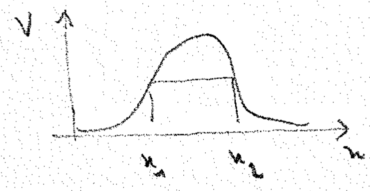
SOLITONS & INSTANTONS IN GAUGE THEORIES

OVERVIEW:

STUDY CORRECTIONS OF $O(e^{-\frac{1}{\hbar}})$ to the quantum amplitudes

$\hbar$ still small but usual pert. theory cannot recover them

basic example in QM: WKB tunneling



$$|T(E)| = \exp\left\{-\frac{1}{\hbar}\int dx\left[2(V-E)\right]^{\frac{1}{2}}\right\}(1+O(\hbar))$$

QFT analog: INSTANTONS $\Leftrightarrow$ finite action sl't's

semiclassical approx in path integral

completely under control in SUSY models

APPLICATIONS:

- mesons spectrum & decay in QCD: U(1) problem, six fermion eff. vertices, DIS
- non pert. dynamics in SUSY theories: conf., chiral symm. breaking, dynamical breaking of SUSY for chiral models
- classifying vacua of SUSY / string theories: S-duality, BH entropy as microstates counting –
- math counterpart: topological (diff) invariants (Donaldson, DT) enumerative geometry (Gromov Witten invariants)

PLAN:

- solitons & instantons in G.T.
- path integral: semiclassical approx & ev. of amplitudes
- instantons in SUSY
- application to $\mathcal{N}=2$ SYM

LECTURE ONE

SOLITONS & INSTANTONS IN GAUGE THEORIES

- Non-singular slt's of non-linear gauge-th. eq's of motion

  (1) finite energy / action

  (2) stability

- General properties:

7) ion (1) => localised in space / (euclidean) space-time

$\downarrow$ new states in the spectrum ("particles" or also extended objects)

$\downarrow$ tunneling between different vacua (WKB) breaks degeneration

REMARK: soliton vs instanton is a dimension dependent concept:

soliton in $d+1$ dimensions <=> instanton in d

② probe NON LINEAR structure of the theory: cannot be found perturbing slt's of linearized eom's

③ mass $\sim \frac{1}{e}$ => very light at strong coupling + elementary d.o.f.'s

[thrustable only if mass formulae not quantum corrected => SUSY!]

I. Classification:

(2) => topological conservation laws: conservation of charge & energy does not come from a symmetry of the Lagrangian but from topological properties of the space of non-singular finite energy/action solutions:

$$A_\mu \to i g^{-1} \partial_\mu g \qquad \text{SURPRISES!}$$

example: gauge field in 2D (abelian)

finite action $\Rightarrow$ $A \xrightarrow[r\to\infty]{} -i g^{-1} dg$ $\quad A = A_\mu dx^\mu \quad \mu = 1,2$

polar coordinates, radial gauge $\Rightarrow$ $A_r = 0 \quad A_\vartheta(r,\vartheta) \xrightarrow[r\to\infty]{} \frac{i}{r} g^{-1} \partial_\vartheta g$

$F \sim \frac{1}{r^2}$ integrable $\qquad g : S^1_\infty \to S^1$

flux:
$$\int_{\mathbb{R}^2} F = \lim_{r\to\infty} \oint_{S^1} A_\vartheta \, r \, d\vartheta = \oint_{S^1_\infty} -i g^{-1} \partial_\vartheta g \, d\vartheta = 2\pi n$$

$n \in \mathbb{Z}$ classifies homotopy class of the map

$$g : S^1_\infty \to S^1$$

trivial map: $\quad g(\vartheta) = 1 \quad 0 \le \vartheta \le 2\pi \quad n = 0$

identity map: $\quad g(\vartheta) = e^{i\vartheta} \quad n = 1$

n-covering map: $\quad g(\vartheta) = e^{in\vartheta}$

$$\pi_1(U(1)) = \pi_1(S^1) = \mathbb{Z} \text{ first homotopy group}$$

Since maps in different homotopy classes cannot be deformed continuously one in another (*), the flux is conserved and moreover is QUANTIZED $\Rightarrow$ cfg space of gauge fields solving eom's, is split into DISCONNECTED COMPONENTS labeled by an integer n.

