N=1 SYM THEORIES (continue...)

Generic unitary gauge group G $\rightarrow$ ϕ^a fund. repr. of G

+

gauge superfield $\rightarrow$ prepotential $V = V^A T_A$

$$V^a{}_b = V^A (T_A)^a{}_b$$

Most general renormalizable action has the following form g = gauge coupling constant

$$S = \int d^4x\, d^4\theta \, \underbrace{\bar{\Phi}_a \left(e^{gV}\right)^a{}_b \phi^b} + \int d^4x\, d^2\theta \, \mathrm{Tr}\left(\tfrac{1}{2} W^\alpha W_\alpha\right)$$

$$+ \int d^4x\, d^2\theta \, P(\phi) + h.c.$$

$$W_\alpha = \bar{D}^2\left(e^{-gV} D_\alpha e^{gV}\right) \qquad P(\phi) = \frac{m}{2}\phi^2 + \frac{\lambda}{3!}\phi^3$$

R-symmetry invariance realised if V transforms as

$$V \rightarrow V'(x,\theta,\bar\theta) = V(x, e^{-i\beta}\theta, e^{i\beta}\bar\theta)$$

If we have a $U(1)$ subgroup, then we can add a Fayet-Iliopoulos term

$$S_{FI} = \int d^4x\, d^4\theta \quad (\xi V) \quad = \int d^4x \quad (\xi D)$$

In fact this is invariant under $V \rightarrow V' = V + i(\bar\Lambda - \Lambda)$

$$\int d^4\theta\, V' = \int d^4\theta \left(V + i(\bar\Lambda - \Lambda)\right) = \int d^4\theta\, V$$

$$\int d^2\theta\, \bar\Lambda = 0 \qquad \int d^2\bar\theta\, \Lambda = 0$$

"Smart trick" to get S in components

$\mathcal{L}$ required to be supergauge invariant

$$\int d^4\theta\, \mathcal{L} \equiv D^2 \bar{D}^2 \mathcal{L} \Big| = \nabla^2 \bar{\nabla}^2 \mathcal{L} \Big|$$

We define covariantly chiral superfields

$$\cancel{\bar{D}_{\dot\alpha} \phi = 0} \longrightarrow \nabla_\alpha \tilde{\phi} = 0 = \bar{D}_{\dot\alpha} \tilde{\phi} \Rightarrow \tilde{\phi} = \phi$$

$$\cancel{D_\alpha \bar{\phi} = 0} \longrightarrow \nabla_\alpha \bar{\tilde{\phi}} = 0 = e^{-V}\left(D_\alpha e^{V} \bar{\tilde{\phi}}\right) \Rightarrow \bar{\tilde{\phi}} = e^{-V} \bar{\phi}$$

$$\bar{\nabla}_{\dot\alpha} \tilde{\phi} = 0 = e^{V} \bar{D}_{\dot\alpha}\left(e^{-V} \tilde{\phi}\right) \Rightarrow \boxed{\tilde{\phi} = e^{V} \phi}$$

$$\bar{\nabla}_{\dot\alpha} \bar{\tilde{\phi}} = 0 = D_\alpha \bar{\tilde{\phi}} \Rightarrow \boxed{\bar{\tilde{\phi}} = \bar{\phi}}$$

+ background field method

Renormalizability

S is renormalizable. We have non-renormalization theorems at work

1) no corrections to chiral integrals $\Rightarrow$ no quantum corrections to the superpotential P

$$P(\phi) = \frac{m}{2}\phi^2 + \frac{\lambda}{3!}\phi^3$$

$$\Rightarrow \quad \phi_R = Z_\phi^{1/2} \phi \qquad \begin{cases} m_R = Z_\phi^{-1} m \\ \lambda_R = Z_\phi^{-3/2} \lambda \end{cases}$$

