SUSY REPRESENTATIONS

Aim of this lecture is to organize particles described by quantum numbers (m, s) into representations of susy algebra (<u>supermultiplets</u>)

Let's go back to our toy model

$$\mathcal{L} = \frac{1}{2}\partial_{\alpha\dot{\alpha}}\varphi\,\partial^{\alpha\dot{\alpha}}\bar{\varphi} - \frac{i}{2}\psi^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$$

$\mathcal{L}$ is invariant under

$$\delta\varphi = -\varepsilon^{\alpha}\psi_{\alpha} \qquad \delta\bar{\varphi} = -\bar{\psi}_{\dot{\alpha}}\bar{\varepsilon}^{\dot{\alpha}}$$

$$\delta\psi_{\alpha} = -2i(\partial_{\alpha\dot{\alpha}}\varphi)\bar{\varepsilon}^{\dot{\alpha}} \qquad \delta\bar{\psi}_{\dot{\alpha}} = -2i\,\varepsilon^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{\varphi}$$

<u>Question</u>: Do (φ, ψ) realise a representation of susy algebra?

$$[\delta_1, \delta_2]\varphi = 2(\varepsilon_1^{\alpha}\bar{\varepsilon}_2^{\dot{\alpha}} - \varepsilon_2^{\alpha}\bar{\varepsilon}_1^{\dot{\alpha}})\underbrace{P_{\alpha\dot{\alpha}}}_{(-i\partial_{\alpha\dot{\alpha}})}\varphi$$

$$[\delta_1, \delta_2]\psi_{\alpha} = \delta_1(-2i\,\partial_{\alpha\dot{\alpha}}\varphi\,\bar{\varepsilon}_2^{\dot{\alpha}}) - 1\leftrightarrow 2$$

$$= -2i\,\partial_{\alpha\dot{\alpha}}(-\varepsilon_1^{\beta}\psi_{\beta})\bar{\varepsilon}_2^{\dot{\alpha}} - 1\leftrightarrow 2$$

$$= -2i\,\varepsilon_1^{\beta}\bar{\varepsilon}_2^{\dot{\alpha}}(\underbrace{\partial_{\alpha\dot{\alpha}}\psi_{\beta}}) - 1\leftrightarrow 2$$

General identity $\underbrace{A_{\alpha}B_{\beta}} - \underbrace{A_{\beta}B_{\alpha}} = \varepsilon_{\alpha\beta}A^{\gamma}B_{\gamma}$

$$\Rightarrow \partial_{\alpha\dot{\alpha}}\psi_{\beta} = \partial_{\beta\dot{\alpha}}\psi_{\alpha} + \varepsilon_{\alpha\beta}\partial^{\gamma}{}_{\dot{\alpha}}\psi_{\gamma}$$

$$= -2i\left(\varepsilon_1^\beta \bar{\varepsilon}_2^{\dot\alpha} - \varepsilon_2^\beta \bar{\varepsilon}_1^{\dot\alpha}\right)\partial_{\beta\dot\alpha}\psi_\alpha$$

$$+\left(-2i\,\varepsilon_1^\beta \bar{\varepsilon}_2^{\dot\alpha}\,\varepsilon_{\alpha\beta}\,\underbrace{\partial^\gamma{}_{\dot\alpha}\psi_\gamma} - 1\leftrightarrow 2\right)$$

<u>on-shell</u> $\partial^\gamma{}_{\dot\alpha}\psi_\gamma = 0$

<u>Conclusion</u>: The susy algebra closes on (φ, ψ) <u>on-shell</u>

We can make susy algebra working off-shell if we include in multiplet one extra scalar F with

$$\mathcal{L} = \frac{1}{2}\partial_{\alpha\dot\alpha}\varphi\,\partial^{\alpha\dot\alpha}\bar\varphi - \frac{i}{2}\psi^\alpha\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} - \frac{1}{4}F\bar{F}$$

and

$$\begin{cases} \delta\varphi = -\varepsilon^\alpha\psi_\alpha & \delta\bar\varphi = -\bar\psi_{\dot\alpha}\bar\varepsilon^{\dot\alpha} \\ \delta\psi_\alpha = -2i\,\partial_{\alpha\dot\alpha}\varphi\,\bar\varepsilon^{\dot\alpha} - \varepsilon_\alpha F & \\ \delta\bar\psi_{\dot\alpha} = -2i\,\varepsilon^\alpha\partial_{\alpha\dot\alpha}\bar\varphi - \bar\varepsilon_{\dot\alpha}\bar F & \qquad (1) \\ \delta F = 2i\left(\partial_{\alpha\dot\alpha}\psi^\alpha\right)\bar\varepsilon^{\dot\alpha} & \\ \delta\bar F = 2i\,\varepsilon^\alpha\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} & \end{cases}$$