$\downarrow$

classical vacua of the f.t.!

it is easy to see that also $\pi_2(S^2) = \mathbb{Z}$; in general $\pi_n(S^n) = \mathbb{Z}$

[remark: restoring $\hbar$ & e one gets $\Phi = \frac{2\pi n}{\hbar e}$

namely $\hbar e \Phi = 2\pi n$

electric charge $\quad$ magnetic flux]

(*) invariance under continuous def.:

$g = e^{i\phi(\theta)}$

$\delta g = i\,\delta\phi\, g$

$$\delta(g^{-1}dg) = \left(-i g^{-1}\delta\phi\, dg + i g^{-1} d(\delta\phi\, g)\right)$$
$$= \left(-i g^{-1}\cancel{\delta\phi}\, dg + i g^{-1}\cancel{\delta\phi}\, \cancel{dg} + i g^{-1}(d\delta\phi) g\right.$$
$$= i\, d(\delta\phi) \quad \text{total derivative !}$$

Thm (Coleman chap 6 p. 204)

Hp: Consider a gauge theory with group G and scalar fields with potential U in $D+1$ dimensions [$D = 2, 3$!]
Assume that the set of zeroes of U can be identified with the coset space G/H where H is the stability group of the zeroes of the potential U (namely the subgroup surviving after spontaneous breaking of symmetry)
[This means that we exclude accidental degeneracies and extra global symmetries]

Ths: the connected components of the space of non-sing finite energy slts is in one-to-one correspondence with $\pi_{D-1}(G/H)$

proof

We work in detail the $D = 2$ case. Polar coord. (r, ϑ). Static cfg ∂ field $= 0$, $A_0 = 0$.
Choose the radial gauge $A_r = 0$ $\quad r \geq 1$ (A_r not well defined at the origin)

$$E \geq \int_1^{\infty} r\,dr \int_0^{2\pi} d\vartheta\, (\partial_r \varphi)^{\dagger}(\partial_r \varphi) + U(\varphi)$$

$$E \text{ finite} \Rightarrow \lim_{r\to\infty} \varphi(r, \vartheta) = \phi_0 \qquad \phi_0 \,/\, \left.\frac{\partial U}{\partial \varphi}\right|_{\phi_0} = 0$$

NOTICE that ϕ can depend on ϑ!

$$\lim_{r\to\infty} e_{\vartheta}\cdot D\varphi = \frac{1}{r}\frac{d}{d\vartheta}\phi + i e A^a_{\vartheta} T^a \phi = 0$$

$$\Rightarrow \lim_{r\to\infty} r A^a_{\vartheta} T^a = -i e^{-1} \phi^{-1} \frac{d}{d\vartheta}\phi$$

ϕ is a map from the circle @ ∞ to the space of vacua G/H

$$\Rightarrow \phi : S^1_{\infty} \longrightarrow G/H$$

remark: $A \sim \frac{1}{r} \Rightarrow F \sim \frac{1}{r^2}$ integrable only in $D = 2, 3$ diverges from $D = 4$ on

BUT:

gauge field in 4D Euclidean:

$$A_\mu \underset{r\to\infty}{\to} i g^{-1} \partial_\mu g \qquad \text{finite action}$$

$$\Pi_3(G)$$

simplest case $G = SU(2) \sim S^3$ (*) $\Rightarrow$ $\Pi_3(S^3) = \mathbb{Z}$ instantons

also for $SU(N)$, $SO(N)$, $Sp(N)$, E in general ANY compact simple Lie group

$$g^{(0)} = 1$$

$$g^{(1)} = (x_4 + i x_i \sigma^i)/r \qquad i = 1,2,3 \quad \sigma^i \text{ Pauli matrices}$$

$$g^{(n)} = [g^{(1)}]^n$$

$$n = \frac{1}{24\pi^2} \int_{S^3_\infty} d^3 S_\mu \, \mathrm{Tr}\left[g^{-1}(\partial_\nu g) g^{-1}(\partial_\rho g) g^{-1}(\partial_\sigma g) \epsilon^{\mu\nu\rho\sigma} \right] \quad (*)$$

