{j^*\ell^*}
\end{aligned}
$$

$$\mathscr{R}^n_{m^*} = \mathscr{R}^i_{m^*n\,i} = \partial_{m^*}\Gamma^i_{ni} = \partial_{m^*}\partial_n\, ln\left(\sqrt{g}\right)$$

Kaehler metrics

$$g : \Gamma(T\mathcal{M},\mathcal{M}) \otimes \Gamma(T\mathcal{M},\mathcal{M}) \to C^\infty(\mathcal{M})$$

$$g(\mathbf{u},\mathbf{w}) = g_{\alpha\beta} u^\alpha w^\beta$$

$$K_{\alpha\beta} = g_{\gamma\beta} J^\gamma_\alpha$$

$$g(J\mathbf{u},J\mathbf{w}) = g(\mathbf{u},\mathbf{w})$$ Hermitian metric

$$K(\mathbf{u},\mathbf{w}) = \frac{1}{2\pi} g(J\mathbf{u},\mathbf{w})$$ Kaehler 2-form

g is hermitian if and only if K is anti-symmetric.

A hermitian metric is of the form:

$$ds^2 = g_{ij^\star} dz^i \otimes d\overline{z}^{j^\star} \Rightarrow K = \frac{i}{2\pi} g_{ij^\star} dz^i \wedge d\overline{z}^{j^*}$$

A hermitian metric is Kaehler iff the Kaehler 2-form is closed

$$\mathrm{d}K = 0$$

Kaehler potential

dK=0 is a differential equation for g_{ij^*} whose general solution in any local chart is given by the following expression:

$$g_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}$$

where $\mathcal{K} = \mathcal{K}^* = \mathcal{K}(z, z^*)$ is a real function of z^i, z^{i^*}

$\mathcal{K}$ is the Kähler potential

$\mathcal{K}'(z, z^{i^*}) = \mathcal{K}(z, z^{i^*}) + f(z) + f^*(z^*)$ gives rise to the same metric

N=1 supersymmetry requires that the scalar fields in the scalar multiplets should be the coordinates of a Kaehler manifold!

Hypergeometry

$$\dim_{\mathbf{R}} \mathcal{QM} = 4m \equiv 4\#\text{of hypermultiplets}$$

We name *Hypergeometry* that pertaining to the hypermultiplet sector, irrespectively whether we deal with global or local $\mathcal{N}=2$ theories. Yet there are two kinds of hypergeometries. Supersymmetry requires the existence of a principal SU(2)–bundle

$$\mathcal{SU} \longrightarrow \mathcal{QM} \tag{3.6.2}$$

The bundle $\mathcal{SU}$ is **flat** in the *rigid supersymmetry case* while its curvature is proportional to the Kähler forms in the *local case*.

$$\text{rigid hypergeometry} \equiv \text{HyperKähler geometry.}$$
$$\text{local hypergeometry} \equiv \text{Quaternionic Kähler geometry}$$

HyperKaehler and Quaternionic

Both a Quaternionic Kähler or a HyperKähler manifold $\mathcal{QM}$ is a $4m$-dimensional real manifold endowed with a metric h:

$$ds^2 = h_{uv}(q) dq^u \otimes dq^v \quad ; \quad u, v = 1, \ldots, 4m$$

and three complex structures

$$(J^x) : \; T(\mathcal{QM}) \longrightarrow T(\mathcal{QM}) \qquad (x = 1, 2, 3)$$

that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} \, \mathbb{1} + \varepsilon^{xyz} J^z$$

and respect to which the metric is hermitian:

$$\forall \mathbf{X}, \mathbf{Y} \in T\mathcal{QM} : \quad h(J^x \mathbf{X}, J^x \mathbf{Y}) = h(\mathbf{X}, \mathbf{Y}) \qquad (x = 1, 2, 3)$$

Triplet of Kaehler 2-forms

one can introduce a triplet of 2-forms

$$K^x = K^x_{uv} dq^u \wedge dq^v \; ; \; K^x_{uv} = h_{uw} (J^x)^w_v$$

$$\nabla K^x \equiv dK^x + \varepsilon^{xyz} \omega^y \wedge K^z = 0$$

In the HyperKaehler case we can choose a frame where the three 2-forms are closed

Defining the $\mathscr{SU}$–curvature by:

$$\Omega^x \equiv d\omega^x + \frac{1}{2} \varepsilon^{xyz} \omega^y \wedge \omega^z$$

$\Omega^x = 0$ HyperKaehler → $\omega^x = 0 \to \mathrm{d}K^x = 0$

$\Omega^x = \lambda K^x$ Quaternionic Kaehler

Holonomy restrictions

As a consequence of the above structure the manifold $\mathcal{QM}$ has a holonomy group of the following type:

$$\begin{aligned}
\mathrm{Hol}(\mathcal{QM}) &= \mathrm{SU}(2) \otimes \mathrm{H} \quad (\text{Quaternionic Kähler}) \\
\mathrm{Hol}(\mathcal{QM}) &= \mathbb{1} \otimes \mathrm{H} \quad (\text{HyperKähler}) \\
\mathrm{H} &\subset \mathrm{Sp}(2\mathrm{m}, \mathbb{R})
\end{aligned}$$

Formalism for hypergeometry

$\{A,B,C = 1,2\}\{\alpha,\beta,\gamma = 1,..,2m\}$ $\quad \mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\varepsilon_{AB} = -\varepsilon_{BA}$

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u(q)dq^u$$

$$\nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \frac{i}{2}\omega^x(\varepsilon\sigma_x\varepsilon^{-1})^A{}_B \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma}\mathbb{C}_{\beta\gamma} = 0$$

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta}\varepsilon_{AB}$$

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \varepsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta}$$

Structure of the curvature 2-form

$$\mathscr{R}^{uv}{}_{ts}\mathscr{U}_u^{\alpha A}\mathscr{U}_v^{\beta B} = -\frac{\mathrm{i}}{2}\Omega_{ts}^x \varepsilon^{AC}(\sigma_x)_C{}^B \mathbb{C}^{\alpha\beta} + \mathbb{R}_{ts}^{\alpha\beta}\varepsilon^{AB}$$

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma}\wedge\Delta^{\delta\beta}\mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}_{ts}^{\alpha\beta}dq^t\wedge dq^s$$ symplectic curvature

$$h^{st}K_{us}^x K_{tw}^y = -\delta^{xy}h_{uw} + \varepsilon^{xyz}K_{uw}^z$$ quaternionic algebra of the Kaehler forms

Moment Maps

Moment maps were born in hamiltonian mechanics but play a distinguished essential role in supersymmetric theories being essential building blocks of the Lagrangian and of the scalar potential. The use of moment-maps in supersymmetric field theories provided the framework of Kaehler and HyperKaehler quotient, one of the most prolific directions in modern differential and algebraic geometry.

They are also essential items in the AdS/CFT correspondence.

Moment Maps

The conception of moment maps has its root in Hamiltonian mechanics where the time-derivative of any dynamical variable can be represented by the Poisson bracket of that variable with the hamiltonian. More generally the action of any vector field $\mathbf{t}$ on functions defined over the phase-space $\mathscr{M}$ can be represented as the Poisson bracket of that function with a generalized hamiltonian $\mathscr{H}_{\mathfrak{t}}$ which is associated with the vector field:

$$\mathbf{t} \equiv t^i(p,q)\frac{\partial}{\partial q^i} + t_i(p,q)\frac{\partial}{\partial p_i}$$
$$\mathbf{t}\mathfrak{f}(p,q) = \{\mathfrak{f}, \mathscr{H}_{\mathfrak{t}}\}$$

The moment map is the map:

$$\mu \; : \; \Gamma[T\mathscr{M}, \mathscr{M}] \to \mathbb{C}[\mathscr{M}]$$
$$\mu[\mathbf{t}] = \mathscr{H}_{\mathfrak{t}}$$

which to every vector field associates its proper hamiltonian.

In the present geometrical context, conceptually very much different from that of dynamical systems which are of no concern to us the focus is on the moment-maps of Killing vectors, associated with isometries of the manifold $\mathscr{M}$.

Moment maps of holomorphic Killing vector fields

Let $g_{ij^\star}$ be the Kähler metric of a Kähler manifold $\mathscr{M}$ and let us assume that $g_{ij^\star}$ admits a non trivial group of continuous isometries $\mathscr{G}$ generated by Killing vectors $k^i_{\mathbf{I}}$ $(\mathbf{I} = 1, \ldots, \dim\mathscr{G})$ that define the infinitesimal variation of the complex coordinates z^i under the group action:

$$z^i \to z^i + \varepsilon^{\mathbf{I}} k^i_{\mathbf{I}}(z)$$

$$\partial_{j^*} k^i_{\mathbf{I}}(z) = 0 \leftrightarrow \partial_j k^{i^*}_{\mathbf{I}}(\bar{z}) = 0$$

$$\nabla_\mu k_\nu + \nabla_\mu k_\nu = 0$$

$$\nabla_i k_j + \nabla_j k_i = 0 \; ; \; \nabla_{i^*} k_j + \nabla_j k_{i^*} = 0$$

$$k^i_{\mathbf{I}} = i g^{ij^*} \partial_{j^*} \mathscr{P}_{\mathbf{I}}, \qquad \mathscr{P}^*_{\mathbf{I}} = \mathscr{P}_{\mathbf{I}}$$

Moment map

In a more intrinsic language

$$\left.\begin{array}{l}\mathscr{L}_{\vec{X}} g = 0 \leftrightarrow \nabla_{(\mu} X_{\nu)} = 0 \\ \mathscr{L}_{\vec{X}} J = 0\end{array}\right\} \Rightarrow 0 = \mathscr{L}_{\vec{X}} K = i_{\vec{X}} dK + d(i_{\vec{X}} K) = d(i_{\vec{X}} K)$$

If $\mathscr{M}$ is simply connected, $d(i_{\vec{X}} K) = 0$ implies the existence of a function $\mathscr{P}_{\vec{X}}$ such that

$$-\frac{1}{2} d\mathscr{P}_{\vec{X}} = i_{\vec{X}} K$$

The function $\mathscr{P}_{\vec{X}}$ is defined up to a constant, which can be arranged so as to make it equivariant:

$$\vec{X}\, \mathscr{P}_{\vec{Y}} = \mathscr{P}_{[\vec{X}, \vec{Y}]}$$

$\mathscr{P}_{\vec{X}}$ constitutes then a *moment map*. This can be regarded as a map

$$\mathscr{P} : \mathscr{M} \longrightarrow \mathbb{R} \otimes \mathbb{G}^*$$

Poisson bracket

$$[k_{\mathbf{I}}, k_{\mathbf{L}}] = f_{\mathbf{IL}}{}^{\mathbf{K}} k_{\mathbf{K}} \qquad \mathscr{P}_{\overrightarrow{X}} = a^{\mathbf{I}} \mathscr{P}_{\mathbf{I}} \qquad \{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\} \equiv 4\pi K(\mathbf{I}, \mathbf{J})$$

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\} = f_{\mathbf{IJ}}{}^{\mathbf{L}} \mathscr{P}_{\mathbf{L}} + C_{\mathbf{IJ}}$$

where $C_{\mathbf{IJ}}$ *is a constant fulfilling the cocycle condition*

$$f_{\mathbf{IM}}{}^{\mathbf{L}} C_{\mathbf{LJ}} + f_{\mathbf{MJ}}{}^{\mathbf{L}} C_{\mathbf{LI}} + f_{\mathbf{JI}}{}^{\mathbf{L}} C_{\mathbf{LM}} = 0$$

If cocycle $C_{\mathbf{IJ}}$ is a coboundary; namely we have

$$C_{\mathbf{IJ}} = f_{\mathbf{IJ}}{}^{\mathbf{L}} C_{\mathbf{L}} \qquad \mathscr{P}_{\mathbf{I}} \to \mathscr{P}_{\mathbf{I}} + C_{\mathbf{I}}$$

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\} = f_{\mathbf{IJ}}{}^{\mathbf{L}} \mathscr{P}_{\mathbf{L}} \qquad \frac{i}{2} g_{ij^*} (k^i_{\mathbf{I}} k^{j^*}_{\mathbf{J}} - k^i_{\mathbf{J}} k^{j^*}_{\mathbf{I}}) = \frac{1}{2} f_{\mathbf{IJ}}{}^{\mathbf{L}} \mathscr{P}_{\mathbf{L}}$$

Final form of the moment map

$$\mathscr{P}_{\mathbf{I}}{}^{x} = -\frac{i}{2}\left(k^{i}_{\mathbf{I}}\partial_{i}\mathscr{K} - k^{\bar{\imath}}_{\mathbf{I}}\partial_{\bar{\imath}}\mathscr{K}\right) + \mathrm{Im}(f_{\mathbf{I}}),$$

where $f_{\mathbf{I}} = f_{\mathbf{I}}(z)$ is a holomorphic transformation on the line-bundle, defining a compensating Kähler transformation:

$$k^{i}_{\mathbf{I}}\partial_{i}\mathscr{K} + k^{\bar{\imath}}_{\mathbf{I}}\partial_{\bar{\imath}}\mathscr{K} = -f_{\mathbf{I}}(z) - \overline{f}_{\mathbf{I}}(\bar{z}).$$

Triholomorphic moment maps

$$\mathbf{k_I} = k^u_{\mathbf{I}} \frac{\partial}{\partial q^u}$$

$$\mathscr{L}_{\mathbf{I}} K^x = \varepsilon^{xyz} K^y W^z_{\mathbf{I}} \ ; \ \mathscr{L}_{\mathbf{I}} \omega^x = \nabla W^x_{\mathbf{I}}$$

$$\mathscr{L}_{\mathbf{I}} W^x_{\mathbf{J}} - \mathscr{L}_{\mathbf{J}} W^x_{\mathbf{I}} + \varepsilon^{xyz} W^y_{\mathbf{I}} W^z_{\mathbf{J}} = f_{\mathbf{IJ}}^{\cdot\cdot\mathbf{L}} W^x_{\mathbf{L}}$$