Exercise: check that $\mathcal{L}$ invariant under (1)

<u>Closure of the algebra</u>

$$[\delta_1, \delta_2]\varphi = \delta_1(-\varepsilon_2^\alpha\psi_\alpha) - 1\leftrightarrow 2$$

$$\rightarrow -\varepsilon_2^\alpha(-\varepsilon_{1\alpha}F) - 1\leftrightarrow 2$$

$$= \left(\varepsilon_2^\alpha \varepsilon_{1\alpha} - \varepsilon_1^\alpha \varepsilon_{2\alpha}\right) F = 0$$

$$[\delta_1, \delta_2]\psi_\alpha = \delta_1\left(-2i\,\partial_{\alpha\dot\alpha}\varphi\,\bar\varepsilon_2^{\dot\alpha} - \varepsilon_{2\alpha}F\right) - 1 \leftrightarrow 2$$

$$= -2i\,\underbrace{\partial_{\alpha\dot\alpha}\left(-\varepsilon_1^\beta\psi_\beta\right)}\bar\varepsilon_2^{\dot\alpha} - \varepsilon_{2\alpha}\,2i\,\partial_{\beta\dot\beta}\psi^\beta\,\bar\varepsilon_1^{\dot\beta} - 1 \leftrightarrow 2$$

$$= \boxed{-2i\,\varepsilon_1^\beta\,\bar\varepsilon_2^{\dot\alpha}\,\partial_{\beta\dot\alpha}\psi_\alpha - 1 \leftrightarrow 2}$$

$$-2i\,\varepsilon_1^\beta\,\bar\varepsilon_2^{\dot\alpha}\,\varepsilon_{\alpha\beta}\,\partial^\gamma{}_{\dot\alpha}\psi_\gamma - 2i\,\varepsilon_{2\alpha}\,\bar\varepsilon_1^{\dot\beta}\,\partial^\beta{}_{\dot\beta}\psi_\beta - 1 \leftrightarrow 2$$

$$\longrightarrow -2i\,\varepsilon_{1\alpha}\,\bar\varepsilon_2^{\dot\alpha}\,\partial^\gamma{}_{\dot\alpha}\psi_\gamma - 2i\,\varepsilon_{2\alpha}\,\bar\varepsilon_1^{\dot\alpha}\,\partial^\gamma{}_{\dot\alpha}\psi_\gamma$$

$$- 1 \leftrightarrow 2 \qquad = 0$$

Finally

$$[\delta_1, \delta_2] F = 2\left(\varepsilon_1^\alpha \bar\varepsilon_2^{\dot\alpha} - \varepsilon_2^\alpha \bar\varepsilon_1^{\dot\alpha}\right)\underbrace{\left(-i\,\partial_{\alpha\dot\alpha}\right)}_{P_{\alpha\dot\alpha}} F$$

F-field satisfies algebraic EoM : $F = 0$ $\Rightarrow$

F is not a physical field. It is <u>AUXILIARY</u>

<u>N=1 SUSY REPRESENTATIONS</u>

(1) <u>MASSIVE REPR.S</u>

Physical states are labeled $|m, s, s_3\rangle$

$s_3 = -s, -s+1, \dots, s$

We choose to work in the <u>rest frame</u>

$$p_\mu = (m, 0, 0, 0) \quad \Rightarrow \quad p_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha}\,p_\mu = m\,(\sigma^0)_{\alpha\dot\alpha}$$

$$= m\,\delta_{\alpha\dot\alpha}$$

$$\Rightarrow \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \qquad \{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2m\,\delta_{\alpha\dot\alpha}$$

$$\rightsquigarrow \{Q_1, \bar{Q}_{\dot 1}\} = 2m \qquad \{Q_2, \bar{Q}_{\dot 2}\} = 2m$$

$\Rightarrow$ 2 Clifford algebras

We define a Clifford vacuum state of spin s : $|s\rangle$

by the cond $\quad Q_1 |s\rangle = 0$

$\quad Q_2 |s\rangle = 0$

State	Spin	# of components
$\vert s\rangle$	s	$2s+1$
$\bar{Q}_{\dot\alpha} \vert s\rangle$	$s+1/2$ $s-1/2$	$\left[2(s+\frac{1}{2})+1\right] +$ $\left[2(s-\frac{1}{2})+1\right] = 4s+2$
$\bar{Q}^2 \vert s\rangle$ $\left(\overset{\shortparallel}{\tfrac{1}{2}\bar{Q}_{\dot\alpha}\bar{Q}^{\dot\alpha}}\right)$	s	$2s+1$

Why do we stop here?