(*) $\forall g \in SU(2)$ $\quad g = a + i b_i \sigma^i$ (**) $\quad a^2 + \sum_i b_i^2 = 1 \Leftrightarrow S^3 \quad a, b_i \in \mathbb{R}$

Exercise: evaluate the integral for $g^{(1)}$. See Coleman p. 289 (A)

REMARKS: (*) valid $\forall$ $SU(N)$ group
$SO(N)$
$Sp(N)$
E

(**) unit quaternions $q = q_4 \mathbb{1} + i q_i \sigma^i \in \mathbb{H}$

$$\bar{q} q = (q_4 - i q_i \sigma^i)(q_4 + i q_j \sigma^j)$$
$$= q_4^2 + q_i q_j \sigma^i \sigma^j = q_4^2 + q_i q_j (\delta_{ij} + \epsilon_{ijk} \sigma^k)$$

REMARKS

1) INTEGRABILITY: In math. phys. solitons have a NARROWER def.

one requires that they are sol's of nonlinear eqs whose # and momenta are preserved in scattering processes ⇒ integrable PDE's

∞ # of conservation laws

E.G.'s: KdV in hydrodynamics (1+1 dimensions) [historically "soliton" comes from this]
Sine Gordon model in QFT

This is not true for gauge solitons in general: the associated set of eq's can often be related to integrable systems BUT the dynamics is not integrable due to radiative corrections.

Notable exceptions in SUSY theories! SW, spin-chains ...

2) the topological classification is not COMPLETE, in the sense that it gives only <u>sufficient</u> criterium. In fact:

a) $\exists$ also stable dynamical non-topological solitons [T.D. Lee first examples]

b) if topological criterium is satisfied still one has to show that $\exists$ solutions to PDE's and construct them.

examples

$\pi_1(G/H)$ vortices

$\pi_2(G/H)$ monopoles

[$\pi_3(G)$ instantons]

II. FINDING THE SLTS

Typically 2^{nd} order non linear PDE's $\Rightarrow$ very hard to solve!
& cannot use p.t.

$\Rightarrow$ try to simplify the task by writing a bound for the energy / action

In 4D Euclidean YM:

$$S_{YM} = -\frac{1}{2g^2}\int \mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) d^4x = \frac{1}{4g^2}\int F^a_{\mu\nu}F^a_{\mu\nu}\, d^4x \qquad \mathrm{Tr}(T^aT^b) = -\frac{1}{2}\delta^{ab}$$

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^a_{\rho\sigma} \qquad \tilde{F}^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = F^a_{\mu\nu}F^a_{\mu\nu} \quad \text{contraction of } \epsilon \text{ tensor}$$

$$\leqslant -\frac{1}{g^2}\int \mathrm{Tr}\,\frac{1}{2}\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)\frac{1}{2}\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right) = -\frac{1}{2g^2}\int\left\{\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) \pm \mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right)\right\}$$

$$\Rightarrow \quad S_{YM} \geqslant \mp\frac{1}{2g^2}\int \mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) d^4x = \frac{8\pi^2}{g^2}|K| \qquad (*)$$

$K \in \mathbb{Z}$ topological invariant of the gauge bundle
(U(N) $\Leftrightarrow$ ch_2 2^{nd} Chern character ?
SO(N) $\Leftrightarrow$ first Pontryagin class
Sp(N) $\Leftrightarrow$?

minima: $\boxed{F_{\mu\nu} = \pm\tilde{F}_{\mu\nu} \quad \text{(anti) instantons}}$

REMARK: for $F_{\mu\nu} = -\tilde{F}_{\mu\nu} \Rightarrow K < 0 \Rightarrow +\frac{1}{2g^2}\int \mathrm{Tr}(F\tilde{F}) = -\frac{1}{4g^2}\int F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = -K > 0$

