Programme

1) Physical motivations for SUSY
   N=1 SUSY in 4D

2) Susy algebra

3) N=1 susy representations – Rep's realized on fields

4) N-extended cases

5) N=1 superspace

6) Wess-Zumino model

8) Susy gauge theories – N=1 abelian & non ab.

9) N-extended susy gauge theories.
   Non renormalization theorems
   N=4 SYM finiteness

10) SSB (spontaneous breaking)

Bibliography

- Sohnius (review)
- Terning (book)
- Bilal (review)
- Argyres (")

  ⋮

Prerequisites

Minkowski 4D spacetime $\eta_{\mu\nu} = (1, -1, -1 \ldots)$

Weyl spinors $\in \left(\frac{1}{2}, 0\right)$ | rep. of Lorentz group LEFT H

Weyl spinors $\in$ $(\frac{1}{2},0)$ LEFT H.
$(0,\frac{1}{2})$ RIGHT H.
} repr.'s of Lorentz group

Left $\rightarrow$ ψ_α $\quad \alpha = 1,2$

Right $\rightarrow$ $\bar{\chi}^{\dot{\alpha}}$ $\quad \dot{\alpha} = \dot{1},\dot{2}$

Dirac spinor $\in (\frac{1}{2},0) \oplus (0,\frac{1}{2}) \longrightarrow \psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$

Weyl basis for γ-matrices $\quad \gamma^\mu = \left(\begin{array}{c|c} 0 & \sigma^\mu \\ \hline \bar{\sigma}^\mu & 0 \end{array}\right)$ $\quad \mu = 0,1,2,3$

$\sigma^\mu \equiv (\mathbb{1}, \sigma^i)$

$\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$

$\sigma^i =$ Pauli matrices

$\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

$$\gamma^\mu \psi = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\alpha}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} & 0 \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} (\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \psi_\alpha \end{pmatrix}$$

Convention $\begin{cases} \varepsilon\psi \equiv \varepsilon^\alpha \psi_\alpha \\ \bar{\varepsilon}\bar{\psi} \equiv \bar{\varepsilon}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \end{cases}$

Rising and lowering spinorial indices :

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$(\varepsilon_{12} = -1)$

$$\boxed{\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma}$$

$$\varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\varepsilon^{\dot\alpha\dot\beta}\,\varepsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}{}_{\dot\gamma}}$$

$$\left\{\begin{array}{ll} \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta & \psi_\beta = \varepsilon_{\beta\alpha}\psi^\alpha \\ \bar\chi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta} & \bar\chi^{\dot\beta} = \varepsilon^{\dot\beta\dot\alpha}\bar\chi_{\dot\alpha} \end{array}\right\}$$

$\Longrightarrow$ ① $\chi^\alpha\psi_\alpha = -\chi_\alpha\psi^\alpha$ check it!

$\bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha}$

② $\chi\psi = \psi\chi$

<u>Useful identities for $\sigma^\mu, \bar\sigma^\mu$</u>

1) $(\sigma^\mu)_{\alpha\dot\alpha}\,(\bar\sigma_\mu)^{\dot\beta\beta} = 2\,\delta_\alpha^{\beta}\,\delta_{\dot\alpha}^{\dot\beta}$

2) $\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}$

$$(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\beta} + (\sigma^\nu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta} = 2\eta^{\mu\nu}\delta_\alpha^{\beta}$$

3) $$\left\{\begin{array}{l} (\bar\sigma^\mu)^{\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\,\varepsilon^{\dot\alpha\dot\beta}\,(\sigma^\mu)_{\beta\dot\beta} \\ (\sigma^\mu)_{\alpha\dot\alpha} = \varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,(\bar\sigma^\mu)^{\dot\beta\beta} \end{array}\right.$$