We could go on applying Q's

$$Q_\alpha \bar{Q}^2 |s\rangle = [Q_\alpha, \bar{Q}^2]|s\rangle = 2 P_{\alpha\dot\alpha} \underbrace{\bar{Q}^{\dot\alpha}|s\rangle}$$

$$Q^2 \bar{Q}^2 |s\rangle = [Q^2, \bar{Q}^2]|s\rangle = 2\Box |s\rangle$$

these are not independent states

Remember that $[M_{\mu\nu}, \bar{Q}_{\dot\alpha}] = \frac{1}{2} (\bar{\sigma}_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}$

$$\Downarrow \ \mu, \nu = 1,2,3$$

$$[\vec{M}, \bar{Q}_{\dot\alpha}] = \frac{1}{2} (\vec{\sigma} \bar{Q})_{\dot\alpha}$$

$$\uparrow$$

$$\bar{Q}_{\dot\alpha} |m\, s\, s_3\rangle = \sum_{s_3'} \Big[c_{s_3 s_3'} \underbrace{|m\ s+\tfrac{1}{2}\ s_3'\rangle} + d_{s_3 s_3'} \underbrace{|m\ s-\tfrac{1}{2}\ s_3'\rangle} \Big]$$

Comments

a) Dimension of representation $= 4(2s+1)$

b) We apply Q_α to $\bar{Q}_{\dot\alpha}|s\rangle$

$$Q_\alpha \underbrace{\bar{Q}_{\dot\alpha} |s\rangle}_{s \pm 1/2} = \{Q_\alpha, \bar{Q}_{\dot\alpha}\} |s\rangle = 2m\, \delta_{\alpha\dot\alpha} \underbrace{|s\rangle}_{s}$$

Physical examples

- massive chiral multiplet (matter)

We start with $s=0$

States	Spin	# comp.
$\|0\rangle$	0	1
$\bar{Q}_{\dot 1}\|0\rangle, \bar{Q}_{\dot 2}\|0\rangle$	$\frac{1}{2}$	2
$\bar{Q}^2\|0\rangle$	0	1

$2+2=4$

- massive vector multiplet (massive vector particles)

Start with $s = \frac{1}{2}$

States	Spin	# comp.
$\vert \frac{1}{2} \rangle$	$1/2$	2
$\bar{Q}_{\dot{1}} \vert \frac{1}{2} \rangle$	1	3
$\bar{Q}_{\dot{2}} \vert \frac{1}{2} \rangle$	0	1
$\bar{Q}^2 \vert \frac{1}{2} \rangle$	$1/2$	2

Dimension $= 4 + 4 = 8$

- massive Weyl multiplet

Start with $S = 3/2$

States	Spin	# comp.
$\vert 3/2 \rangle$	$3/2$	4
$\bar{Q}_{\dot{\alpha}} \vert \frac{3}{2} \rangle$	2	5
	1	3
$\bar{Q}^2 \vert \frac{3}{2} \rangle$	$3/2$	4

② MASSLESS REPR'S

let's work in a ref frame where

$$p_\mu = (E, 0, 0, E)$$

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2 (\sigma^\mu)_{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} = 2E (\mathbb{1} + \sigma^3)_{\alpha\dot{\alpha}}$$

$$\Rightarrow \boxed{\{Q_1, \bar{Q}_{\dot{1}}\} = 4E}$$

$$\{Q_2, \bar{Q}_{\dot{2}}\} = 0 \qquad \{Q_1, \bar{Q}_{\dot{2}}\} = 0$$

Particles are classified in terms of λ = helicity

Helicity operator $\Gamma = \vec{L} \cdot \frac{\vec{P}}{E}$

$$\uparrow$$

$$\vec{L} = (L_1, L_2, L_3) = (M_{23}, M_{31}, M_{12})$$

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$$

$$W^0 = \frac{1}{2} \varepsilon^{0ijk} P_i M_{jk} = \vec{P} \cdot \vec{L} = E\,\Gamma$$