HyperKaehler

$$\mathscr{Q}\mathscr{M} \longrightarrow L^x{}_y(q) \in \mathrm{SO}(3)$$

$$K^{x\prime} = L^x{}_y(q) K^y$$

$$\mathscr{L}_{\mathbf{I}} K^{x\prime} = 0$$

$$\mathbf{k_I}\, K^x = -\mathrm{d}\mathcal{P}^x$$

Quaternionic Kaehler

$$\mathscr{L}_{\mathbf{I}} \Omega^x = \varepsilon^{xyz} \Omega^y W^z_{\mathbf{I}} \ ; \ \mathscr{L}_{\mathbf{I}} \omega^x = \nabla W^x_{\mathbf{I}}$$

$$\mathbf{i_I} K^x = -\nabla \mathscr{P}^x_{\mathbf{I}} \equiv -(d\mathscr{P}^x_{\mathbf{I}} + \varepsilon^{xyz} \omega^y \mathscr{P}^z_{\mathbf{I}})$$

Triholomorphic Equivariance

Equivariance

$$\vec{X} \circ \mathscr{P}_{\vec{Y}} = \mathbf{i}_{\vec{X}} \nabla \mathscr{P}_{\vec{Y}} = \mathscr{P}_{[\vec{X},\vec{Y}]}$$

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\}^x = f^{\mathbf{K}}{}_{\mathbf{IJ}} \mathscr{P}^x_{\mathbf{K}}$$

HyperKaehler

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\}^x \equiv 2K^x(\mathbf{I},\mathbf{J})$$

Quaternionic

$$\{\mathscr{P}_{\mathbf{I}}, \mathscr{P}_{\mathbf{J}}\}^x \equiv 2K^x(\mathbf{I},\mathbf{J}) - \lambda\, \varepsilon^{xyz}\, \mathscr{P}^y_{\mathbf{I}}\, \mathscr{P}^z_{\mathbf{J}}$$

Special Geometry

Special Kaehler Geometry is an entire new chapter of complex geometry created by supergravity with profound relations with the deformation theory of complex structures and Kaehler structures of Calabi-Yau three-folds. It also provides new visions on symmetric spaces and their structures.

Hodge Kaehler manifolds

Consider a *line bundle* $\mathscr{L} \xrightarrow{\pi} \mathscr{M}$ over a Kähler manifold $\mathscr{M}$. By definition this is a holomorphic vector bundle of rank $r = 1$. For such bundles the only available Chern class is the first:

$$c_1(\mathscr{L}) = \frac{i}{2}\overline{\partial}\left(h^{-1}\partial h\right) = \frac{i}{2}\overline{\partial}\,\partial \log h$$

$$c_1(\mathscr{L}) = \frac{i}{2}\overline{\partial}\,\partial \log \parallel \xi(z) \parallel^2$$

where $\parallel \xi(z) \parallel^2 = h(z,\overline{z})\overline{\xi}(\overline{z})\xi(z)$ denotes the norm of the holomorphic section

A Kähler manifold $\mathscr{M}$ is a Hodge manifold if and only if there exists a line bundle $\mathscr{L} \xrightarrow{\pi} \mathscr{M}$ such that its first Chern class equals the cohomology class of the Kähler two-form K:

$$\boxed{c_1(\mathscr{L}) = [\mathrm{K}]}$$

In local terms this means that there is a holomorphic section $\xi(z)$ such that we can write

$$\mathrm{K} = \frac{i}{2} g_{ij^\star}\, dz^i \wedge d\overline{z}^{j^\star} = \frac{i}{2}\overline{\partial}\,\partial \log \parallel \xi(z) \parallel^2$$

Connection on the line bundle

since the fibre metric h can be identified with the exponential of the Kähler potential we obtain:

$$\theta = \partial \mathcal{K} = \partial_i \mathcal{K} dz^i \; ; \; \overline{\theta} = \overline{\partial} \mathcal{K} = \partial_{i^\star} \mathcal{K} d\overline{z}^{i^\star}$$

$$\nabla_i \Phi = (\partial_i + \tfrac{1}{2} p \partial_i \mathcal{K}) \Phi \; ; \; \nabla_{i^*} \Phi = (\partial_{i^*} - \tfrac{1}{2} p \partial_{i^*} \mathcal{K}) \Phi$$

$$\tilde{\Phi} = e^{-p\mathcal{K}/2} \Phi \, .$$

$$\nabla_i \tilde{\Phi} = (\partial_i + p \partial_i \mathcal{K}) \tilde{\Phi} \; ; \; \nabla_{i^*} \tilde{\Phi} = \partial_{i^*} \tilde{\Phi}$$

Special Kaehler manifolds

Definition 1 *A Hodge Kähler manifold is* **Special Kähler (of the local type)** *if there exists a completely symmetric holomorphic 3-index section* W_{ijk} *of* $(T^\star \mathscr{M})^3 \otimes \mathscr{L}^2$ *(and its antiholomorphic conjugate* $W_{i^* j^* k^*}$*) such that the following identity is satisfied by the Riemann tensor of the Levi–Civita connection:*

$$\partial_{m^*} W_{ijk} = 0 \quad \partial_m W_{i^* j^* k^*} = 0$$

$$\nabla_{[m} W_{i]jk} = 0 \quad \nabla_{[m} W_{i^*] j^* k^*} = 0$$

$$\mathscr{R}_{i^* j \ell^* k} = g_{\ell^* j} g_{k i^*} + g_{\ell^* k} g_{j i^*} - e^{2\mathscr{K}} W_{i^* \ell^* s^*} W_{tkj} g^{s^* t}$$

$$C_{ijk} = W_{ijk} e^{\mathscr{K}} \quad ; \quad C_{i^\star j^\star k^\star} = W_{i^\star j^\star k^\star} e^{\mathscr{K}}$$

Special Kaehler 2nd definition

Let $\mathscr{L} \xrightarrow{\pi} \mathscr{M}$ denote the complex line bundle whose first Chern class equals the cohomology class of the Kähler form K of an n-dimensional Hodge–Kähler manifold $\mathscr{M}$. Let $\mathscr{S}\mathscr{V} \longrightarrow \mathscr{M}$ denote a holomorphic flat vector bundle of rank $2n+2$ with structural group $\mathrm{Sp}(2\mathrm{n}+2,\mathbb{R})$. Consider tensor bundles of the type $\mathscr{H} = \mathscr{S}\mathscr{V} \otimes \mathscr{L}$. A typical holomorphic section of such a bundle will be denoted by Ω and will have the following structure:

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad \Lambda, \Sigma = 0, 1, \ldots, n$$

By definition the transition functions between two local trivializations $U_i \subset \mathscr{M}$ and $U_j \subset \mathscr{M}$ of the bundle $\mathscr{H}$ have the following form:

$$\begin{pmatrix} X \\ F \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} X \\ F \end{pmatrix}_j$$

where f_{ij} are holomorphic maps $U_i \cap U_j \to \mathbb{C}$ while M_{ij} is a constant $\mathrm{Sp}(2\mathrm{n}+2,\mathbb{R})$ matrix.

2nd definition continued

Let $\mathrm{i}\langle\ |\ \rangle$ be the compatible hermitian metric on $\mathscr{H}$

$$\mathrm{i}\langle\Omega\,|\,\overline{\Omega}\rangle \equiv -\mathrm{i}\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \overline{\Omega}$$

Definition 2 *We say that a Hodge–Kähler manifold* $\mathscr{M}$ *is* **special Kähler** *if there exists a bundle* $\mathscr{H}$ *of the type described above such that for some section* $\Omega \in \Gamma(\mathscr{H},\mathscr{M})$ *the Kähler two form is given by:*

$$\mathrm{K} = \frac{\mathrm{i}}{2}\partial\overline{\partial}\log\left(\mathrm{i}\langle\Omega\,|\,\overline{\Omega}\rangle\right) = \frac{i}{2}g_{ij^*}\,dz^i \wedge d\overline{z}^{j^*}$$

Kaehler potential

$$\mathscr{K} = -\log\left(\mathrm{i}\langle\Omega\,|\,\overline{\Omega}\rangle\right) = -\log\left[\mathrm{i}\left(\overline{X}^\Lambda F_\Lambda - \overline{F}_\Sigma X^\Sigma\right)\right]$$

Relation between the two definitions

$$V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{\mathscr{K}/2}\Omega = e^{\mathscr{K}/2}\begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}$$

$$1 = \mathrm{i}\langle V\,|\overline{V}\rangle = \mathrm{i}\left(\overline{L}^\Lambda M_\Lambda - \overline{M}_\Sigma L^\Sigma\right)$$

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i \mathscr{K}\right) V \equiv \begin{pmatrix} f_i^\Lambda \\ h_{\Sigma|i} \end{pmatrix}$$

$$\overline{U}_{i^\star} = \nabla_{i^\star}\overline{V} = \left(\partial_{i^\star} + \frac{1}{2}\partial_{i^\star}\mathscr{K}\right)\overline{V} \equiv \begin{pmatrix} \overline{f}_{i^\star}^\Lambda \\ \overline{h}_{\Sigma|i^\star} \end{pmatrix}$$

$$\nabla_{i^\star} V = \left(\partial_{i^\star} - \frac{1}{2}\partial_{i^\star}\mathscr{K}\right) V = 0$$

$$\nabla_i U_j = \mathrm{i} C_{ijk}\, g^{k\ell^\star}\,\overline{U}_{\ell^\star}$$

$$\begin{aligned} \nabla_i V &= U_i \\ \nabla_i U_j &= \mathrm{i} C_{ijk} g^{k\ell^\star} U_{\ell^\star} \\ \nabla_{i^\star} U_j &= g_{i^\star j} V \\ \nabla_{i^\star} V &= 0 \end{aligned}$$

The integrability conditions of the equations here on the left reproduces the statement on the Riemann tensor occurring in the first definition

Symplectic transformations of the period matrix

The vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ in special geometry

$$\overline{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\overline{L}^\Sigma \quad ; \quad h_{\Sigma|i} = \mathcal{N}_{\Lambda\Sigma} f_i^\Sigma$$

$$f_I^\Lambda = \begin{pmatrix} f_i^\Lambda \\ \overline{L}^\Lambda \end{pmatrix} \quad ; \quad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \overline{M}_\Lambda \end{pmatrix}$$

$$\boxed{\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^I{}_\Sigma}$$

a symplectic embedding of the isometry group $\mathcal{SK}_n$

$$\mathrm{U}_{\mathcal{SK}} \mapsto \mathrm{Sp}(2\mathrm{n}+2,\mathbb{R})$$

$$\xi \in \mathrm{U}_{\mathcal{SK}} \qquad \xi \mapsto \Lambda_\xi \equiv \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix}$$

$$\Lambda_\xi^T \underbrace{\begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}}_{\equiv\mathbb{C}} \Lambda_\xi = \underbrace{\begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}}_{\mathbb{C}}$$

The transformation

$$\Omega\left(\xi \cdot z\right) = \Lambda_\xi \, \Omega\left(z\right)$$

$$\mathcal{N}\left(\xi \cdot z, \xi \cdot \bar{z}\right) = \left(C_\xi + D_\xi \, \mathcal{N}(z,\bar{z})\right) \, \left(A_\xi + B_\xi \, \mathcal{N}(z,\bar{z})\right)^{-1}$$

The matrix N has a positive definite imaginary part and generalizes the notion of upper complex plane to what is known as the upper Siegel plane. The linear fractional transformations with symplectic matrices map the upper Siegel plane into itself just as SL(2,R) maps the Poincaré Lobachevsky plane into itself

Quotient singularities and Kaehler quotient resolutions

Finite Group Theory has an important bearing on the geometry of singular algebraic surfaces when considering orbifolds of the type C^n/Γ.

The resolutions of these singularities is done by means of Kaehler or HyperKaehler quotients.

Gravitational instantons, the ALE-manifold are obtained in these way from C^2/Γ where Γ is a finite subgroup of SU(2). Hence we have a new incarnation of the ADE classification.

These structures are very important for the study of the $AdS4/CFT_3$ correspondence

HyperKaehler quotients

Given a HyperKähler manifold $\mathscr{S}$ which admits a Lie group $\mathscr{G}$ of triholomorphic isometries, the *HyperKähler quotient* is a procedure that provides a way to construct from $\mathscr{S}$ a lower-dimensional HyperKähler manifold $\mathscr{M}$, as follows. Let $\mathfrak{Z}^* \subset \mathbb{G}^*$ be the dual of the center of the Lie algebra $\mathbb{G}$. For each $\zeta \in \mathbb{R}^3 \otimes \mathfrak{Z}^*$ the level set of the momentum map

$$\mathscr{N} \equiv \bigcap_i \mathscr{P}_i^{-1}(\zeta^i) \subset \mathscr{S},$$

$$\mathscr{M} = \mathscr{N} / \mathscr{G}$$

$$\dim \mathscr{N} = \dim \mathscr{S} - 3 \dim \mathscr{G}$$

$$\dim \mathscr{M} = \dim \mathscr{S} - 4 \dim \mathscr{G}$$

The standard use of the HyperKähler quotient is that of obtaining non trivial HyperKähler manifolds starting from a flat $4n$ real-dimensional manifold $\mathbb{R}^{4n}$ acted on by a suitable group $\mathscr{G}$ generating triholomorphic isometries

Complexified Killing fields & Kaehler potential

$$\mathbf{V} = I\mathbf{Y} = V^a \mathbf{k}_a$$

$$\pi: \quad s \in \mathscr{S} \longrightarrow e^{-V} s \in \mathscr{P}^{-1}(\zeta)$$

Solve for V $\quad \mathscr{P}(e^{-V} s) = \zeta$

$$\hat{\mathscr{K}} = \mathscr{K}|_{\mathscr{N}} + \mathbf{V}^a \zeta_a$$

ALE Manifolds

$$\mathfrak{R}^{ab}_{ALE} = \frac{1}{2}\,\varepsilon^{abcd}\,\mathfrak{R}^{cd}_{ALE}$$

Actually ALE manifolds are all Ricci flat and constitute vacuum solutions of Einstein equations after Wick rotation. In this sense ALE-manifolds are gravitational instantons in the same way as the connections with a self dual field strength are gauge instantons.