- <u>Weyl projectors</u>

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$P_L = \left(\frac{1-\gamma^5}{2}\right) \Longrightarrow P_L\psi = \psi_L \equiv \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$$

$$= \left(\frac{1+\gamma^5}{2}\right) \Longrightarrow P_R\psi = \psi_R \equiv \begin{pmatrix} 0 \\ \bar{\chi}^{\dot\alpha} \end{pmatrix}$$

- <u>Dirac conjugate</u>

$$\bar{\Psi} \equiv \psi^\dagger\gamma^0 \underset{\uparrow}{=} \left((\psi_\alpha)^* \quad (\bar{\chi}^{\dot\alpha})^* \right) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \left((\bar{\chi}^{\dot\alpha})^* \; (\psi_\alpha)^* \right)$$

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} \qquad\qquad = \left(-\chi^\alpha \quad -\bar{\Psi}_{\dot\alpha} \right)$$

$$\left.\begin{aligned} (\psi_\alpha)^* &= -\bar{\Psi}_{\dot\alpha} \\ (\bar{\chi}^{\dot\alpha})^* &= - \end{aligned}\right\} \qquad \begin{aligned} \bar{\Psi}^{\dot\alpha} &= i\,(\bar\sigma^2)^{\dot\alpha\alpha}\,(\psi_\alpha)^* \\ \chi_\alpha &= i\,(\sigma^2)_{\alpha\dot\alpha}\,(\bar{\chi}^{\dot\alpha})^* \end{aligned}$$

- <u>Charge conjugated spinors</u>

$$\psi^c \equiv -i\gamma^0\gamma^2\,(\bar{\Psi})^T = i\gamma^2\psi^*$$

$$= \begin{pmatrix} 0 & i\bar\sigma^2 \\ i\bar\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} (\psi_\alpha)^* \\ (\bar{\chi}^{\dot\alpha})^* \end{pmatrix} = \begin{pmatrix} i\sigma^2(\bar{\chi}^{\dot\alpha})^* \\ i\bar\sigma^2(\psi_\alpha)^* \end{pmatrix}$$

$$= \begin{pmatrix} \chi_\alpha \\ \bar{\Psi}^{\dot\alpha} \end{pmatrix}$$

<u>Majorana spinors</u> : $\psi^c \equiv \psi$

Majorana spinors : $\psi^c \equiv \psi$

$$\begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} \equiv \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} \Rightarrow \bar{\chi}^{\dot\alpha} = \bar{\psi}^{\dot\alpha}$$

$$\Rightarrow \boxed{\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix}} \qquad \bar{\psi}^{\dot\alpha} = i(\bar{\sigma}^2)^{\dot\alpha\alpha}\psi^*_\alpha$$

Tensorial algebra

4D vector $V_\mu = (V_0, V_1, V_2, V_3)$

$$\downarrow$$

$$(1) \qquad V_{\alpha\dot\alpha} = \underset{\sim}{\frac{1}{\sqrt{2}}} (\sigma^\mu)_{\alpha\dot\alpha} V_\mu = (V_{1\dot1}, V_{1\dot2}, V_{2\dot1}, V_{2\dot2})$$

Check the explicit relation between $V_{\alpha\dot\alpha}$ and V_μ

If we use $(\sigma_\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\mu)^{\dot\alpha\beta} = 4\,\delta_\alpha^{\ \beta}$ we can invert eq (1)

$$(2) \qquad V_\mu = \frac{1}{\sqrt{2}} (\bar{\sigma}_\mu)^{\dot\alpha\alpha} V_{\alpha\dot\alpha}$$

For spacetime coords and derivatives

$$x^{\alpha\dot\alpha} = \frac{1}{2}(\bar{\sigma}_\mu)^{\dot\alpha\alpha} x^\mu \qquad x^\mu = \frac{1}{2}(\sigma^\mu)_{\alpha\dot\alpha} x^{\alpha\dot\alpha}$$

$$\partial_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu \qquad \partial_\mu = (\bar{\sigma}_\mu)^{\dot\alpha\alpha}\partial_{\alpha\dot\alpha}$$

$$\rightarrow \qquad \partial_{\alpha\dot\alpha} x^{\beta\dot\beta} = \delta_\alpha^{\ \beta}\,\delta_{\dot\alpha}^{\ \dot\beta}$$

$$\partial^{\dot\alpha\alpha}\partial_{\beta\dot\alpha} = \delta_\beta^\alpha \Box \qquad \Box = \partial^\mu \partial_\mu$$

$$\boxed{\Box = \frac{1}{2}\partial^{\dot\alpha\alpha}\partial_{\alpha\dot\alpha}}$$