States $\to$ $|E, \lambda\rangle$ eigenstates of W_0

$$W_0 |E, \lambda\rangle = E\lambda\, |E, \lambda\rangle$$

We want to study what happens when we apply $Q, \bar{Q}$ to a given $|E, \lambda\rangle$

$$W_0 \left(Q_\alpha |E, \lambda\rangle \right)$$

$$W_0 \left(\bar{Q}_{\dot{\alpha}} |E, \lambda\rangle \right)$$

From superpoincaré algebra:

$$[W_0, Q_\alpha] = -\frac{1}{2} E \,(\sigma^3 Q)_\alpha$$

$$[W_0, \bar{Q}_{\dot{\alpha}}] = \frac{1}{2} E \,(\sigma^3 \bar{Q})_{\dot{\alpha}}$$

$$W_0 \left(Q_\alpha |E, \lambda\rangle \right) = Q_\alpha \left(W_0 |E, \lambda\rangle \right) + [W_0, Q_\alpha] |E, \lambda\rangle$$

$$= E\lambda \left(Q_\alpha |E, \lambda\rangle \right) - \frac{1}{2} E \left((\sigma^3 Q)_\alpha |E, \lambda\rangle \right)$$

$$= E\left(\lambda \mathbb{1} - \frac{1}{2}\sigma^3\right)_\alpha{}^\beta Q_\beta |E,\lambda\rangle$$

$$W_0\left(\bar{Q}_{\dot\alpha}|E,\lambda\rangle\right) = E\left(\lambda\mathbb{1} + \frac{1}{2}\sigma^3\right)_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}|E,\lambda\rangle$$

$$Q_1|E,\lambda\rangle = |E,\lambda-\tfrac{1}{2}\rangle \qquad \bar{Q}_{\dot 1}|E,\lambda\rangle = |E,\lambda+\tfrac{1}{2}\rangle$$

<u>NB</u>: $\{Q_2, \bar{Q}_{\dot 2}\} = 0$

$$\Downarrow$$

$\forall$ any $|E,\lambda\rangle$ $\quad \langle E,\lambda|\{Q_2,\bar{Q}_{\dot 2}\}|E,\lambda\rangle = 0$

$$\Updownarrow$$

$$\boxed{Q_2|E,\lambda\rangle = 0} \quad \boxed{\bar{Q}_{\dot 2}|E,\lambda\rangle = 0}$$

Applying Q_1 many times we will decrease λ down to λ_0 s.t.

$$Q_1|E,\lambda_0\rangle = 0$$

To build a massless repr: we start with $|E,\lambda_0\rangle$

s.t. $\quad Q_1|E,\lambda_0\rangle = 0 \qquad Q_2|E,\lambda_0\rangle = \bar{Q}_{\dot 2}|E,\lambda_0\rangle = 0$

State	<u>Helicity</u>	
$\|E,\lambda_0\rangle$	λ_0	1 scalar
		+
$\bar{Q}_{\dot 1}\|E,\lambda_0\rangle$	$\lambda_0 + 1/2$	1 fermion

CPT invariance of a physical theory requires to include a massless multiplet together with its CPT-conjugate with opposite helicities

Simplest example: case $\lambda_0 = 0$

$$|E,0\rangle \quad 0 \qquad \oplus \qquad |E,0\rangle_c \quad 0$$

$$\bar{Q}_{\dot{1}} |E,0\rangle \quad 1/2 \qquad\qquad (\bar{Q}_{\dot{1}} |E,0\rangle)_c \quad -1/2$$

helicity	# comp.
1/2	1
0	1+1
-1/2	1

$\boxed{\text{N-EXTENDED SUSY}}$

We generalise superpoincaré algebra by introducing N supercharges $\begin{cases} Q_\alpha^a & a = 1, 2, \dots N \\ \bar{Q}_{\dot{\alpha}}^a \end{cases}$