Only logarithmically divergent terms

2) $$\int d^4x\, d^2\theta\, \mathrm{Tr}\left(W^\alpha W_\alpha\right) = \int d^4x\, d^4\theta\, \mathrm{Tr}\left(e^{-gV} D^\alpha e^{gV} \bar{D}^2 \left(e^{-gV} D_\alpha e^{gV}\right)\right)$$

$\Rightarrow$ nontrivial renormalization of V

$$V_R = Z_V^{1/2}\, V$$

But one can prove that e^{gV} is not renormalized

$\Rightarrow$ $g_R = Z_g\, g$ but $Z_g = Z_V^{-1/2}$

N-EXTENDED SYM

$N=2$ and $N=4$ SYM

1) Field content and superspace description
2) Invariant actions
3) Non-renormalization theorems

N=2 SYM

2 chiral supercharges $\{Q_{\alpha i}, \bar{Q}_{\dot\alpha}^{\,j}\} = 2\,\delta_i^{\,j}\, P_{\alpha\dot\alpha}$ $\quad i,j=1,2$

2 possible susy multiplets:

1) Vector multiplet $\rightarrow$ $\left(1, \frac{1}{2}, \frac{1}{2}, 0, 0\right)$

2) Scalar multiplet (Hyper multiplet) $\rightarrow$ $\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right)$

We realise these multiplets in terms of $N=1$ supermultiplets

1) Vector multiplet

$$\left.\begin{array}{ll} N=1 \text{ vector} & (\quad , \lambda, A_\mu) \\ N=1 \text{ chiral} & (\varphi, \psi, \quad) \end{array}\right\}$$

(λ, ψ): SO(2) doublet

N=1 chiral is in the adjoint of G

$\varphi, A_\mu \Rightarrow SU(2)$ singlets whereas $\chi_{\alpha i} \equiv (\lambda_\alpha, \psi_\alpha)$
$\underline{\underline{i=1,2}}$ (isospin index)

2) Hypermultiplet made by 2 chiral $N=1$ multiplets

Q_1, Q_2 $\qquad Q_i = (q^i, \psi_i, \dots)$

$$\left\{ \begin{array}{l} \left(\begin{bmatrix} q_1 \\ q_2^\dagger \end{bmatrix}, \begin{array}{l} \psi_1 \quad \dots \\ \psi_2^\dagger \quad \dots \end{array} \right) \end{array} \right\}$$

$SU(2)$ doublet

$$\begin{pmatrix} \psi_1 \\ \psi_2^\dagger \end{pmatrix} \to \text{Dirac fermion}$$

Invariant actions

1) Vector action

Since ϕ is in the adjoint of G its transf. read

$$\phi \to \phi' = e^{i\Lambda} \phi \, e^{-i\Lambda}$$

$$S = \mathrm{Tr} \left[\int d^4x \, d^4\theta \, e^{-V} \bar{\Phi} e^V \phi + \int d^4x \, d^2\theta \, \tfrac{1}{2} W^\alpha W_\alpha \right]$$

Covariantly chiral fields $\left(\tilde{\phi} = e^V \phi \, , \, \tilde{\bar{\phi}} = e^{-V} \bar{\Phi} \right)$

S is invariant under

$$\begin{cases} \delta \phi = - W^\alpha \nabla_\alpha \xi - i \left[\bar{\nabla}^2 (\nabla^\alpha \varepsilon) \nabla_\alpha + (\nabla^\alpha \varepsilon) \, i W_\alpha \right] \phi \\ \delta \nabla_\alpha = - \nabla_\alpha \left(i \xi \bar{\Phi} - \bar{W}^{\dot{\beta}} \bar{\nabla}_{\dot{\beta}} \varepsilon \right) \end{cases}$$

$\xi, \varepsilon =$ parameter superfields

If we set $\xi = 0$ and $\varepsilon^\alpha = \bar{D}^2 D^\alpha \varepsilon \Big|$

Can we construct a $N=2$ superspace where we can build up the vector multiplet as a $N=2$ superfield ?