The first instance of an ALE manifold was found by Eguchi and Hanson 1979

The fascination of ALE manifolds is that they happen to be in one-to-one correspondence with the finite subgroups $\Gamma \subset \mathrm{SU}(2)$ and are similarly classified by the

The base manifold of the gravitational instanton has a boundary at infinity which, rather than a pure 3–sphere is:

$$\mathbb{S}^3/\Gamma$$

Except for the singular point

$$ALE_\Gamma \sim \mathbb{C}^2/\Gamma \qquad ; \qquad \Gamma \subset \mathrm{SU}(2)$$

$$ALE_\Gamma = \mathbb{C}^{2|\Gamma|}//_{HK}\mathscr{F}_{|\Gamma|-1}$$

Crepant resolutions

Definition 2.1 *The **canonical line bundle** $K_{\mathbb{V}}$ over a complex algebraic variety $\mathbb{V}$ of complex dimension n is the bundle of holomorphic $(n,0)$-forms $\Omega^{(n,0)}$ defined over $\mathbb{V}$.*

Definition 2.2 *An orbifold $\mathbb{V}/\Gamma$ of an algebraic variety modded by the action of a finite group is named **Gorenstein** if the isotropy subgroup $H_p \subset \Gamma$ of every point $p \in \mathbb{V}$ has a trivial action on the canonical bundle $K_{\mathbb{V}}$.*

Definition 2.3 *A resolution of singularities $\pi : \mathbb{W} \to \mathbb{X} \equiv \mathbb{V}/\Gamma$ is named **crepant**, if $K_{\mathbb{W}} = \pi^{\star} K_{\mathbb{X}}$. In particular this implies that the first Chern class of the resolved variety vanishes ($c_1(T\mathbb{W}) = 0$), if it vanishes for the orbifold, namely if $c_1(T\mathbb{X}) = 0$.*

In the case $\mathbb{V} = \mathbb{C}^n$, a resolution of quotient singularity:

$$\pi : \mathbb{W} \to \mathbb{C}^n/\Gamma \tag{2.12}$$

is crepant if the resolved variety $\mathbb{W}$ has vanishing first Chern class, namely it is a Calabi-Yau q–fold.

The Gorenstein condition plus the request that there should be a crepant resolution restricts the possible Γ.s to be subgroups of $\mathrm{SL}(\mathrm{n},\mathbb{C})$.

Concept of aging

Suppose that Γ (a finite group) acts in a holomorphic linear way on $\mathbb{C}^n$. Consider an element $\gamma \in \Gamma$ whose action is the following:

$$\gamma.\vec{z} = \underbrace{\begin{pmatrix} \dots & \dots & \dots \\ \vdots & \vdots & \vdots \\ \dots & \dots & \dots \end{pmatrix}}_{\mathcal{Q}(\gamma)} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

Since in a finite group all elements have a finite order, $\exists r \in \mathbb{N}$, such that $\gamma^r = \mathbf{1}$. We define the age of an element in the following way. Let us diagonalize $D(\gamma)$, namely compute its eigenvalues. They will be as follows:

$$(\lambda_1, \dots, \lambda_n) = \exp\left[\frac{2\pi i}{r} a_i\right] \quad ; \quad r > a_i \in \mathbb{N} \quad i = 1, \dots, n$$

We define:

$$\text{age}(\gamma) = \frac{1}{r} \sum_{i=1}^{n} a_i$$

Theorem 12.1 *Let $Y \to \mathbb{C}^3/\Gamma$ be a crepant resolution of the Gorenstein singularity. Then we have the following relation between the de-Rham cohomology groups of the resolved smooth variety Y and the ages of Γ:*

$$\dim H^{2k}(Y) = \#\ \textit{of age k conjugacy classes of}\ \Gamma$$

Example with L168 and its subgroups

Conjugacy class of L_{168}	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$	$\mathscr{C}_6$
representative of the class	$\mathbf{e}$	R	S	TSR	T	SR
order of the elements in the class	1	2	3	4	7	7
age	0	1	1	1	1	2
number of elements in the class	1	21	56	42	24	24

Conjugacy Class of G_{21}	C_1	C_2	C_3	C_4	C_5
representative of the class	e	$\mathscr{Y}$	$\mathscr{X}^2\mathscr{Y}\mathscr{X}\mathscr{Y}^2$	$\mathscr{Y}\mathscr{X}^2$	$\mathscr{X}$
order of the elements in the class	1	7	7	3	3
age	0	2	1	1	1
number of elements in the class	1	3	3	7	7

$\mathbb{C}^2/\Gamma$ singularities

$\Gamma.$	$W_\Gamma(u,w,z)$	$\mathscr{R} = \frac{\mathbb{C}[u,w,z]}{\partial W}$	$\vert\mathscr{R}\vert$	$\#c.\ c.$	$\tau \equiv \chi - 1$
A_k	$u^2 + w^2 - z^{k+1}$	$\{1, z, \ldots, z^{k-1}\}$	k	$k+1$	k
D_{k+2}	$u^2 + w^2 z + z^{k+1}$	$\{1, w, z, w^2, z^2, \ldots, z^{k-1}\}$	$k+2$	$k+3$	$k+2$
$E_6 = \mathscr{T}$	$u^2 + w^3 + z^4$	$\{1, w, z, wz, z^2, wz^2\}$	6	7	6
$E_7 = \mathscr{O}$	$u^2 + w^3 + wz^3$	$\{1, w, z, w^2, z^2, wz, w^2 z\}$	7	8	7
$E_8 = \mathscr{I}$	$u^2 + w^3 + z^5$	$\{1, w, z, z^2, wz, z^3, wz^2, wz^3\}$	8	9	8

$$W_\Gamma^{ALE}(u,w,z;\mathbf{t}) = W_\Gamma(u,w,z) + \sum_{i=1}^{r} t_i\, \mathscr{P}^{(i)}(u,w,z)$$

$$r \equiv \dim \mathscr{R}_\Gamma$$

$$\mathscr{R}_\Gamma = \frac{\mathbb{C}[u,w,z]}{\partial W_\Gamma}$$

For ALE$_\Gamma$ manifolds

$$\dim H^{(1,1)} = r \equiv \#\text{ non trivial conjugacy classes of }\Gamma$$

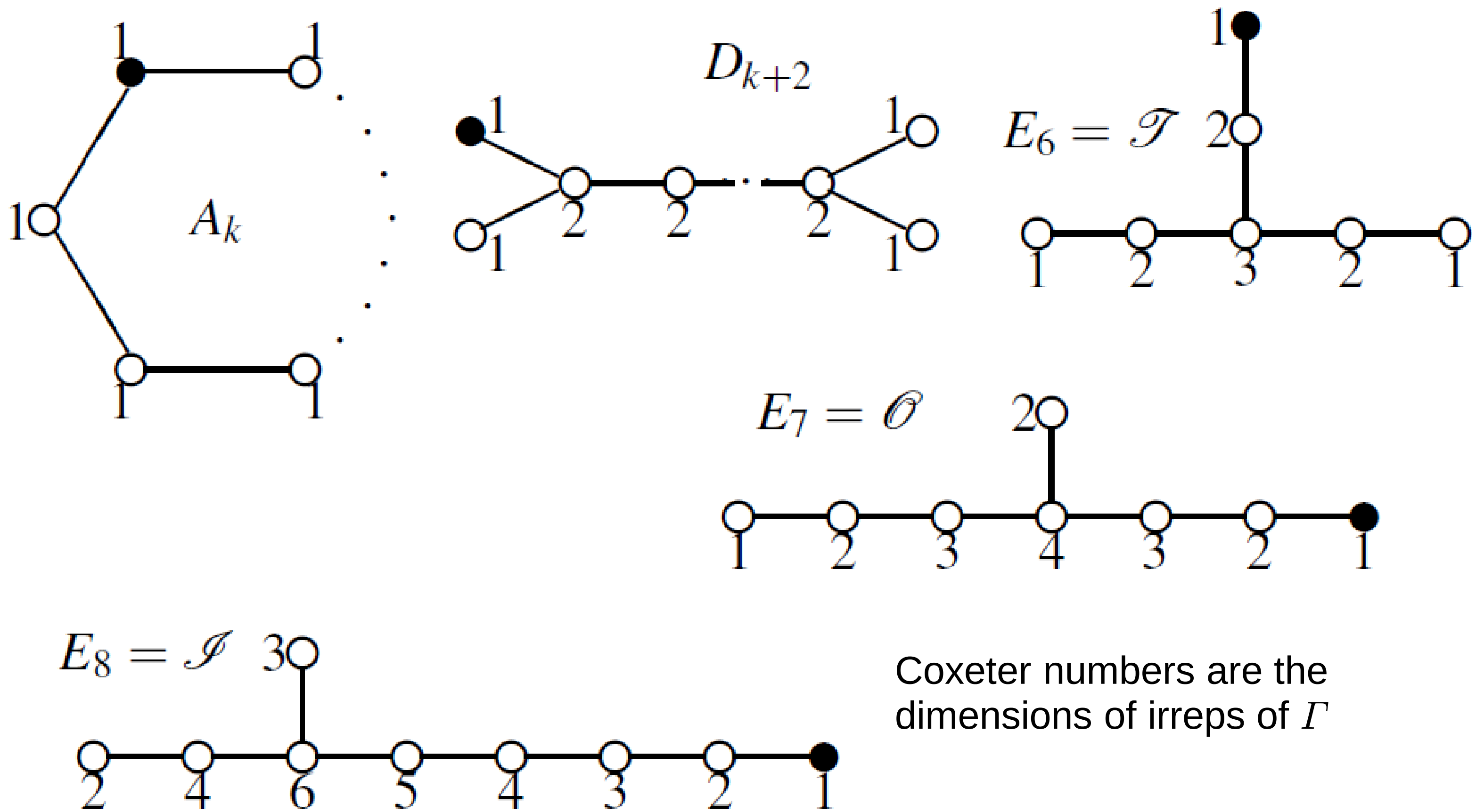
$$\dim \mathscr{R}_\Gamma = r$$

The McKay correspondence for $\mathbb{C}^2/\Gamma$

$$\begin{aligned} D &= \bigoplus_{\mu=1}^{r} a_\mu D_\mu \\ a_\mu &= \frac{1}{g}\sum_{i=1}^{r} g_i \chi_i^{(D)} \chi_i^{(\mu)\star} \end{aligned} \qquad \mathscr{Q}\otimes D_\mu = \bigoplus_{\nu=0}^{r} A_{\mu\nu} D_\nu$$

$\overline{c}_{\mu\nu} = 2\delta_{\mu\nu} - A_{\mu\nu}$ *extended Cartan matrix* of the *extended Dynkin diagram* → *simple roots* $\{\alpha_1 \ldots\ldots \alpha_r\}$ + highest root $\alpha_0 = \sum_{i=1}^{r} n_i \alpha_i$

The McKay Dynkin Graphs



Coxeter numbers are the dimensions of irreps of Γ

Kronheimer's construction

Given any finite subgroup of $\Gamma \subset \mathrm{SU}(2)$, we consider a space $\mathscr{P}$ whose elements are two-vectors of $|\Gamma| \times |\Gamma|$ complex matrices: $p \in \mathscr{P} = (A, B)$. The action of an element $\gamma \in \Gamma$ on the points of $\mathscr{P}$ is the following:

$$\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} u_\gamma & i\overline{v}_\gamma \\ i v_\gamma & \overline{u}_\gamma \end{pmatrix} \begin{pmatrix} R(\gamma) A R(\gamma^{-1}) \\ R(\gamma) B R(\gamma^{-1}) \end{pmatrix}$$

Regular representation

$$R(\gamma) e_\delta = e_{\gamma \cdot \delta} \quad \forall \gamma, \delta \in \Gamma$$

$$\mathscr{P} \simeq \mathrm{Hom}\,(R, \mathscr{Q} \otimes R)$$

Γ - invariant subspace

$$\mathscr{S} \equiv \{ p \in \mathscr{P} / \forall \gamma \in \Gamma, \gamma \cdot p = p \}$$

$$\mathscr{S} \simeq \mathrm{Hom}_\Gamma (R, \mathscr{Q} \otimes R)$$

$$\forall \gamma \in \Gamma : \quad \begin{pmatrix} u_\gamma & i\overline{v}_\gamma \\ i v_\gamma & \overline{u}_\gamma \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} R(\gamma^{-1}) A R(\gamma) \\ R(\gamma^{-1}) B R(\gamma) \end{pmatrix}$$

Schur's Lemma + McKay

$$\mathscr{S} = \bigoplus_{\mu,\nu} A_{\mu,\nu} \mathrm{Hom}(\mathbb{C}^{n_\mu}, \mathbb{C}^{n_\nu}) \ .$$

$$\dim_{\mathbb{C}} \mathscr{S} = 2\sum_{\mu} n_\mu^2 = 2|\Gamma| \ .$$

$$\Theta = \mathrm{Tr}(\mathrm{d}p^\dagger \wedge \mathrm{d}p) = \begin{pmatrix} \mathrm{i}\,\mathbf{K} & \mathrm{i}\overline{\mathbf{\Omega}} \\ \mathrm{i}\mathbf{\Omega} & -\mathrm{i}\,\mathbf{K} \end{pmatrix}$$

$$\mathbf{K} = -\mathrm{i}\left[\mathrm{Tr}\left(\mathrm{d}A^\dagger \wedge \mathrm{d}A\right) + \mathrm{Tr}\left(\mathrm{d}B^\dagger \wedge \mathrm{d}B\right)\right] \equiv \mathrm{i} g_{\alpha\overline{\beta}}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^{\overline{\beta}}$$

$$ds^2 = g_{\alpha\overline{\beta}}\, \mathrm{d}q^\alpha \otimes \mathrm{d}q^{\overline{\beta}}$$

$$\mathbf{\Omega} = 2\mathrm{Tr}\left(\mathrm{d}A \wedge \mathrm{d}B\right) \equiv \Omega_{\alpha\beta}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta$$