Q_α^a realize fund. repr. of $SU(N)$

The most general N-extended algebra is

$$\{Q_\alpha^a, \bar{Q}_{b\dot{\alpha}}\} = 2\,\delta_b^a\, P_{\alpha\dot{\alpha}}$$

$$\{Q_\alpha^a, Q_\beta^b\} = \varepsilon_{\alpha\beta}\, Z^{ab} \qquad (Z^{ab} = -Z^{ba})$$

$$\{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} = \varepsilon_{\dot{\alpha}\dot{\beta}}\, \bar{Z}_{ab} \qquad Z, \bar{Z} = \text{central charges}$$

$$[T^a{}_b, Q_\alpha^c] = -i\left(\delta_b^c\, Q_\alpha^a - \frac{1}{N}\delta_b^a\, Q_\alpha^c\right)$$

$$[T^a{}_b, Q_{\dot\alpha c}] = i\left(\delta^c_c\, Q_{\dot\alpha b} - \frac{1}{N}\,\delta^{\cdot\cdot}_b\, Q_{\dot\alpha c}\right)$$

$$[T^a{}_b, T^c{}_d] = i\left(\delta^a_d\, T^c_b - \delta^c_b\, T^a_d\right)$$

$SU(N) \to$ R-symmetry $+$ $U(1)$ (multiplying by a phase)

$\Rightarrow$ R-symmetry $= SU(N) \times U(1) \to U(N)$

Except for special case of $N=4$: in this case R-symmetry is $SU(4)$

<u>Construct</u> <u>representations</u>

① massive rep's $\to$ go to rest frame $p_\mu = (m, 0, 0, 0)$

$$\{Q^a_\alpha, \bar Q_{b\dot\alpha}\} = 2m\, \delta^a_b\, \delta_{\alpha\dot\alpha} \qquad a, b = 1, \ldots N$$

N replicas of Clifford algebra $\times 2$

We define Clifford vacuum $\quad Q^a_\alpha |s\rangle = 0$

<u>State</u>	<u>Spin</u>
$\vert s\rangle$	s
$\bar Q_{a\dot\alpha} \vert s\rangle$	$s \pm \frac{1}{2}$
$\bar Q_{b\dot\beta} \bar Q_{a\dot\alpha} \vert s\rangle$	$s+1$
$\vdots$	
$\bar Q_{1\dot\alpha} \bar Q_{2\dot\beta} \bar Q_{3\dot\gamma} \cdots \bar Q_{N\dot\varepsilon} \vert s\rangle$	$\boxed{s + N/2}$
$\bar Q_1^2 \bar Q_{2\dot\beta} \bar Q_{3\dot\gamma} \cdots \bar Q_{N\dot\varepsilon} \vert s\rangle$	$s + \frac{(N-1)}{2}$

$Q_1 \; Q_{2\dot\beta} \; Q_{3\dot\gamma} \cdots Q_{N\dot\varepsilon} |s\rangle \qquad s + \frac{(N-1)}{2}$

$\vdots$

$\bar{Q}_1^2 \; \bar{Q}_2^2 \cdots \bar{Q}_N^2 |s\rangle \qquad s$

(2) Massless case $\rightarrow \quad p_\mu = (E, 0, 0, E)$

$$\{ Q^a_1 , \bar{Q}_{b\dot{1}} \} = 4E \, \delta^a_b \qquad N\text{-clifford algebras}$$

We choose a ground state $\quad Q^a_1 |E, \lambda_0\rangle = 0$

$$Q^a_2 |E,\lambda_0\rangle = \bar{Q}_{a\dot{2}} |E, \lambda_0\rangle = 0$$

<u>States</u>	<u>helicity</u>	<u># comp.</u>
$\vert E,\lambda_0\rangle$	λ	$1 = \binom{N}{0}$
$\bar{Q}_{a\dot{i}} \vert E,\lambda_0\rangle$	$\lambda + 1/2$	$N = \binom{N}{1}$
$\bar{Q}_{a\dot{i}} \bar{Q}_{b\dot{i}} \vert E,\lambda_0\rangle$	$\lambda + 1$	$\binom{N}{2}$
$\vdots$		$\vdots$
$\bar{Q}_{1\dot{i}} \bar{Q}_{2\dot{i}} \cdots \bar{Q}_{N\dot{i}} \vert E,\lambda_0\rangle$	$\lambda + N/2$	$1 = \binom{N}{N}$

<u>Important examples</u>

<u>$N=4$</u> Vector multiplet $\qquad \lambda_0 = -1$

<u>helicity</u>	# components
-1	1
$-1/2$	4
$-$	6

$-1/2$	4
0	6
$1/2$	4
1	1

<u>$N=8$</u> supergravity multiplet $\lambda_0 = -2$

helicity	# comp.
-2	
$-3/2$	
-1	
$-1/2$	
0	
$1/2$	
1	
$3/2$	
2	