Useful identities :

$$A_\alpha B_\beta - A_\beta B_\alpha = \varepsilon_{\alpha\beta} A^\gamma B_\gamma$$

$$\bar A_{\dot\alpha} \bar B_{\dot\beta} - \bar A_{\dot\beta} \bar B_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta} \bar A^{\dot\gamma} \bar B_{\dot\gamma}$$

MOTIVATIONS FOR SUSY

Comprehension of elementary particles and their fundamental interaction is based on

symmetry principles

1) One set of symmetry principles are used to classify particles according to a given set of quantum numbers

2) Gauge invariance principle allows to describe fund. interactions.

① Particle classification

Noether th. $\rightarrow$ n invariances $\Rightarrow$ n conserved currents

$$\partial_\mu J_i^\mu = 0$$

$$\Downarrow$$

$$Q_i = \int d^3x \, J^0_i$$

We can use the corresponding eigenvalues to classify particles.

<u>Poincaré invariance</u>

Momentum $\partial_\mu P^\mu = 0$ $\quad P^\mu$ = translation generators

Pauli-Lubanski vector W_μ $\quad \partial_\mu W^\mu = 0$

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} P^\nu \underbrace{M^{\rho\sigma}}_{\text{Lorentz generators}}$$

Casimir ops $\rightarrow$ $P^2 \rightarrow m^2$ ($\equiv$ mass^2 of particles)

$W^2 \rightarrow s$ ($\equiv$ spin of particles)

<u>Internal symmetries</u> $\rightarrow$ Isospin $SU(2)$

flavor $SU(n)$

Hypercharge

Charge $U(1)$

$\vdots$

Poincaré algebra:

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho})$$

Internal symmetry algebra $\longmapsto$ $\{T_i\}_{i=1\dots n}$

$$[T_i, T_j] = i f_{ijk} T_k$$

structure constants

$$\phi \to \phi' = \underbrace{e^{i\lambda^i T_i}} \phi$$

If we have a theory for scalar and fermion

$$\mathcal{L} = \underbrace{\partial_\mu \varphi \, \partial^\mu \bar{\varphi}}_{\mathcal{L}_\varphi} + \underbrace{i \bar{\psi} \gamma^\mu \partial_\mu \psi}_{\mathcal{L}_\psi}$$

$\mathcal{L}_\varphi, \mathcal{L}_\psi$ will be separately invariant under Poincaré and internal symmetry since transfs will never mix φ with ψ, being the corresponding parameters ordinary commuting numbers.

Can we generalise $\mathcal{L}$ in such a way that it becomes invariant under a set of transfs given by $(P_\mu, M_{\nu\rho}, T_i)$ s.t. we may have nontrivial $[\ ,\ ]$ between $(P_\mu, M_{\nu\rho})$ and T_i

The answer is <u>NO</u> (COLEMANN – MANDULA THEOREM)
(no-go theorem)

Analicity properties of the relativistic S-matrix imply that most general symmetry group for a physical theory is of the form

$$\{\text{Poincaré}\} \times G_{\text{internal}}$$

$$\Updownarrow$$

$$[P_\mu, T_i] = 0 \qquad [M_{\mu\nu}, T_i] = 0$$

It follows: $[P^2, T_i] = 0$

$[W_\mu, T_i] = 0 \quad [W^2, T_i] = 0$

Fields belonging to the same rep. of the internal group G have to have

- the same mass
- the same momentum
- the same spin

Introducing SUSY by a down-top construction

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \bar{\varphi} + \frac{i}{4} \bar{\Psi} \gamma^\mu \partial_\mu \Psi$$

$$= \frac{1}{2} \partial_{\alpha\dot{\alpha}} \varphi \partial^{\alpha\dot{\alpha}} \bar{\varphi} - \frac{i}{2} \psi^\alpha \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \qquad \text{(check it!)}$$