Solution of invariance constraints for A_k

Decomposition into irreps (one)-dim.

$$R(e_1) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\widehat{R}(e_1) \equiv S^{-1} R(e_1) S = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & v & 0\dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & v^{k-1} & 0 \\ 0 & 0 & \dots & 0 & v^k \end{pmatrix}$$

$$v^{k+1} = 1.$$

$$A = \begin{pmatrix} 0 & u_0 & 0 & \cdots & 0 \\ 0 & 0 & u_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & u_{k-1} \\ u_k & 0 & 0 & \cdots & 0 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & 0 & \cdots & \cdots & v_k \\ v_0 & 0 & \cdots & \cdots & 0 \\ 0 & v_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & v_{k-1} & 0 \end{pmatrix}$$

The gauge group

$$\forall g \in \mathrm{SU}(|\Gamma|) \quad , \quad g \quad : \quad \begin{pmatrix} A & iB^\dagger \\ iB & A^\dagger \end{pmatrix} \longmapsto \begin{pmatrix} gAg^{-1} & igB^\dagger g^{-1} \\ igBg^{-1} & gA^\dagger g^{-1} \end{pmatrix}$$

Let $\mathscr{F} \subset \mathrm{SU}(|\Gamma|)$ be the subgroup of the above group which *commutes with the action of* Γ

$$\mathscr{F} = \bigotimes_{\mu=1}^{r+1} \mathrm{U}(\mathrm{n}_\mu) \bigcap \mathrm{SU}(|\Gamma|)$$

$$\dim \mathscr{F} = \sum_\mu n_\mu^2 - 1 = |\Gamma| - 1 \qquad \mathscr{M}_\zeta \equiv \mu^{-1}(\zeta) // _{HK} \mathscr{F}$$

$$\dim_{\mathbb{R}} \mathscr{M}_\zeta = \dim_{\mathbb{R}} \mathrm{Hom}_\Gamma (R, \mathscr{Q} \otimes R) - 4 \dim_{\mathbb{R}} \mathscr{F} = 4|\Gamma| - (4|\Gamma| - 1) = 4$$

Moment maps

$$f = \mathrm{diag}(e^{i\varphi_0}, e^{i\varphi_1}, \dots, e^{i\varphi_k}) \ ; \ \sum_{i=0}^{k} \varphi_i = 0 \, .$$

$$\begin{aligned} \mu_3(p) &= -i\left([A, A^\dagger] + [B, B^\dagger]\right) \\ \mu_+(p) &= ([A, B]) \ . \end{aligned}$$

$$\mu_A = \mathrm{Tr}\,(q^\dagger\,\delta_A\,p) \ \equiv \ \mathrm{Tr}\begin{pmatrix} f_A\,\mu_3(p) & f_A\,\mu_-(p) \\ f_A\,\mu_+(p) & f_A\,\mu_3(p) \end{pmatrix}$$

$$\mu_A = \begin{pmatrix} \mathfrak{P}^3_A & \mathfrak{P}^-_A \\ \mathfrak{P}^+_A & -\mathfrak{P}^3_A \end{pmatrix}$$

$$\begin{aligned} \mathfrak{P}^3_A &= -\mathrm{i}\left[\mathrm{Tr}\left(\left[A\,,A^\dagger\right]\,f_A\right) + \mathrm{Tr}\left(\left[B^\dagger\,,B\right]\,f_A\right)\right] \\ \mathfrak{P}^+_A &= \mathrm{Tr}\left([A\,,B]\,f_A\right) \end{aligned}$$

$$\begin{aligned} \mathfrak{P}^3_A &= |u^0|^2 - |u^k|^2 - |v_0|^2 + |v_k|^2 + \left(|u^{A-1}|^2 - |u^A|^2 - |v_{A-1}|^2 + |v_A|^2\right) \\ \mathfrak{P}^+_A &= u^0 v_0 - u^k v_k + \left(u^{A-1} v_{A-1} - u^A\,v_A\right) \quad , \quad (A = 1, \dots, k) \end{aligned}$$

Eguchi Hanson space

$$\sigma_1 = -\frac{1}{2}(d\theta\cos(\psi) + d\phi\sin(\theta)\sin(\psi))$$
$$\sigma_2 = \frac{1}{2}(d\theta\sin(\psi) - d\phi\sin(\theta)\cos(\psi))$$
$$\sigma_3 = -\frac{1}{2}(d\phi\cos(\theta) + d\psi)$$

$$d\sigma_i = \varepsilon_{ijk}\,\sigma_j \wedge \sigma_k$$

$$W(r) = \sqrt{1 - \left(\frac{a}{r}\right)^2} \equiv \mathscr{W}(r/a)$$

$$ds^2_{EH} = W(r)^{-2}dr^2 + r^2\left(\sigma_1^2 + \sigma_2^2\right) + r^2 W(r)^2\sigma_3^2$$
$$= \frac{1}{4}\left(\frac{(r^4 - a^4)(d\phi\cos(\theta) + d\psi)^2}{r^2} + \frac{4dr^2}{1 - \frac{a^4}{r^4}} + r^2\left(d\phi^2\sin^2(\theta) + d\theta^2\right)\right)$$

It is far than obvious that this is a HyperKaehler metric ! Yet it is and it is a HyperKaehler quotient!

The complex structure

$$\zeta^1 = \frac{G(\rho)\left(e^{i(\theta+\phi)} + ie^{i\theta} + e^{i\phi} - i\right) e^{-\frac{1}{2}i(\theta-\psi+\phi)}}{2\sqrt{2}}$$

$$\zeta^2 = \frac{G(\rho)\left(e^{i(\theta+\phi)} - ie^{i\theta} + e^{i\phi} + i\right) e^{-\frac{1}{2}i(\theta-\psi+\phi)}}{2\sqrt{2}}$$

$$G(\rho) = \sqrt[4]{\rho^4 - 1}$$

$$\widehat{ds^2_{EH}} = \frac{\partial}{\partial \zeta^i} \frac{\overline{\partial}}{\partial \overline{\zeta}^{j^\star}} \mathcal{K}_{EH}\, d\zeta^i \otimes d\overline{\zeta}^{j^\star}$$

$$\mathcal{K}_{EH} = \sqrt{\tau^2+1} - \log\left(\sqrt{\tau^2+1}+1\right) + \log(\tau)$$

$$\tau \equiv |\zeta^1|^2 + |\zeta^2|^2$$

Relation with the Kronheimer approach

In this case it is convenient to redefine:

$$U = \{u_0, v_1\}$$
$$V = \{v_0, u_1\}$$

$$\mathfrak{P}^3 = |u_0|^2 - |v_0|^2 + |v_1|^2 - |u_1|^2$$
$$\mathfrak{P}^+ = u^0\, v_0 - u^1\, v_1$$

$$\mathfrak{P}^3 = \mathscr{P}^3(U,V) \equiv \sum_{i=1}^{2} |U_i|^2 - \sum_{i=1}^{2} |V_i|^2$$

$$\mathfrak{P}^+ = \mathscr{P}^+(U,V) \equiv \sum_{i=1}^{2} U_i V_i$$

$$\mathrm{U}(1) \quad : \quad (U,V) \quad \Longrightarrow \quad \left(e^{i\varphi}U\,,\,e^{-i\varphi}V\right)$$

$$\mathscr{F}^c \quad : \quad (U,V) \quad \Longrightarrow \quad \left(e^{-\mathbf{V}}U\,,\,e^{\mathbf{V}}V\right)$$

$$\ell = \mathscr{P}^3\left(e^{-\mathbf{V}}U, e^{\mathbf{V}}V\right)$$
$$\mathfrak{s} = \mathscr{P}^+\left(e^{-\mathbf{V}}U, e^{\mathbf{V}}V\right) = \mathscr{P}^+(U,V)$$

As stated in the second line of the above equation the holomorphic part of the moment-map is invariant under the action of the complexified group.

The Kaehler potential retrieved

$$U_1 = V_2 \qquad U_1 = V_2 \equiv \tfrac{1}{2} z^1 \quad ; \quad U_2 = V_1 \equiv \tfrac{1}{2} z^2$$

$$\mathbf{V} = -\log\left[\frac{\ell \pm \sqrt{\ell^2 + 4|U|^2\,|V^2|}}{2|V^2|}\right] = -\log\left[\frac{\ell \pm \sqrt{\ell^2 + |z|^4}}{2|z|^2}\right]$$

$$\mathcal{K}|_{\mathcal{N}} = e^{-2\mathbf{V}}\,|U|^2 + e^{2\mathbf{V}}\,|V|^2 = \sqrt{\ell^2 + |z|^4}$$

$$\mathcal{K}_M = \sqrt{\ell^2 + |z|^4} - \ell \log\left[\frac{\ell \pm \sqrt{\ell^2 + |z|^4}}{2|z|^2}\right]$$

For $\ell = 1$, identifying $z^i = \zeta^i$ we see that the Kähler potential already constructed

The c-map and the c*-map

The c-map and the c*-map are purely mathematical structures put into evidence by supergravity. In alliance with the Tits Satake projection they provide very important tools in the study of the following two problems:

1. Gauging of supergravity theories and study of their vacua
2. Black Hole solutions of Supergravity Theories

In any case they provide new quality and new visions in geometry.

Approaching the c-map

The main object of study in the present section are those Quaternionic Kähler manifolds that are in the image of the c-map.[5] This latter

$$\text{c-map} \ : \ \mathscr{SK}_n \Longrightarrow \mathscr{QM}_{4n+4}$$

is a universal construction that starting from an arbitrary Special Kähler manifold $\mathscr{SK}_n$ of complex dimension n, irrespectively whether it is homogenoeus or not, leads to a unique Quaternionic Kähler manifold $\mathscr{QM}_{4n+4}$ of real dimension $4n+4$ which contains $\mathscr{SK}_n$ as a submanifold.

Definition *Let $\mathcal{SK}_n$ be a special Kähler manifold whose complex coordinates we denote by z^i and whose Kähler metric we denote by $g_{ij^\star}$. Let moreover $\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})$ be the symmetric period matrix defined* above *, introduce the following set of $4n+4$ coordinates:*

$$\{q^u\} \equiv \underbrace{\{U,a\}}_{2\ real} \bigcup \underbrace{\underbrace{\{z^i\}}_{n\ complex}}_{2n\ real} \bigcup \underbrace{\mathbf{Z} = \{Z^\Lambda, Z_\Sigma\}}_{(2n+2)\ real}$$

Let us further introduce the following $(2n+2)\times(2n+2)$ matrix $\mathcal{M}_4^{-1}$:

$$\mathcal{M}_4^{-1} = \left(\begin{array}{c|c} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1} \\ \hline -\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} & \mathrm{Im}\mathcal{N}^{-1} \end{array}\right)$$

which depends only on the coordinate of the Special Kähler manifold. The c-map image of $\mathcal{SK}_n$ is the unique Quaternionic Kähler manifold $\mathcal{QM}_{4n+4}$ whose coordinates are the q^u defined in (4.3.2) and whose metric is given by the following universal formula

XX

$$ds^2_{\mathcal{QM}} = \frac{1}{4}\left(dU^2 + 4 g_{ij^\star}\, dz^j\, d\bar{z}^{j^\star} + e^{-2U}\left(da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z}\right)^2 - 2\,e^{-U}\, d\mathbf{Z}^T\, \mathcal{M}_4^{-1}\, d\mathbf{Z}\right)$$

The items of hypergeometry

$$\text{sign}\left[ds^2_{\mathcal{QM}}\right] = \Big(\underbrace{+,\dots,+}_{4+4n}\Big)$$