$\boxed{\text{FIRST ORDER EQ'S !}}$

automatically solve 2nd order eqs

$$D_\mu F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} D_\mu F_{\rho\sigma} = 0 \quad \text{Bianchi id. !}$$

identity:

$$\mathrm{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) = \partial_\mu \underbrace{\epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left(A_\nu F_{\rho\sigma} - \frac{2}{3} A_\nu A_\rho A_\sigma\right)}_{\text{CS term } J_\mu}$$

$$\Rightarrow \int_{\mathbb{R}^4} \mathrm{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu})\, d^4x = \int_{S^3_\infty} d^3S\, \hat{\varepsilon}_\mu J_\mu$$

using $A_\mu \to g \partial_\mu g^{-1} \Rightarrow A_\mu F_{\nu\rho} \to 0$ and we get

$$-\frac{2}{3} \int d^3S\, \hat{\varepsilon}_\mu \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left(g\partial_\nu g^{-1}\, g\partial_\rho g^{-1}\, g\partial_\sigma g^{-1}\right) = 16\pi^2 K$$

$$\Pi_3(G) = \mathbb{Z}$$

then (✱) is shown

Explicit sltn for $k=1$ $G = SU(2)$ [BPST '75]

$$A_\mu^{NS} = -\sigma_{\mu\nu} \frac{(x-x_0)_\nu}{(x-x_0)^2 + \rho^2} = \frac{(x-x_0)^2}{(x-x_0)^2+\rho^2}\, g \partial_\mu g^{-1}$$

$$g = \frac{(x-x_0)_\mu \sigma_\mu}{|x-x_0|}$$

$$(\square) \quad F_{\mu\nu}^{NS} = 2\sigma_{\mu\nu} \frac{\rho^2}{[(x-x_0)^2+\rho^2]^2}$$

EXERCISE: SHOW ($\square$)

Interlude : Hodge star operator

$\Omega^p(M, \mathrm{ad}P)$ p-forms valued in ad P (also End E)
↑ vector bundle

$* : \Omega^p \to \Omega^{d-p}$ $\dim M = d$

i.e.

$$\omega = \frac{1}{p!}\, \omega_{\mu_1 \dots \mu_p}\, dx^1 \wedge \dots \wedge dx^p$$

$$*\omega = \frac{\sqrt{|g|}}{p!(d-p)!}\, \epsilon_{\mu_1 \dots \mu_d}\, g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p}\, \omega_{\nu_1 \dots \nu_p}\, dx^{\mu_{p+1}} \dots dx^{\mu_d}$$

$$\epsilon_{\mu_1 \dots \mu_d} = \mathrm{sign}(P)$$

$$*^2 = \begin{cases} (-)^{p(d-p)} & \text{Euclidean (Riemannian) check} \\ (-)^{p(d-p)+1} & \text{Minkowski (Lorentzian)} \quad \det \eta = -1 \end{cases}$$

$$\Pi^{\pm} = \frac{1}{2}(1 \pm *) \Rightarrow \Omega^2(M, \mathrm{ad}P) = \Omega^{2,+}(M, \mathrm{ad}P) \oplus \Omega^{2,-}(M, \mathrm{ad}P)$$

A connection F_A curvature two form $F_A = F_A^+ + F_A^-$

$F = A\,dA + A^2$

$$S_{YM} \sim \|F\|^2 = -\int \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}) \sqrt{|g|}\, d^4x = -\int \mathrm{Tr}(F \wedge *F)$$

$$S_{YM} \geq \frac{1}{2g^2} \int \mathrm{Tr}(F \wedge F) = \frac{1}{2g^2} \int d\, \mathrm{Tr}\left(A\, F - \frac{1}{3} A^3\right)$$

$$= \frac{1}{2g^2} \int \mathrm{Tr}\Big(\underbrace{A\,dA + \frac{2}{3} A^3}_{\text{Chern-Simons term}}\Big)$$

$$\delta A = \tau \Rightarrow \delta F = d_A \tau \qquad \delta \int \mathrm{Tr}(F \wedge F) = \int \mathrm{Tr}(F \wedge d_A \tau) = \int \mathrm{Tr}(d_A F \wedge \tau)$$

$= 0$

$\Rightarrow$ on instanton slts $\quad S_{YM} = -\frac{1}{2g^2}\int \mathrm{Tr}(F\wedge F)$

$\uparrow$ independent of metric!

$\Rightarrow$ zero energy-momentum tensor.

Δ EXERCISE: check with a direct calculation on $\mathbb{R}^4$

TUNNELING & ϑ VACUA

Wick rotation to Minkowski $\quad *^2 = -1$

$$F_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

both d^4x & $F\tilde{F}$ produce i factor $\Rightarrow S_{YM}^{inst}$ stays invariant!