We can define $N=2$ superspace $\longrightarrow \left(x^{\alpha\dot\alpha}, \theta^\alpha, \bar\theta^{\dot\alpha}, \tilde\theta^\alpha, \bar{\tilde\theta}^{\dot\alpha} \right)$

Vector $N=2$ superfield

$$\Psi(x, \theta, \bar\theta, \tilde\theta, \bar{\tilde\theta}) = \underbrace{\phi(\tilde y, \theta)} + \tilde\theta^\alpha W_\alpha(\tilde y, \theta) + \tilde\theta^2 G(\tilde y, \theta)$$

where $\tilde y = x^{\alpha\dot\alpha} + i\theta^\alpha \bar\theta^{\dot\alpha} + i\tilde\theta^\alpha \bar{\tilde\theta}^{\dot\alpha}$

ϕ = chiral $N=1$ superfield

W_α = vector $N=1$ superfield

and

$$G(\tilde y, \theta) = \int d^2\bar\theta \; \phi^\dagger(\tilde y - i\theta\bar\theta, \theta, \bar\theta) \, e^{-gV(\tilde y - i\theta\bar\theta, \theta, \bar\theta)}$$

The original action S becomes

$$S = \mathrm{Tr}\int d^4x \, d^2\theta \, d^2\tilde\theta \; \frac{1}{2}\Psi^2 \qquad \Longleftarrow$$

$N=2$ vector action is <u>holomorphic</u>

More generally

$$S = \mathrm{Tr}\int d^4x \, d^2\theta \, d^2\tilde\theta \; F(\Psi)$$

2) Hypermultiplet action

2 $N=1$ chirals Q^1, Q^2 $\quad (\bar Q_1, \bar Q_2)$

$$S = \int d^4x \, d^4\theta \; \bar Q_i Q^i$$

We might add $\quad S = \int d^4x \, d^2\theta \; Q^i m_{ij} Q^j$

these actions are invariant under

$$\delta Q^i = -\left(\bar{D}^2\bar{\xi}\ \varepsilon^{ij}\ \bar{Q}_j\right) - i\bar{D}^2\left[(D^\alpha\varepsilon)D_\alpha Q^i + (D^2\varepsilon)Q^i\right]$$

$\varepsilon^{12} = 1$

Most general SYM action which is allowed by $N=2$?

Generic unitary group G $\Longrightarrow$ vector multiplets
($N=1$ vector + chiral in adj repr.)

Matter in fund repr of G $\quad \underbrace{Q_1^a \quad \bar{Q}_{2a}} \rightarrow Q^a \quad Q_a$

$$S = \int d^4x\, d^4\theta \Big[\mathrm{Tr}\left(e^{-gV}\bar{\Phi}\, e^{gV}\phi\right) + \mathrm{Tr}\, \bar{Q}_a \left(e^{gV}\right)^a{}_b\, Q^b$$
$$\mathrm{Tr}\, Q_a \left(e^{gV}\right)^a{}_b\, \bar{Q}^b \Big]$$

$$+ \int d^4x\, d^2\theta\, \mathrm{Tr}\left(\tfrac{1}{2}W^\alpha W_\alpha\right) + \int d^4 d^2\theta\, W + h.c.$$

where $\quad W = -mQ^a Q_a + g\, Q^a \underbrace{\phi_a{}^b} Q_b$

$N=4$ SYM

Only 1 possible multiplet

$N=4$ multiplet
$$\begin{cases} 1 \text{ vector field (gauge)} & s=1 \\ 4 \text{ chiral fermions} & (s=1/2) \\ 6 \text{ scalars} & (s=0 \end{cases}$$

1 $N=2$ vector multiplet (1 $N=1$ vector + 1 adj chiral)

1 $N=2$ hypermultiplet (1 $N=1$ chiral + 1 $N=1$ antich.)
in the adj of gauge group