The HyperKähler two-forms and the $\mathfrak{su}(2)$-connection

$$\Phi = da + \mathbf{Z}^T \,\mathbb{C}\, \mathrm{d}\mathbf{Z}$$

$$\omega = \frac{\mathrm{i}}{\sqrt{2}} \sum_{x=1}^{3} \omega^x \gamma_x$$

$$\mathbf{K} = \frac{\mathrm{i}}{\sqrt{2}} \sum_{x=1}^{3} K^x \sigma_x$$

$$\{\gamma_x , \gamma_y\} = \delta^{xy} \mathbf{1}_{2\times 2}$$

$$\gamma_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

General formulae for su(2)

$$\omega = \begin{pmatrix} -\frac{\mathrm{i}}{2}\,\mathcal{Q} - \frac{\mathrm{i}}{4}\,e^{-U}\,\Phi & e^{-\frac{U}{2}}\,V^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} \\ -e^{-\frac{U}{2}}\,\overline{V}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} & \frac{\mathrm{i}}{2}\,\mathcal{Q} + \frac{\mathrm{i}}{4}\,e^{-U}\,\Phi \end{pmatrix}$$

$$\mathbf{K} \equiv d\omega + \omega \wedge \omega = \begin{pmatrix} \mathfrak{u} & \mathfrak{v} \\ -\overline{\mathfrak{v}} & -\mathfrak{u} \end{pmatrix}$$

$$\mathfrak{u} = -\mathrm{i}\frac{1}{2}K - \frac{1}{8}dS \wedge d\overline{S} - e^{-U}\,V^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} \wedge \overline{V}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z} - \frac{1}{4}e^{-U}\,\mathrm{d}\mathbf{Z}^T \wedge \mathbb{C}\,\mathrm{d}\mathbf{Z}$$

$$\mathfrak{v} = e^{-\frac{U}{2}}\left(DV^T \wedge \mathbb{C}\,\mathrm{d}\mathbf{Z} - \frac{1}{2}dS \wedge V^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right)$$

$$\overline{\mathfrak{v}} = e^{-\frac{U}{2}}\left(D\overline{V}^T \wedge \mathbb{C}\,\mathrm{d}\mathbf{Z} - \frac{1}{2}d\overline{S} \wedge \overline{V}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right)$$

$$\mathcal{Q} = \frac{\mathrm{i}}{2}\left(\partial_i \mathcal{K}\,dz^i - \partial_{i^\star}\mathcal{K}\,d\overline{z}^{i^\star}\right)$$

$$\begin{aligned} dS &= dU + \mathrm{i}\,e^{-U}\left(da + \mathbf{Z}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right) \\ d\overline{S} &= dU - \mathrm{i}\,e^{-U}\left(da + \mathbf{Z}^T\,\mathbb{C}\,\mathrm{d}\mathbf{Z}\right) \\ DV &= dz^i\,\nabla_i V \\ D\overline{V} &= d\overline{z}^{i^\star}\,\nabla_{i^\star} V \end{aligned}$$

c-map and triholomorphic moment maps

$$K^x = -\mathrm{i}4\sqrt{2}\,\mathrm{Tr}\,(\gamma^x\,\mathbf{K}) \equiv K^x_{uv}\,dq^u \wedge dq^v$$

$$J^{x|s}_u = K^x_{uv}\,h^{vs}$$

$$J^{x|s}_u\,J^{y|v}_s = -\delta^{xy}\,\delta^v_u + \varepsilon^{xyz}\,J^{z|v}_u$$

The above formulae are not only the general proof that the Riemaniann manifold $\mathcal{QM}$ defined by the metric XX is indeed a Quaternionic Kähler manifold, but, what is most relevant, they also provide an algorithm to write in terms of Special Geometry structures the tri-holomorphic moment map of the principal isometries possessed by $\mathcal{QM}$.

Types of isometries of the quaternionic manifold in the c-map image

a) The isometries of the $(2n+3)$–dimensional Heisenberg algebra $\mathbb{H}\mathrm{eis}$ which is always present and is universal for any $(4n+4)$–dimensional Quaternionic Kähler manifold in the image of the c-map. We describe it below.

b) All the isometries of the pre-image Special Kähler manifold $\mathcal{SK}_n$ that are promoted to isometries of the image manifold in a way described below.

c) The additional $2n+4$ isometries that occur only when $\mathcal{SK}_n$ is a symmetric space and such, as a consequence, is also the c-map image $\mathcal{QM}_{4n+4}$. We will discuss these isometries in section (4.3.4).

General Form of the Killing Vector

$$\begin{aligned}
\mathbf{k} &= k^u(q)\,\partial_u \\
&= k^\diamond \frac{\partial}{\partial U} + k^i \frac{\partial}{\partial z^i} + k^{i^\star} \frac{\partial}{\partial \bar{z}^{i^\star}} + k^\bullet \frac{\partial}{\partial a} + k^\alpha \frac{\partial}{\partial \mathbf{Z}^\alpha} \\
&\equiv k^\diamond\,\partial_\diamond + k^i\,\partial_i + k^{i^\star}\,\partial_{i^\star} + k^\bullet\,\partial_\bullet + k^\alpha\,\partial_\alpha
\end{aligned}$$

Killing vectors of Heis

$$Z^\alpha \mapsto Z^\alpha + \Lambda^\alpha \quad ; \quad a \mapsto a - \Lambda^T \mathbb{C}\mathbf{Z}$$

$$\begin{aligned}\overrightarrow{\mathbf{k}}_{[\Lambda]} &= \Lambda^\alpha \overrightarrow{\mathbf{k}}_\alpha \\ &= \Lambda^\alpha \partial_\alpha - \Lambda^T \mathbb{C}\mathbf{Z}\,\partial_\bullet\end{aligned}$$

$$\mathbf{i}_{[\Lambda]}\mathbf{K} \equiv \begin{pmatrix} \mathbf{i}_{[\Lambda]}\mathfrak{u} & \mathbf{i}_{[\Lambda]}\mathfrak{v} \\ -\mathbf{i}_{[\Lambda]}\overline{\mathfrak{v}} & -\mathbf{i}_{[\Lambda]}\mathfrak{u} \end{pmatrix} = \mathrm{d}\mathfrak{P}_{[\Lambda]} + \left[\omega\,,\,\mathfrak{P}_{[\Lambda]}\right]$$

The moment maps solving the equation are

$$\mathfrak{P}_{[\Lambda]} = \begin{pmatrix} -\frac{\mathrm{i}}{4}\,e^{-U}\,\Lambda^T\mathbb{C}\mathbf{Z} & \frac{1}{2}\,e^{-\frac{U}{2}}\,\Lambda^T C V \\ -\frac{1}{2}\,e^{-\frac{U}{2}}\,\Lambda^T C\overline{V} & \frac{\mathrm{i}}{4}\,e^{-U}\,\Lambda^T\mathbb{C}\mathbf{Z} \end{pmatrix}$$

Moment map for the central charge

$$a \mapsto a + \varepsilon \qquad \varepsilon \overrightarrow{\mathbf{k}}_{[\bullet]} = \varepsilon\, \partial_\bullet$$

$$\mathbf{i}_{[\bullet]} \mathbf{K} = \mathrm{d}\mathfrak{P}_{[\bullet]} + \left[\omega \,, \mathfrak{P}_{[\bullet]}\right]$$

$$\mathfrak{P}_{[\bullet]} = \begin{pmatrix} -\frac{\mathrm{i}}{8}\, e^{-U} & 0 \\ 0 & \frac{\mathrm{i}}{8}\, e^{-U} \end{pmatrix}$$

Killing vectors of the special Kaehler and their moments maps

$$k^i_{\mathbf{I}}(z)\,\partial_i\,\Omega(z) = \exp\left[f_{\mathbf{I}}(z)\right]\,\mathfrak{T}_{\mathbf{I}}\,\Omega(z)$$

$$z^i \mapsto z^i + k^i_{\mathbf{I}}(z) \quad ; \quad \mathbf{Z} \mapsto \mathbf{Z} + \mathfrak{T}_{\mathbf{I}}\mathbf{Z}$$

$$\mathbf{k}_{\mathbf{I}} = k^i_{\mathbf{I}}(z)\,\partial_i + k^{i^\star}_{\mathbf{I}}(\bar{z})\,\partial_{i^\star} + \left(\mathfrak{T}_{\mathbf{I}}\right)^{\alpha}{}_{\beta}\,\mathbf{Z}^{\beta}\,\partial_{\alpha}$$

$$\mathfrak{P}_{[\mathbf{I}]} = \begin{pmatrix} \frac{\mathrm{i}}{4}\left(\mathscr{P}_{\mathbf{I}} + \frac{1}{2}\,e^{-U}\,\mathbf{Z}^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\mathbf{Z}\right) & -\frac{1}{2}\,e^{-U/2}\,V^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\mathbf{Z} \\ \frac{1}{2}\,e^{-U/2}\,\overline{V}^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\mathbf{Z} & -\frac{\mathrm{i}}{4}\left(\mathscr{P}_{\mathbf{I}} + \frac{1}{2}\,e^{-U}\,\mathbf{Z}^T\,\mathbb{C}\,\mathfrak{T}_{\mathbf{I}}\mathbf{Z}\right) \end{pmatrix}$$

Symmetric manifolds in the image of the c-map

When the Special Kähler manifold $\mathcal{SK}_n$ is a symmetric coset space, it turns out that the metric XX is actually the symmetric metric on an enlarged symmetric coset manifold

$$\mathcal{QM}_{4n+4} = \frac{\mathrm{U}_Q}{\mathrm{H}_Q} \supset \frac{\mathrm{U}_{\mathcal{SK}}}{\mathrm{H}_{\mathcal{SK}}}$$

Naming $\Lambda[\mathfrak{g}]$ the $\mathbf{W}$-representation of any finite element of the $\mathfrak{g} \in \mathrm{U}_{\mathcal{SK}}$ group, we have that the matrix $\mathcal{M}_4(z,\bar{z})$ transforms as follows:

$$\mathcal{M}_4\left(\mathfrak{g}\cdot z, \mathfrak{g}\cdot \bar{z}\right) = \Lambda\left[\mathfrak{g}\right] \mathcal{M}_4\left(z,\bar{z}\right)\right] \Lambda^T\left[\mathfrak{g}\right]$$

$$\mathfrak{g}_z \cdot z_0 = z$$

$$\mathcal{M}_4^{-1}(z,\bar{z}) = \Lambda^T\left[\mathfrak{g}_z^{-1}\right] \mathcal{M}_4^{-1}(z_0,\bar{z}_0)\right] \Lambda\left[\mathfrak{g}_z^{-1}\right]$$

This allows to introduce a set of $4n+4$ vielbein defined in the following way:

$$E^I_{\mathcal{QM}} = \frac{1}{2}\left\{ dU\, , \underbrace{e^i(z)}_{2n}, e^{-U}\left(da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z}\right), \underbrace{e^{-\frac{U}{2}}\Lambda\left[\mathfrak{g}_z^{-1}\right] \mathrm{d}\mathbf{Z}}_{2n+2} \right\}$$

$$ds^2_{\mathcal{QM}} = E^I_{\mathcal{QM}}\, \mathfrak{q}_{IJ}\, E^J_{\mathcal{QM}}$$

Identifying U_Q via its solvable Lie algebra

$$\mathfrak{q}_{IJ} = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & \delta_{ij} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -2\,\mathscr{M}_4^{-1}(z_0,\bar{z}_0) \end{array}\right)$$

$$dE^I_{\mathscr{Q}\mathscr{M}} - \frac{1}{2} f^I{}_{JK}\, E^J_{\mathscr{Q}\mathscr{M}} \wedge E^K_{\mathscr{Q}\mathscr{M}} = 0$$

where $f^I{}_{JK}$ are the structure constants of a solvable Lie algebra $\mathfrak{A}$ which can be identified as follows:

$$\mathfrak{A} = Solv\left(\frac{\mathrm{U}_{\mathscr{Q}}}{\mathrm{H}_{\mathscr{Q}}}\right)$$

Universal Heisenber algebras

Inspecting eq.s (1.7.19) we immediately realize that the Lie Algebra $\mathbb{U}_Q$ contains two universal Heisenberg subalgebras of dimension $(2n+3)$, namely:

$$\mathbb{U}_{\mathcal{Q}} \supset \mathbb{H}\mathrm{eis}_1 = \mathrm{span}_{\mathbb{R}} \left\{ \mathbf{W}^{1\alpha}, \mathbb{Z}_1 \right\} \quad ; \quad \mathbb{Z}_1 = L_+ \equiv L^1 + L^2$$
$$\left[\mathbf{W}^{1\alpha}, \mathbf{W}^{1\beta} \right] = -\frac{1}{2} \mathbb{C}^{\alpha\beta} \mathbb{Z}_1 \quad ; \quad \left[\mathbb{Z}_1, \mathbf{W}^{1\beta} \right] = 0$$

$$\mathbb{U}_{\mathcal{Q}} \supset \mathbb{H}\mathrm{eis}_2 = \mathrm{span}_{\mathbb{R}} \left\{ \mathbf{W}^{2\alpha}, \mathbb{Z}_2 \right\} \quad ; \quad \mathbb{Z}_2 = L_- \equiv L^1 - L^2$$
$$\left[\mathbf{W}^{2\alpha}, \mathbf{W}^{2\beta} \right] = -\frac{1}{2} \mathbb{C}^{\alpha\beta} \mathbb{Z}_2 \quad ; \quad \left[\mathbb{Z}_2, \mathbf{W}^{2\beta} \right] = 0$$

The Tits Satake Projection

Although originally introduced in the mathematical literature, the Tits Satake projection reveals its profund meaning in the supergravity context, in particular in connection with the c-map and the c*-map.

It allows to arrange manifolds into universality classes that provide a useful classification for supergravity models and the exploration of their general properties.