[in fact does NOT depend on metric $\Rightarrow$ neither on its signature!]

$e^{-\frac{1}{\hbar}S_{YM}^{inst}}$ tunneling amplitude

∞ disconnected vacua $|k\rangle$ with zero energy

(anti) inst $\Leftrightarrow$ large gauge transf. changing $k\to k+1$

breaks degeneracy of vacua $\Rightarrow$ vacuum eigenstate of $T|k\rangle \Rightarrow |k+1\rangle$

$$T|vac\rangle = e^{i\varphi}|vac\rangle \quad\Rightarrow\quad |vac\rangle = \sum_k e^{ik\vartheta}|k\rangle$$

YM in $|k=0\rangle$ plus $\boxed{i\vartheta\int \mathrm{Tr}\,F\wedge F}$ in fact $T|vac\rangle = e^{-i\vartheta}|vac\rangle$

in the action which gives precisely

$e^{ik\vartheta}$ for $|k\rangle$

usually (especially in SUSY theories) $\boxed{\tau = i\frac{4\pi^2}{g^2} + \frac{\vartheta}{2\pi}}$ ϑ-angle

NOTICE that the ϑ-term distinguishes between inst & anti inst

MONOPOLES

dim. reduction: $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$ take only zero modes on S^1 and neglect all nonzero one's

$$A_\mu \to A_i, \; A_4 = \Phi$$

$$F_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \to D_i\Phi = \pm\epsilon_{ijk}F_{jk}$$ Bogomolny eq's for monopole

Bogomolny bound

$$M_{\text{monop.}} = \int d^3x \, \frac{1}{g^2}\mathrm{Tr}\left(F_{ij}^2 + (D_i\phi)^2 \right)$$

$$= \int d^3x \, \frac{1}{g^2}\,\mathrm{Tr}\left((\epsilon_{ijk}F_{jk} \pm D_i\phi)^2 \mp 2\,\epsilon_{ijk}F_{jk}D_i\phi \right)$$

$$\geq \mp\frac{1}{g^2}\int_{\mathbb{R}^3} \partial_i \, \mathrm{Tr}\left(\epsilon_{ijk}F_{jk}\phi\right) \qquad \boxed{\text{BPS bound}}$$

$$\phi \xrightarrow[r\to\infty]{} a_\lambda H^\lambda \qquad H^\lambda \text{ Cartan generators of } G$$

$$\langle\phi\rangle = \langle\phi(\vartheta,\varphi)\rangle$$

maps $\langle\phi\rangle : S^2_\infty \longrightarrow G/H \quad \Rightarrow \quad \pi_2(G/H)$

e.g. $G = SU(N)$, $H = U(1)^{N-1}$ $\Rightarrow \pi_2\left(SU(N)/U(1)^{N-1} \right) \simeq \pi_1\left(U(1)^{N-1}\right) \simeq \mathbb{Z}^{N-1}$

monopoles carry magnetic charge under $N-1$ abelian factors of $SU(N)$.

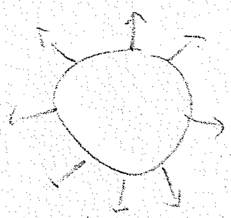
simplest example : 't Hooft Polyakov (Prasad Sommerfield)

$G = SU(2)$

$$\phi = \frac{\hat{r}_a}{r} \sigma^a \left(a r \coth(ar) - 1 \right)$$

$$A_i = - \epsilon_{aib} \frac{\hat{r}^b \sigma^a}{r} \left(1 - \frac{ar}{\operatorname{sh} ar} \right)$$

$\phi \to a \hat{r}^a \sigma^a$ hedgehog



winding one

Instantons, monopoles & Hopf fibrations

magnetic monopole field better seen in unitary gauge $\langle \phi \rangle = a_\lambda H^\lambda =$ const

in this gauge

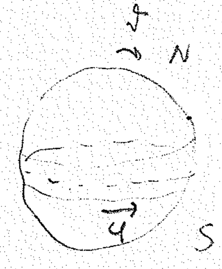
$$B_i = \operatorname{diag}(g_1 \dots g_N) \frac{\hat{r}_i}{4\pi r^2} \qquad g_\alpha : \text{magnetic charges}$$

gauge potential

$$A^N_\varphi = \frac{1 - \cos\vartheta}{4\pi r \operatorname{sen}\vartheta} g_\lambda H^\lambda$$