- Poincaré invariance
- Phase transf's invariance $\quad \varphi \to e^{i\alpha} \varphi \quad \alpha \in \mathbb{R}$ (constant)

$\psi \to e^{i\beta} \psi \quad \beta \in \mathbb{R}$

- New transf parameters $\varepsilon_\alpha \quad (\bar{\varepsilon}_{\dot{\alpha}} = -\varepsilon_\alpha^*)$ (Weyl constant spinors)

$$(3) \begin{cases} \delta\varphi = -\varepsilon^\alpha \psi_\alpha & \delta\bar{\varphi} = -\bar{\psi}_{\dot{\alpha}} \bar{\varepsilon}^{\dot{\alpha}} \\ \delta\psi_\alpha = -2i(\partial_{\alpha\dot{\alpha}} \varphi)\bar{\varepsilon}^{\dot{\alpha}} & \delta\bar{\psi}_{\dot{\alpha}} = -2i(\partial_{\alpha\dot{\alpha}} \bar{\varphi})\varepsilon^\alpha \end{cases}$$

$$\delta\mathcal{L} = \frac{1}{2} \partial_{\alpha\dot{\alpha}} (\delta\varphi) \partial^{\alpha\dot{\alpha}} \bar{\varphi} + \frac{1}{2} \partial_{\alpha\dot{\alpha}} \varphi \partial^{\alpha\dot{\alpha}} (\delta\bar{\varphi})$$

$$- \frac{i}{2} (\delta\psi^\alpha) \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} - \frac{i}{2} \psi^\alpha \partial_{\alpha\dot{\alpha}} (\delta\bar{\psi}^{\dot{\alpha}})$$

$$= \frac{1}{2}\partial_{\alpha\dot\alpha}\left(-\varepsilon^\beta\psi_\beta\right)\partial^{\dot\alpha\alpha}\bar\varphi + \frac{1}{2}\partial_{\alpha\dot\alpha}\varphi\,\partial^{\dot\alpha\alpha}\left(-\bar\psi_{\dot\beta}\bar\varepsilon^{\dot\beta}\right)$$

$$- \frac{i}{\cancel{2}}\left(\cancel{2}i\,\partial^{\dot\alpha\dot\beta}\varphi\,\bar\varepsilon_{\dot\beta}\right)\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} - \frac{i}{\cancel{2}}\psi^\alpha\partial_{\alpha\dot\alpha}\left(\cancel{2}i\,\partial^{\beta\dot\alpha}\bar\varphi\,\varepsilon_\beta\right)$$

$$\partial^{\dot\alpha\beta}\partial_{\alpha\dot\alpha} = \Box\,\delta^\beta_\alpha$$

$$= \varepsilon^\beta\psi_\beta\Box\bar\varphi + \Box\varphi\,\bar\psi_{\dot\beta}\bar\varepsilon^{\dot\beta}$$

$$- \Box\varphi\,\bar\varepsilon_{\dot\beta}\bar\psi^{\dot\beta} - \psi^\beta\varepsilon_\beta\Box\bar\varphi = 0$$

If we <u>assume</u> $\{\varepsilon_\alpha, \psi_\beta\} = \{\varepsilon_\alpha, \bar\psi_{\dot\beta}\} = 0$

$\{\varepsilon_\alpha, \varepsilon_\beta\} = 0$

We formally think of transfs (3) as due to the action of a set of generators Q_α, $\bar Q_{\dot\alpha}$ s.t.