Structure of the Tits Satake projection

$$\mathscr{S}\mathscr{H} \stackrel{\text{Tits-Satake}}{\Longrightarrow} \mathscr{S}\mathscr{H}_{\text{TS}}$$

1. π_{TS} is a projection operator, so that several different manifolds $\mathscr{S}\mathscr{H}_i$ ($i = 1,\dots,r$) have the same image $\pi_{\text{TS}}(\mathscr{S}\mathscr{H}_i)$.
2. π_{TS} preserves the rank of $\mathscr{G}_M$ namely the dimension of the maximal Abelian semisimple subalgebra (Cartan subalgebra) of $\mathscr{G}_M$.
3. π_{TS} maps special homogeneous into special homogeneous manifolds. Not only. It preserves the two classes of manifolds discussed above, namely maps *special Kähler* into *special Kähler* and maps *Quaternionic* into *Quaternionic*
4. π_{TS} commutes with c–map, so that we obtain the following commutative diagram:

$$\begin{array}{ccc} \text{Special Kähler} & \stackrel{c\text{-map}}{\Longrightarrow} & \text{Quaternionic-Kähler} \\ \pi_{\text{TS}} \Downarrow & & \pi_{\text{TS}} \Downarrow \\ (\text{Special Kähler})_{\text{TS}} & \stackrel{c\text{-map}}{\Longrightarrow} & (\text{Quaternionic-Kähler})_{\text{TS}} \end{array}$$

TS commutes also with the c*-map

$$\begin{array}{ccc} \text{Special Kähler} & \stackrel{c^\star\text{-map}}{\Longrightarrow} & \text{Pseudo-Quaternionic-Kähler} \\ \pi_{\mathrm{TS}} \Downarrow & & \pi_{\mathrm{TS}} \Downarrow \\ (\text{Special Kähler})_{\mathrm{TS}} & \stackrel{c^\star\text{-map}}{\Longrightarrow} & (\text{Pseudo-Quaternionic-Kähler})_{\mathrm{TS}} \end{array}$$

$$\Pi_{\mathrm{TS}} \quad : \quad \Delta_{\mathbb{G}} \mapsto \Delta_{\mathrm{TS}}\,; \qquad \Delta_{\mathbb{G}} \stackrel{\alpha_\perp = 0}{\longmapsto} \overline{\Delta} \underset{\text{multiplicities}}{\stackrel{\text{deleting}}{\longmapsto}} \Delta_{\mathrm{TS}}\,.$$

$$\Delta_{\mathrm{TS}} = \text{root system of } \mathbb{G}_{\mathrm{TS}}\,, \qquad \mathbb{G}_{\mathrm{TS}} \subset \mathbb{G}_R\,.$$

The concept of Paint Group

We saw that each real form G_R of a Lie algebra is in one-to-one correspondence with a symmetric space $M=G_R/H$. This latter singles out a solvable Lie algebra as we have seen. The group of external Automorphisms of Solv(M) is the Paint Group.

$$\mathbb{G}_{\text{paint}} = \text{Aut}_{\text{Ext}}\left[\text{Solv}\left(\mathscr{M}\right)\right] \qquad \text{Aut}_{\text{Ext}}\left[\text{Solv}\left(\mathscr{M}\right)\right] \equiv \frac{\text{Aut}\left[\text{Solv}\left(\mathscr{M}\right)\right]}{\text{Solv}\left(\mathscr{M}\right)},$$

$$\text{Solv}\left(\mathscr{M}\right) = \text{maximally split} \Leftrightarrow \text{Aut}_{\text{Ext}}\left[\text{Solv}\left(\mathscr{M}\right)\right] = \emptyset.$$

The sub Tits Satake algebra and the long roots

Let r be the rank of $\mathrm{Solv}(\mathscr{M})$, namely the number of its Cartan generators H_i and n the number of its nilpotent generators $\mathscr{W}_\alpha$, namely the number of generalized roots α. The whole set of Cartan generators H_i, plus a subset of p nilpotent generators $\mathscr{W}_{\alpha^\ell}$ associated with roots α^ℓ that we name *long*, close a solvable subalgebra $\mathrm{Solv}_{\mathrm{subTS}} \subset \mathrm{Solv}(\mathscr{M})$ that is made of singlets under the action of the paint Lie algebra $\mathbb{G}_{\mathrm{paint}}$, *i.e.*

$$\begin{aligned} \mathrm{Solv}_{\mathrm{subTS}} &= \mathrm{span}\,\{H_i, \mathscr{W}_{\alpha^\ell}\} \\ [\mathrm{Solv}_{\mathrm{subTS}}\,, \mathrm{Solv}_{\mathrm{subTS}}] &\subset \mathrm{Solv}_{\mathrm{subTS}}, \\ \forall X \in \mathbb{G}_{\mathrm{paint}}\,, \forall \Psi \in \mathrm{Solv}_{\mathrm{subTS}} &: \; [X\,,\Psi] = 0. \end{aligned}$$

We name $\mathrm{Solv}_{\mathrm{subTS}}$ the *sub Tits-Satake algebra*. By definition $\mathrm{Solv}_{\mathrm{subTS}}$ has the same rank as the original solvable algebra $\mathrm{Solv}(\mathscr{M})$. In all possible cases, it is the solvable Lie algebra of a symmetric maximally split coset $\mathbb{G}_{\mathrm{subTS}}/\mathbb{H}_{\mathrm{subTS}}$. In this way, eventually, we have the notion of a semisimple Lie algebra $\mathbb{G}_{\mathrm{subTS}}$.

By definition G_{subTS} commutes with the Paint Group.
The long roots are those whose projections are singlets under the Paint Group.

The short roots

Considering the orthogonal decomposition of the original solvable Lie algebra with respect to its *sub Tits-Satake algebra*:

$$\mathrm{Solv}\,(\mathscr{M}) = \mathrm{Solv}_{\mathrm{subTS}} \oplus \mathbb{K}_{\mathrm{short}}.$$

$$\mathbb{K}_{\mathrm{short}} = \bigoplus_{\wp=1}^{q} \mathbb{D}\left[\mathscr{P}^{+}_{\wp}, \mathbf{Q}_{\wp}\right],$$

$\mathbb{D}\left[\mathscr{P}^{+}_{\wp}, \mathbf{Q}_{\wp}\right] = \mathscr{P}^{+}_{\wp} \otimes \mathbf{Q}_{\wp}$ is the tensor product:

of an irreducible module $\mathbf{Q}_{\wp}$ (i.e. representation) of the compact paint algebra $\mathbb{G}_{\mathrm{paint}}$ with an irreducible module $\mathscr{P}^{+}_{\wp}$ of the solvable sub Tits-Satake algebra $\mathrm{Solv}_{\mathrm{subTS}}$.

$$\mathscr{P}^{+}_{\wp} = \bigoplus_{s=1}^{n_{\wp}} \mathbb{W}[\alpha^{(\wp,s)}], \qquad n_{\wp} = \dim \mathscr{P}^{+}_{\wp},$$

where each $\mathbb{W}[\alpha^{(\wp,s)}]$ is an eigenspace of the CSA of $\mathbb{G}_{\mathrm{subTS}}$

$$\forall H_i \in \mathrm{CSA}\,(\mathrm{Solv}\,(\mathscr{M}))\,, \forall \Psi \in \mathbb{W}[\alpha^{(\wp,s)}] \otimes \mathbf{Q}_{\wp} \quad : \quad [H_i\,, \Psi] = \alpha_i^{(\wp,s)}\,\Psi.$$

A general structure

The decomposition of $\mathbb{K}_{\text{short}}$ mentioned above has actually a general form depending on the rank. We will discuss this here for the quaternionic-Kähler manifolds.

r = 4) $\mathbb{G}_{\text{subTS}} = \mathrm{SO}(4,4)$ $\mathscr{P}_{\mathbf{8}_\mathbf{v}}, \mathscr{P}_{\mathbf{8}_\mathbf{s}}, \mathscr{P}_{\mathbf{8}_{\overline{\mathbf{s}}}}$, Denoting the: half spaces by $\mathbf{4}^+_{\mathbf{v},\mathbf{s},\overline{\mathbf{s}}}$,

$$\mathbb{K}_{\text{short}} = \left(\mathbf{4}^+_{\mathbf{v}}, \mathbf{Q}_{\mathbf{v}}\right) \oplus \left(\mathbf{4}^+_{\mathbf{s}}, \mathbf{Q}_{\mathbf{s}}\right) \oplus \left(\mathbf{4}^+_{\overline{\mathbf{s}}}, \mathbf{Q}_{\overline{\mathbf{s}}}\right),$$

where $\mathbf{Q}_{\mathbf{v},\mathbf{s},\overline{\mathbf{s}}}$ are three different irreducible modules of $\mathbb{G}_{\text{paint}}$

We analyse an example where $\mathbb{G}_{\text{paint}} = \mathrm{SO}(8)$

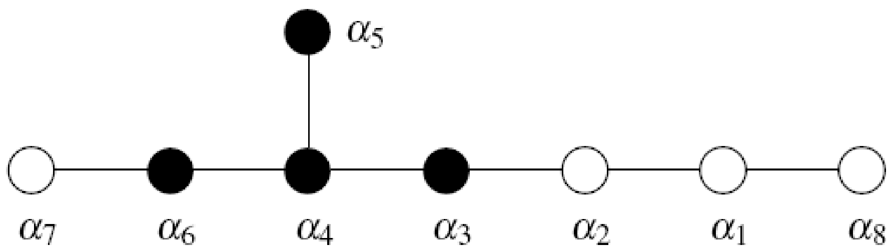
$$\mathbf{Q}_{\mathbf{v}} = \mathbf{8}_{\mathbf{v}} \quad ; \quad \mathbf{Q}_{\mathbf{s}} = \mathbf{8}_{\mathbf{s}} \quad ; \quad \mathbf{Q}_{\overline{\mathbf{s}}} = \mathbf{8}_{\overline{\mathbf{s}}}$$

A primary example $E_{(8,-24)}$

$$\Delta_{E_8} \equiv \left\{ \begin{array}{lr} \pm\varepsilon_i \pm \varepsilon_j & (i \neq j)\ \mathbf{112} \\ \underbrace{\pm\tfrac{1}{2}\varepsilon_1 \pm \tfrac{1}{2}\varepsilon_2 \pm \tfrac{1}{2}\varepsilon_3 \pm \tfrac{1}{2}\varepsilon_4 \pm \tfrac{1}{2}\varepsilon_5 \pm \tfrac{1}{2}\varepsilon_6 \pm \tfrac{1}{2}\varepsilon_7 \pm \tfrac{1}{2}\varepsilon_8}_{\text{even number of minus signs}} & \mathbf{128} \\ \hline & \mathbf{240} \end{array} \right\}$$

$$\Pi_{\mathrm{TS}} \quad : \quad \frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_{7(-133)} \times \mathrm{SU}(2)} \longrightarrow \frac{\mathrm{F}_{4(4)}}{\mathrm{USp}(6) \times \mathrm{SU}(2)}$$

The primary example continued



$$G_{\text{paint}} = SO(8)$$

$$\alpha_\perp = \left\{\alpha^4\,,\,\alpha^5\,,\,\alpha^6\,,\,\alpha^3\right\} \quad ; \quad \alpha_\| = \left\{\alpha^1\,,\,\alpha^2\,,\,\alpha^7\,,\,\alpha^8\right\}$$

and the projection TS immediately yields the following restricted root system:

$$\Delta_{\text{TS}} = \left\{ \begin{array}{ccr} \pm\varepsilon_i \pm \varepsilon_j & (i \neq j \quad ; \quad i,j = 1,2,3,8) & \mathbf{24} \\ \pm\varepsilon_i & (i = 1,2,3,8) & \mathbf{8} \\ \pm\frac{1}{2}\varepsilon_1 \pm \frac{1}{2}\varepsilon_2 \pm \frac{1}{2}\varepsilon_3 \pm \frac{1}{2}\varepsilon_8 & & \mathbf{16} \\ \hline & & \mathbf{48} \end{array} \right\},$$

Decompositions

$$\mathrm{SO}(4,4) \times \mathrm{SO}(8) \subset \mathrm{E}_{8(-24)},$$

$$\mathbf{248} \stackrel{\mathrm{SO}(4,4)\times\mathrm{SO}(8)}{\longrightarrow} (1,\mathbf{28}) \oplus (\mathbf{28}^{\mathrm{nc}},\mathbf{1}) \oplus (\mathbf{8_v}^{\mathrm{nc}},\mathbf{8_v}) \oplus (\mathbf{8_s}^{\mathrm{nc}},\mathbf{8_s}) \oplus (\mathbf{8_{\overline{s}}}^{\mathrm{nc}},\mathbf{8_{\overline{s}}})$$

$$\mathrm{F}_{4(4)} \times \mathrm{G}_{2(-14)} \subset \mathrm{E}_{8(-24)}$$

$$\mathbf{248} \stackrel{\mathrm{F}_{4(4)}\times\mathrm{G}_{2(-14)}}{\longrightarrow} (\mathbf{52},\mathbf{1}) \oplus (\mathbf{1},\mathbf{14}) \oplus (\mathbf{26},\mathbf{7})$$

Special Kähler $\mathcal{SK}_n$	Quaternionic $\mathcal{QM}_{4n+4}$	Tits Satake projection of Quater. $\mathcal{QM}_{\mathrm{TS}}$
$\frac{\mathrm{U(s+1,1)}}{\mathrm{U(s+1)\times U(1)}}$	$\frac{\mathrm{U(s+2,2)}}{\mathrm{U(s+2)\times U(2)}}$	$\frac{\mathrm{U(3,2)}}{\mathrm{U(3)\times U(2)}}$
$\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}}$	$\frac{\mathrm{G_{(2,2)}}}{\mathrm{SU(2)\times SU(2)}}$	$\frac{\mathrm{G_{(2,2)}}}{\mathrm{SU(2)\times SU(2)}}$
$\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}} \times \frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}}$	$\frac{\mathrm{SO(3,4)}}{\mathrm{SO(3)\times SO(4)}}$	$\frac{\mathrm{SO(3,4)}}{\mathrm{SO(3)\times SO(4)}}$
$\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}} \times \frac{\mathrm{SO(p+2,2)}}{\mathrm{SO(p+2)\times SO(2)}}$	$\frac{\mathrm{SO(p+4,4)}}{\mathrm{SO(p+4)\times SO(4)}}$	$\frac{\mathrm{SO(5,4)}}{\mathrm{SO(5)\times SO(4)}}$
$\frac{\mathrm{Sp(6)}}{\mathrm{U(3)}}$ $\frac{\mathrm{SU(3,3)}}{\mathrm{SU(3)\times SU(3)\times U(1)}}$ $\frac{\mathrm{SO^\star(12)}}{\mathrm{SU(6)\times U(1)}}$ $\frac{\mathrm{E_{(7,-25)}}}{\mathrm{E_{(6,-78)}\times U(1)}}$	$\frac{\mathrm{F_{(4,4)}}}{\mathrm{Usp(6)\times SU(2)}}$ $\frac{\mathrm{E_{(6,-2)}}}{\mathrm{SU(6)\times SU(2)}}$ $\frac{\mathrm{E_{(7,-5)}}}{\mathrm{SO(12)\times SU(2)}}$ $\frac{\mathrm{E_{(8,-24)}}}{\mathrm{E_{(7,-133)}\times SU(2)}}$	$\frac{\mathrm{F_{(4,4)}}}{\mathrm{Usp(6)\times SU(2)}}$