In $N=1$ language $\rightarrow$ 1 $N=1$ vector superfield
+
3 $N=1$ chiral superfields ϕ^i $i=1,2,3$

Invariant action

$$S = \int d^8z \; \mathrm{Tr}\left(e^{-gV}\bar{\Phi}_i e^{gV}\phi^i\right) + \int d^6z \; \mathrm{Tr}\, W^2$$

$$+ \frac{1}{3!}\int d^6z \; \underbrace{i f_{ijk}}_{\text{structure constants of } G} \phi^i [\phi^j, \phi^k] + h.c.$$

ϕ^i realise fund repr of $SU(3) \subset SU(4)$ (R-symmetry)

How do we realise $SU(4)$ on the fields?

Fermions $\quad \chi_I = (\underbrace{\psi_i}_{\text{from } \phi^i}, \underbrace{\lambda}_{\text{fermion in } W_\alpha})$ $\quad I = 1,2,3,4$

χ_I is in fund. repr. of $SU(4)$

For scalars: φ_i $\quad i=1,2,3$

Define
$$\begin{cases} M_{ij} = -\frac{1}{2}\varepsilon_{ijk}\varphi_k^* \\ M_{4i} = -M_{i4} = \frac{1}{2}\varphi_i \end{cases}$$

Renormalization properties

N=2 SYM

$$\int d^8 z \, \mathrm{Tr}\left[e^{-gV}\bar{\chi}\, e^{gV}\chi\right] + \int d^8 z \, \bar{\Phi}_a \left(e^{gV}\right)^a{}_b \, \phi^b$$

$$+ \int d^8 z \, \phi_a \left(e^{gV}\right)^a{}_b \, \bar{\Phi}^b \quad + \int d^6 z \, \mathrm{Tr}\, W^2$$

$$+ \int d^6 z \left[- m \phi^a \phi_a + \underset{\approx}{g'} \, \phi^a \chi_a{}^b \phi_b \right] + \mathrm{h.c.}$$

Most general ren. scheme:

$$V_R = Z_V^{1/2} V \qquad\qquad g_R = Z_g \, g$$

$$\phi_{Ra} = Z_\phi^{1/2} \phi_a \qquad\qquad g'_R = Z'_g \, g'$$

$$\phi_R^a = Z'^{\,1/2}_\phi \, \phi^a \qquad\qquad m \to m + \delta m$$

$$\chi_R = Z_\chi^{1/2} \chi$$

- But non renorm theorems $\Longrightarrow$

$$\left.\begin{aligned} & Z_V^{1/2} Z_g = 1 \\ & Z'_g \, Z_\phi^{1/2} Z'^{\,1/2}_\phi Z_\chi^{1/2} = 1 \\ & \left(1 + \frac{\delta m}{m}\right) Z_\phi^{1/2} Z'^{\,1/2}_\phi = 1 \end{aligned}\right] \quad (1)$$

- Require N=2 susy $\Longrightarrow$ $\left.\begin{aligned} & Z'_g = Z_g \\ & Z_\chi = Z_V \end{aligned}\right\}$
  $(g' = g)$
  (χ and V belong to same N=2 mult.)

Plugging into (1) $\Longrightarrow$ $\Big\{ \; Z_g = Z'_g = Z_V^{-1/2}$

(same N=2 mult.)

Plugging into (1) $\Rightarrow$
$$\begin{cases} Z_g = Z'_g = Z_V^{-1/2} \\ Z_\phi^{1/2} \, Z_\phi'^{1/2} = 1 \\ \delta m = 0 \end{cases}$$

2 independent ren. functions Z_V, Z_ϕ

- N=4 susy

everything belongs to the same multiplet (V, χ, ϕ^i) $\Rightarrow$ $Z_\chi = Z_\phi = Z_{\phi'} = Z_V$

$$\Downarrow$$

$$Z_\phi = Z'_\phi = Z_\chi = Z_V = 1$$

$$\boxed{N=4 \text{ is finite}}$$

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