$$\underset{\varepsilon^\alpha}{\uparrow} \qquad \underset{\bar\varepsilon^{\dot\alpha}}{\uparrow}$$

$$\delta\varphi = i\left(\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)\varphi$$

$$\delta\psi \qquad \text{"} \qquad \psi$$

For instance : $\bar Q^{\dot\alpha}\varphi = 0$

$Q_\alpha\varphi = i\psi_\alpha$ (from transfs (3))

Let's construct the corresponding algebra

$$\begin{cases}\delta_1\varphi = i\left(\varepsilon_1^\alpha Q_\alpha + \bar\varepsilon_{1\dot\alpha}\bar Q^{\dot\alpha}\right)\varphi \\ \delta_2\varphi \qquad \varepsilon_2 \qquad \bar\varepsilon_2\end{cases} \tag{4}$$

$$[\delta_1, \delta_2]\varphi \equiv \left(\delta_1(\delta_2\varphi) - \delta_2(\delta_1\varphi)\right)$$

$$= \delta_1 (-\varepsilon_2^{\alpha} \psi_\alpha) - 1 \leftrightarrow 2 = -\varepsilon_2^{\alpha} \delta_1 \psi_\alpha - 1 \leftrightarrow 2$$

(3)

$$= -\varepsilon_2^{\alpha} \left(-2i (\partial_{\alpha\dot\beta} \varphi) \bar\varepsilon_1^{\dot\beta} \right) - 1 \leftrightarrow 2$$

$$= \underbrace{-2\left(\varepsilon_2^{\alpha} \bar\varepsilon_1^{\dot\alpha} - \varepsilon_1^{\alpha} \bar\varepsilon_2^{\dot\alpha}\right)}_{a^{\alpha\dot\alpha}} \underbrace{(-i\partial_{\alpha\dot\alpha})}_{P_{\alpha\dot\alpha}} \varphi$$

$$= 2\left(\varepsilon_1^{\alpha} \bar\varepsilon_2^{\dot\alpha} - \varepsilon_2^{\alpha} \bar\varepsilon_1^{\dot\alpha}\right) (-i\partial_{\alpha\dot\alpha}) \varphi \qquad (5)$$

Alternatively, using definitions (4) we have

$$[\delta_1, \delta_2] \varphi = \varepsilon_1^{\alpha} \varepsilon_2^{\beta} \{Q_\alpha, Q_\beta\} \varphi + \bar\varepsilon_{1\dot\alpha} \bar\varepsilon_{2\dot\beta} \{\bar Q^{\dot\alpha}, \bar Q^{\dot\beta}\} \varphi + \varepsilon_1^{\alpha} \bar\varepsilon_{2\dot\beta} \{Q_\alpha, \bar Q^{\dot\beta}\} \varphi + \bar\varepsilon_{1\dot\alpha} \varepsilon_2^{\beta} \{\bar Q^{\dot\alpha}, Q_\beta\} \varphi \qquad (6)$$

Comparing (5) and (6) we are led to

$$\{Q_\alpha, Q_\beta\} = 0 \qquad \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0$$

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = 2(i\partial_{\alpha\dot\beta}) \varphi = \pm 2 P_{\alpha\dot\beta} \varphi \qquad (7)$$

Moreover:

- Jacobi identity $\Rightarrow [P_\mu, Q_\alpha] = [P_\mu, \bar Q_{\dot\beta}] = 0$
  (try!)

- the 2 generators Q_α and $\bar Q_{\dot\alpha}$ are Weyl spinors
  s.t. $\begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix} \equiv$ Majorana spinor in 4D

$\Downarrow$

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2} (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[M_{\mu\nu}, \bar{Q}_{\dot\alpha}] = \frac{1}{2} (\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta} \qquad (8)$$

<u>Conclusion</u>: Poincaré + $(Q_\alpha, \bar{Q}_{\dot\alpha})$ with non trivial (anti) comm. rules (7), (8) is a closed "algebra" $\Longrightarrow$

$\boxed{N=1 \text{ SUPERALGEBRA}}$