Special Kähler $\mathcal{SK}_n$	Pseudo-Quaternionic $\mathcal{QM}^{\star}_{4n+4}$	Tits Satake proj. of Pseudo Quater. $\mathcal{QM}^{\star}_{\mathrm{TS}}$
$\frac{\mathrm{U(s+1,1)}}{\mathrm{U(s+1)\times U(1)}}$	$\frac{\mathrm{U(s+2,2)}}{\mathrm{U(s+1,1)\times U(1,1)}}$	$\frac{\mathrm{U(3,2)}}{\mathrm{U(2,1)\times U(1,1)}}$
$\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}}$	$\frac{\mathrm{G_{(2,2)}}}{\mathrm{SU(1,1)\times SU(1,1)}}$	$\frac{\mathrm{G_{(2,2)}}}{\mathrm{SU(1,1)\times SU(1,1)}}$
$\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}} \times \frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}}$	$\frac{\mathrm{SO(3,4)}}{\mathrm{SO(2,1)\times SO(2,2)}}$	$\frac{\mathrm{SO(3,4)}}{\mathrm{SO(1,2)\times SO(2,2)}}$
$\frac{\mathrm{SU(1,1)}}{\mathrm{U(1)}} \times \frac{\mathrm{SO(p+2,2)}}{\mathrm{SO(p+2)\times SO(2)}}$	$\frac{\mathrm{SO(p+4,4)}}{\mathrm{SO(p+2,2)\times SO(2,2)}}$	$\frac{\mathrm{SO(5,4)}}{\mathrm{SO(3,2)\times SO(2,2)}}$
$\frac{\mathrm{Sp(6)}}{\mathrm{U(3)}}$ $\frac{\mathrm{SU(3,3)}}{\mathrm{SU(3)\times SU(3)\times U(1)}}$ $\frac{\mathrm{SO^{\star}(12)}}{\mathrm{SU(6)\times U(1)}}$ $\frac{\mathrm{E_{(7,-25)}}}{\mathrm{E_{(6,-78)}\times U(1)}}$	$\frac{\mathrm{F_{(4,4)}}}{\mathrm{Sp(6)\times SU(1,1)}}$ $\frac{\mathrm{E_{(6,-2)}}}{\mathrm{SU(3,3)\times SU(1,1)}}$ $\frac{\mathrm{E_{(7,-5)}}}{\mathrm{SO^{\star}(12)\times SU(1,1)}}$ $\frac{\mathrm{E_{(8,-24)}}}{\mathrm{E_{(7,-25)}\times SU(1,1)}}$	$\frac{\mathrm{F_{(4,4)}}}{\mathrm{Sp(6)\times SU(1,1)}}$

Supergravity relevant symmetric spaces

An analysis of supergravity models according to the classification of their scalar manifolds.

General properties of these manifolds

1. The A_1 root-system associated with the $\mathfrak{sl}(2,\mathbb{R})_E$ algebra in the decomposition GOLD is made of $\pm\psi$ where ψ is the highest root of $\mathbb{U}_{D=3}$.
2. Out of the r simple roots α_i of $\mathbb{U}_{D=3}$ there are $r-1$ that have grading zero with respect to ψ and just one α_W that has grading 1:

$$\begin{aligned} (\psi\,,\alpha_i) &= 0 \qquad i \neq W \\ (\psi\,,\alpha_W) &= 1 \end{aligned} \tag{5.5.1}$$

3. The only simple root α_W that has non vanishing grading with respect ψ is just the highest weight of the symplectic representation $\mathbf{W}$ of $\mathbb{U}_{D=4}$ to which the vector fields are assigned.
4. The Dynkin diagram of $\mathbb{U}_{D=4}$ is obtained from that of $\mathbb{U}_{D=3}$ by removing the dot corresponding to the special root α_W.
5. Hence we can arrange a basis for the simple roots of the rank r algebra $\mathbb{U}_{D=3}$ such that:

$$\begin{aligned} \alpha_i &= \{\overline{\alpha}_i, 0\} \quad ; \quad i \neq W \\ \alpha_W &= \left\{\overline{\mathbf{w}}_h, \frac{1}{\sqrt{2}}\right\} \\ \psi &= \left\{\mathbf{0}, \sqrt{2}\right\} \end{aligned} \tag{5.5.2}$$

where $\overline{\alpha}_i$ are $(r-1)$–component vectors representing a basis of simple roots for the Lie algebra $\mathbb{U}_{D=4}$, $\overline{\mathbf{w}}_h$ is also an $(r-1)$–vector representing the *highest weight* of the representation $\mathbf{W}$.

Structure of the root systems according to the golden split and theTits Satake projection

This means that the entire root system and the Cartan subalgebra of the $\mathbb{U}_{D=3}$ Lie algebra can be organized as follows:

$$\begin{array}{lclll}
\pm\psi & = & \pm\left(\mathbf{0},\sqrt{2}\right) & ; & & 2 \\
\pm\hat{\alpha} & = & \pm\left(\alpha,\sqrt{2}\right) & ; & 2\times\#\text{ of roots} = & 2n_r \\
\pm\hat{w} & = & \pm\left(w,\frac{\sqrt{2}}{2}\right) & ; & 2\times\#\text{ of weights} = & 2\times\dim\mathbf{W} \\
\mathscr{H}^i & \in \mathrm{CSA}\subset\mathbb{U}_{D=4} & & ; & \mathrm{rank}\mathbb{U}_{D=4} = & r \\
\mathscr{H}^\psi & & & & & 1 \\
\hline
 & & & & \dim\mathbb{U}_{D=4} & = 3+\dim\mathbb{U}_{D=3}+2\times\dim\mathbf{W}
\end{array}$$

$$\begin{array}{c}
\mathrm{adj}(\mathbb{U}_{D=3}) = \mathrm{adj}(\mathbb{U}_{D=4})\oplus\mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E)\oplus W_{(2,W)} \\
\Downarrow \\
\mathrm{adj}(\mathbb{U}^{TS}_{D=3}) = \mathrm{adj}(\mathbb{U}^{TS}_{D=4})\oplus\mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E)\oplus W_{(2,W^{TS})}
\end{array}$$

$$\Pi^{TS} \quad : \quad \psi \to \psi^{TS}$$

Classification of SUGRA symmetric spaces (non exotic)

1. The $\mathcal{N}=8$ supergravity theory, which is the maximal one in $D=4$,
2. The $\mathcal{N}=2$ supergravity theory with a single vector multiplet and non-vanishing Yukawa coupling(model 2).
3. The $\mathcal{N}=4$ supergravity theory with 5 vector multiplets (model 11).
4. The $\mathcal{N}=4$ supergravity theory with 6 vector multiplets which is obtained compactifying a type II theory on a $\mathrm{T}^6/\mathbb{Z}_2$ orbifold (model 12).
5. The $\mathcal{N}=2$ theory with two vector multiplets and non vanishing Yukawa couplings, usually called the st-model (model 14).
6. The $\mathcal{N}=2$ theory with three vector multiplets and non vanishing Yukawa couplings, usually called the stu-model (model 15).

Next we have two universality classes, each containing an infinite number of elements. They are

1. The $\mathcal{N}=4$ supergravity theory with $n=6+p$ vector multiplets ($p\geq 1$), (model 13).
2. The $\mathcal{N}=2$ supergravity theory with $n=3+p$ vector multiplets ($p\geq 1$) and non vanishing Yukawa couplings (model 16).

A most interesting TS class

We still have the very interesting 4-element universality class whose maximally split representative corresponds to the maximally split special Kähler manifold $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SU}(3)\times \mathrm{U}(1)}$. This class contains the models 3, 4, 5, 6 distinguished by quite peculiar Paint groups. We will thoroughly analyze the structure of this class.

3	$\frac{\mathrm{Sp}(6,\mathrm{R})}{\mathrm{SU}(3)\times \mathrm{U}(1)}$	$\frac{\mathrm{F}_{4(4)}}{\mathrm{Sp}(6,\mathrm{R})\times \mathrm{SL}(2,\mathrm{R})}$	$\frac{\mathrm{Sp}(6,\mathrm{R})}{\mathrm{SU}(3)\times \mathrm{U}(1)}$	$\frac{\mathrm{F}_{4(4)}}{\mathrm{Sp}(6,\mathrm{R})\times \mathrm{SL}(2,\mathrm{R})}$	1	1	$\mathcal{N}=2$ $n=6$
4			$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times \mathrm{SU}(3)\times \mathrm{U}(1)}$	$\frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(3,3)\times \mathrm{SL}(2,\mathrm{R})}$	$\mathrm{SO}(2)\times \mathrm{SO}(2)$	1	$\mathcal{N}=2$ $n=9$
5			$\frac{\mathrm{SO}^\star(12)}{\mathrm{SU}(6)\times \mathrm{U}(1)}$	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}^\star(12)\times \mathrm{SL}(2,\mathrm{R})}$	$\mathrm{SO}(3)\times \mathrm{SO}(3)$ $\times \mathrm{SO}(3)$	$\mathrm{SO}(3)_{\mathrm{d}}$	$\mathcal{N}=6$ $\mathcal{N}=2$ n=16
6			$\frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_{6(-78)}\times \mathrm{U}(1)}$	$\frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_{7(-25)}\times \mathrm{SL}(2,\mathrm{R})}$	$\mathrm{SO}(8)$	$G_{2(-14)}$	$\mathcal{N}=2$ $n=27$

The exotic models

#	TS D=4	TS D=3	coset D=4	coset D=3	Paint Group	subP Group	susy
1_e	bc_1	bc_2	$\frac{\mathrm{SU(p+1,1)}}{\mathrm{SU(p+1)\times U(1)}}$	$\frac{\mathrm{SU(p+2,2)}}{\mathrm{SU(p+1,1)\times SL(2,R)_{h^\star}}}$	$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(\mathrm{p})$	$\mathrm{U}(\mathrm{p}-1)$	$\mathcal{N}=2$ n=p+1
2_e	bc_3	bc_4	$\frac{\mathrm{SU(p+1,3)}}{\mathrm{SU(p+1)\times SU(3)\times U(1)}}$	$\frac{\mathrm{SU(p+2,4)}}{\mathrm{SU(p+1,2)\times SU(1,2)\times U(1)}}$	$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(\mathrm{p})$	$\mathrm{U}(\mathrm{p}-1)$	$\mathcal{N}=3$ n=p+1
3_e	bc_1	bc_2	$\frac{\mathrm{SU(5,1)}}{\mathrm{SU(5)\times U(1)}}$	$\frac{\mathrm{E}_{6(-14)}}{\mathrm{SO^\star(10)\times SO(2)}}$	$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(4)$	$\mathrm{U}(3)$	$\mathcal{N}=5$

Classification tab 1 non exotic

#	TS D=4	TS D=3	coset D=4	coset D=3	Paint Group	subP Group	susy
1	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO^\star(16)}$	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO^\star(16)}$	1	1	$\mathcal{N}=8$
2	$\frac{SU(1,1)}{U(1)}$	$\frac{G_{2(2)}}{SL(2,R)\times SL(2,R)}$	$\frac{SU(1,1)}{U(1)}$	$\frac{G_{2(2)}}{SL(2,R)\times SL(2,R)}$	1	1	$\mathcal{N}=2$ n=1
3	$\frac{Sp(6,R)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{Sp(6,R)\times SL(2,R)}$	$\frac{Sp(6,R)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{Sp(6,R)\times SL(2,R)}$	1	1	$\mathcal{N}=2$ $n=6$
4			$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(2)}}{SU(3,3)\times SL(2,R)}$	$SO(2)\times SO(2)$	1	$\mathcal{N}=2$ $n=9$
5			$\frac{SO^\star(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO^\star(12)\times SL(2,R)}$	$SO(3)\times SO(3)$ $\times SO(3)$	$SO(3)_d$	$\mathcal{N}=6$ $\mathcal{N}=2$ n=16
6			$\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times SL(2,R)}$	$SO(8)$	$G_{2(-14)}$	$\mathcal{N}=2$ $n=27$

Classification tab 2 non exotic

#	TS D=4	TS D=3	coset D=4	coset D=3	Paint Group	subP Group	susy
7	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(2,1)}{SO(2)}$	$\frac{SO(4,3)}{SO(2,2)\times SO(2,1)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,1)}{SO(6)}$	$\frac{SO(8,3)}{SO(6,2)\times SO(2,1)}$	SO(5)	SO(4)	$\mathcal{N}=4$ n=1
8	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(3,2)}{SO(3)\times SO(2)}$	$\frac{SO(5,4)}{SO(3,2)\times SO(2,2)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,2)}{SO(6)\times SO(2)}$	$\frac{SO(8,4)}{SO(6,2)\times SO(2,2)}$	SO(4)	SO(3)	$\mathcal{N}=4$ n=2
9	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(4,3)}{SO(4)\times SO(3)}$	$\frac{SO(6,5)}{SO(4,2)\times SO(2,3)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,3)}{SO(6)\times SO(3)}$	$\frac{SO(8,5)}{SO(6,2)\times SO(2,3)}$	SO(3)	SO(2)	$\mathcal{N}=4$ n=3
10	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(5,4)}{SO(5)\times SO(4)}$	$\frac{SO(7,6)}{SO(5,2)\times SO(2,4)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,4)}{SO(6)\times SO(4)}$	$\frac{SO(8,6)}{SO(6,2)\times SO(2,4)}$	SO(2)	1	$\mathcal{N}=4$ n=4
11	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,5)}{SO(6)\times SO(5)}$	$\frac{SO(8,7)}{SO(6,2)\times SO(2,5)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,5)}{SO(6)\times SO(5)}$	$\frac{SO(8,7)}{SO(6,2)\times SO(2,5)}$	1	1	$\mathcal{N}=4$ n=5
12	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,6)}{SO(6)\times SO(6))}$	$\frac{SO(8,8)}{SO(6,2)\times SO(2,6)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,6)}{SO(6)\times SO(6))}$	$\frac{SO(8,8)}{SO(6,2)\times SO(2,6)}$	1	1	$\mathcal{N}=4$ n=6
13	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,7)}{SO(6)\times SO(7))}$	$\frac{SO(8,9)}{SO(6,2)\times SO(2,7)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(6,6+p)}{SO(6)\times SO(6+p))}$	$\frac{SO(8,8+p)}{SO(6,2)\times SO(2,6+p)}$	SO(p)	SO(p − 1)	$\mathcal{N}=4$ n=6+p

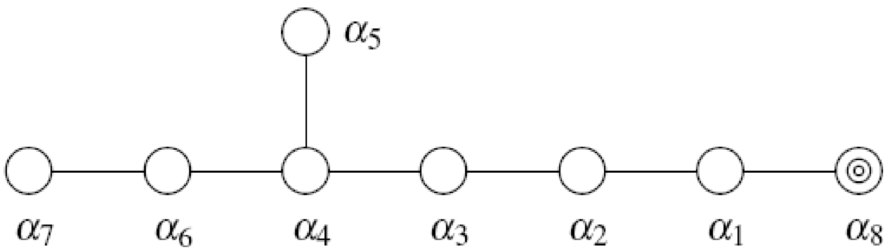
Classification tab 3: non exotic

#	TS D=4	TS D=3	coset D=4	coset D=3	Paint Group	subP Group	susy
14	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(2,1)}{SO(2)}$	$\frac{SO(4,3)}{SO(2,2)\times SO(2,1)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(2,1)}{SO(2)}$	$\frac{SO(4,3)}{SO(2,2)\times SO(2,1)}$	1	1	$\mathcal{N}=2$ n=2
15	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$	$\frac{SO(4,4)}{SO(2,2)\times SO(2,2)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$	$\frac{SO(4,4)}{SO(2,2)\times SO(2,2)}$	1	1	$\mathcal{N}=2$ n=3
16	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(2,3)}{SO(2)\times SO(3)}$	$\frac{SO(4,5)}{SO(2,2)\times SO(2,3)}$	$\frac{SL(2,\mathbb{R})}{O(2)} \times \frac{SO(2,2+p)}{SO(2)\times SO(2+p)}$	$\frac{SO(4,4+p)}{SO(2,2)\times SO(2,2+p)}$	SO(p)	SO(p − 1)	$\mathcal{N}=2$ n=3+p

N=8

$$\mathrm{adj}\,\mathrm{E}_{8(8)} = \mathrm{adj}\,\mathrm{E}_{7(7)} \oplus \mathrm{adj}\,\mathrm{SL}(2,\mathbb{R})_{\mathrm{E}} \oplus (\mathbf{2},\mathbf{56})$$

$$\psi = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7 + 2\alpha_8$$



$$\begin{aligned}
\alpha_1 &= \{1,-1,0,0,0,0,0,0\} &&= \{\overline{\alpha}_1,0\} \\
\alpha_2 &= \{0,1,-1,0,0,0,0,0\} &&= \{\overline{\alpha}_2,0\} \\
\alpha_3 &= \{0,0,1,-1,0,0,0,0\} &&= \{\overline{\alpha}_3,0\} \\
\alpha_4 &= \{0,0,0,1,-1,0,0,0\} &&= \{\overline{\alpha}_4,0\} \\
\alpha_5 &= \{0,0,0,0,1,-1,0,0\} &&= \{\overline{\alpha}_5,0\} \\
\alpha_6 &= \{0,0,0,0,1,1,0,0\} &&= \{\overline{\alpha}_6,0\} \\
\alpha_7 &= \{-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{\sqrt{2}},0\} &&= \{\overline{\alpha}_7,0\} \\
\alpha_8 &= \{-1,0,0,0,0,0,-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}\} &&= \{\mathbf{w}_h,\tfrac{1}{\sqrt{2}}\}
\end{aligned}$$

$$\mathbf{w}_h = \{-1,0,0,0,0,0,-\frac{1}{\sqrt{2}}\}$$

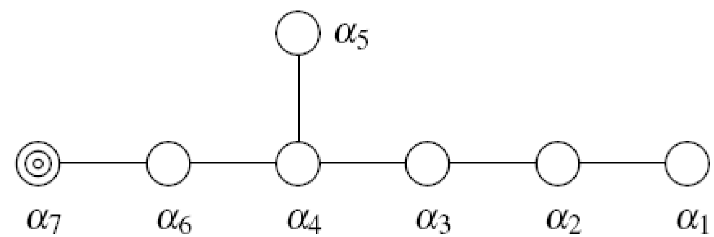
$$\psi = \{0,0,0,0,0,0,0,\sqrt{2}\}$$

N=6

$$\mathscr{S}\mathscr{K}_{N=6} \equiv \frac{\mathrm{SO}^{\star}(12)}{\mathrm{SU}(6)\times \mathrm{U}(1)}$$

$$\mathrm{adj}\,\mathrm{E}_{7(-5)} = \mathrm{adj}\,\mathrm{SO}^{\star}(12) \oplus \mathrm{adj}\,\mathrm{SL}(2,\mathbb{R})_{\mathrm{E}} \oplus (\mathbf{2},\mathbf{32}_s)$$

$$\psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7$$



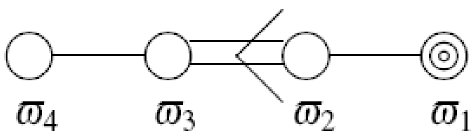
$$\begin{aligned}
\alpha_1 &= \{1,-1,0,0,0,0,0\} &&= \{\overline{\alpha}_1,0\}\\
\alpha_2 &= \{0,1,-1,0,0,0,0\} &&= \{\overline{\alpha}_2,0\}\\
\alpha_3 &= \{0,0,1,-1,0,0,0\} &&= \{\overline{\alpha}_3,0\}\\
\alpha_4 &= \{0,0,0,1,-1,0,0\} &&= \{\overline{\alpha}_4,0\}\\
\alpha_5 &= \{0,0,0,0,1,-1,0\} &&= \{\overline{\alpha}_5,0\}\\
\alpha_6 &= \{0,0,0,0,1,1,0\} &&= \{\overline{\alpha}_6,0\}\\
\alpha_7 &= \{-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{\sqrt{2}}\} &&= \{\overline{\mathbf{w}}_h,\tfrac{1}{\sqrt{2}}\}
\end{aligned}$$

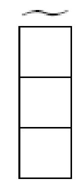
$$\psi = \{0,0,0,0,0,0,\sqrt{2}\}$$

$$\mathbf{w}_h = \{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\}$$

Tits Satake of the N=6 theory F_4

$$\begin{array}{rcl} \mathrm{adj}(E_{7(-5)}) & = & \mathrm{adj}(SO^\star(12)) \oplus \mathrm{adj}(SL(2,\mathbb{R})_E) \oplus (\mathbf{2},\mathbf{32}_s) \\ & \Downarrow & \\ \mathrm{adj}(F_{4(4)}) & = & \mathrm{adj}(Sp(6,\mathbb{R}) \oplus \mathrm{adj}(SL(2,\mathbb{R})_E) \oplus (\mathbf{2},\mathbf{14}') \end{array}$$

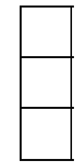


$\dim_{Sp(6,\mathbb{R})}$  $= \mathbf{14'}$

$$\psi = 2\varpi_1 + 3\varpi_2 + 4\varpi_3 + 2\varpi_4$$
$$(\psi\,,\,\varpi_1) = 2 \quad ; \quad (\psi\,,\,\varpi_i) = 0 \quad i \neq 1$$

N=5

$$\mathcal{M}_{\mathcal{N}=5,D=4} = \frac{\mathrm{SU}(1,5)}{\mathrm{SU}(5)\times \mathrm{U}(1)}$$

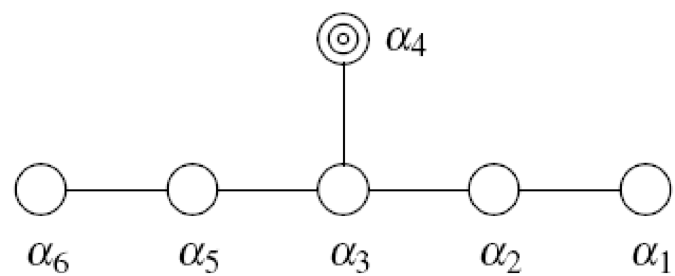
$\dim_{\mathrm{SU}(1,5)}$  $= 20$

$$\mathrm{adj}(\mathrm{E}_{6(-14)}) = \mathrm{adj}(\mathrm{SU}(1,5) \oplus \mathrm{adj}(\mathrm{SL}(2,\mathbb{R})_{\mathrm{E}}) \oplus (\mathbf{2},\mathbf{20})$$

$$\psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$$

$$(\psi\,,\alpha_4) = 1 \quad ; \quad (\psi\,,\alpha_i) = 0 \quad i \neq 4$$

$$\alpha_1 = \left\{0,0,-\tfrac{\sqrt{3}}{2},\tfrac{1}{2\sqrt{5}},\sqrt{\tfrac{6}{5}},0\right\} = \{\overline{\alpha}_1,0\}$$
$$\alpha_2 = \left\{-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{6}},\tfrac{2}{\sqrt{3}},0,0,0\right\} = \{\overline{\alpha}_2,0\}$$
$$\alpha_3 = \left\{\sqrt{2},0,0,0,0,0\right\} = \{\overline{\alpha}_3,0\}$$
$$\alpha_4 = \left\{-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{6}},-\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{5}},-\sqrt{\tfrac{3}{10}},\tfrac{1}{\sqrt{2}}\right\} = \left\{\mathbf{w}_h,\tfrac{1}{\sqrt{2}}\right\}$$
$$\alpha_5 = \left\{-\tfrac{1}{\sqrt{2}},-\sqrt{\tfrac{3}{2}},0,0,0,0\right\} = \{\overline{\alpha}_4,0\}$$
$$\alpha_6 = \left\{0,\sqrt{\tfrac{2}{3}},-\tfrac{1}{2\sqrt{3}},-\tfrac{\sqrt{5}}{2},0,0\right\} = \{\overline{\alpha}_5,0\}$$



Detailed study of the F_4 univ. class

As an illustration we analyze one universality class

The su(3,3) case

$$\mathbb{G}_{\text{subTS}} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sp}(6,\mathbb{R}) = \mathbb{G}_{\text{TS}}$$

$$\mathbf{14'} \stackrel{\mathbb{G}_{\text{subTS}}}{\Longrightarrow} (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{2},\mathbf{2},\mathbf{2})$$

$\mathfrak{su}(3,3)$ **model**

$$\mathbb{G}_{\text{paint}} = \mathfrak{so}(2) \oplus \mathfrak{so}(2) \qquad \mathbb{G}_{\text{subpaint}} = 0$$

$$\mathbf{20} \stackrel{\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}}{\Longrightarrow} (2, q_1|\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (2, q_2|\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (2, q_3|\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (1, 0|\mathbf{2},\mathbf{2},\mathbf{2})$$

$$\mathbf{20} \stackrel{\mathbb{G}_{\text{subpaint}} \oplus \mathbb{G}_{\text{TS}}}{\Longrightarrow} \mathbf{6} \oplus \mathbf{14}$$

$$\mathbf{6} \stackrel{\mathbb{G}_{\text{subTS}}}{\Longrightarrow} (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})$$

The SO*(12) case

$$\mathbb{G}_{\text{subpaint}} = \mathfrak{so}(3)_{\text{diag}}$$

$$\mathbb{G}_{\text{paint}} = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$

$$\mathbf{32}_s \overset{\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}}{\Longrightarrow} (\underline{2},\underline{2},\underline{1}|\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\underline{2},\underline{1},\underline{2}|\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\underline{1},\underline{1},\underline{2}|\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\underline{1},\underline{1},\underline{1}|\mathbf{2},\mathbf{2},\mathbf{2})$$

$$\mathbf{32}_s \overset{\mathbb{G}_{\text{TS}} \oplus \mathbb{G}_{\text{subpaint}}}{\Longrightarrow} (\mathbf{6}|\underline{3}) \oplus (\mathbf{14}'|\underline{1})$$

$$\mathbb{G}_{\text{subTS}} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sp}(6,\mathbb{R}) = \mathbb{G}_{\text{TS}}$$

$$\mathbf{14}' \overset{\mathbb{G}_{\text{subTS}}}{\Longrightarrow} (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{2},\mathbf{2},\mathbf{2})$$

The $E_{(7,-25)}$ case

$$\mathbb{G}_{\text{paint}} = \mathfrak{so}(8)$$

$$\mathbb{G}_{\text{subpaint}} = \mathfrak{g}_{2(-14)}$$

$$\mathbf{56} \stackrel{\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}}{\Longrightarrow} (\mathbf{8}_v|\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{8}_s|\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{8}_c|\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1}|\mathbf{2},\mathbf{2},\mathbf{2})$$

$$\mathbf{8}_{v,s,c} \stackrel{\mathfrak{g}_{2(-14)}}{\Longrightarrow} \mathbf{7} \oplus \mathbf{1}$$

$$\mathbf{56} \stackrel{\mathbb{G}_{\text{TS}} \oplus \mathbb{G}_{\text{subpaint}}}{\Longrightarrow} (\mathbf{6}|\mathbf{7}) \oplus (\mathbf{14}'|\mathbf{1})$$

$$\mathbb{G}_{\text{subTS}} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sp}(6,\mathbb{R}) = \mathbb{G}_{\text{TS}}$$

$$\mathbf{14}' \stackrel{\mathbb{G}_{\text{subTS}}}{\Longrightarrow} (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{2},\mathbf{2},\mathbf{2})$$