(1)

$$A^S_\varphi = \frac{1 + \cos\vartheta}{4\pi r \operatorname{sen}\vartheta} g_\lambda H^\lambda$$



S^2

transition funct

$$A^N_i = U(\partial_i + A^S_i) U^{-1} \qquad U = \exp\left(-i g_\lambda H^\lambda \frac{\varphi}{2\pi} \right)$$

$$U(\varphi + 2\pi) = U(\varphi) \Rightarrow \exp[i g_\lambda H^\lambda] = 1 \Rightarrow g_\lambda = 2\pi \mathbb{Z}$$

Dirac quantization

(1) is an $U(1) \sim S^1$ bundle over S^2; charge 1 $\Leftrightarrow$ Hopf bundle

$$U(1) \sim S^1 \to S^3 \qquad SU(2) \sim S^3 \to S^7 \qquad S^7 \to S^{15}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$S^2 \qquad\qquad S^4 \qquad\qquad S^8$$

monopole instanton "8d inst"?

division algebra

Hopf map

(A)

$$S_3 = \{ (z^0, z^1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1 \}$$

map $\quad (z^0, z^1) \to \{ \lambda (z^0, z^1) \mid \lambda \in \mathbb{C} \}$

$\lambda (z^0, z^1)$ $\quad |\lambda| = 1$ mapped to a single point of $\mathbb{C}P^1 \simeq S^2$

$z^0 = u^1 + i u^2$

$z^1 = u^3 + i u^4$

$S^2 : (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1$ $\qquad$ Hopf $\quad \xi^1 = 2(u^1 u^3 + u^2 u^4)$

$$\xi^2 = 2(u^2 u^3 - u^1 u^4)$$

$$\xi^3 = (u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2$$

$$Z = \frac{\xi^1 + i\xi^2}{1 - \xi^3} = \frac{z^0}{z^1}$$ invariant under $(z^0, z^1) \to \lambda (z^0, z^1)$

$\lambda \in U(1)$

$$W = \frac{\xi^1 - i\xi^2}{1 + \xi^3} = \frac{z^1}{z^0}$$

$$z = \frac{1}{w}$$

Hopf bundle:

local triv. $\quad \left(z^0/z^1, \frac{z^1}{|z^1|} \right)$ $\quad N$

$\left(\frac{z^1}{z^0}, \frac{z^0}{|z^0|} \right)$ $\quad S$ $\qquad$ eq. $|z^0| = |z^1| = \frac{1}{\sqrt{2}}$

$$G_{NS} = \frac{\sqrt{2}\, z^0}{\sqrt{2}\, z^1} = \xi^1 + i\xi^2 \in U(1)$$ circle going once

$\pi_1(U(1)) = \mathbb{Z}$

quaternions

$$S^7 = \{ (q_0, q_1) \in \mathbb{H}^2 \mid |q_0|^2 + |q_1|^2 = 1 \}$$

Hopf map

$(q_0, q_1) \to \lambda (q_0, q_1) \qquad \lambda \in \mathbb{H} \quad |\lambda| = 1 \qquad S^3$

$$\begin{array}{c} S^3 \to S^7 \\ \downarrow \\ S^4 \end{array}$$

any point of S^7 is projected to a single point of S^4

$\lambda \in \mathbb{H}$ $|\lambda| = 1$ $SU(2)$ group $SU(2)$ bundle on S^4 w charge 1

namely $k=1$ $SU(2)$ instanton!

Self-duality eq's in complex coordinates & reduction to Hitchin's system

Introduce cplx coordinates on $\mathbb{R}^4 \simeq \mathbb{C}^2$

$$z = x_1 + i x_2$$
$$w = x_3 + i x_4$$

then it is easy to show that

$$\Omega^2_+ = \Omega^{(2,0)} \oplus \Omega^0_\omega$$

$$\Omega^2_- = \Omega^{(1,1)}_\omega$$

in fact

(*) $$dz\,dw = (dx_1 + i dx_2)(dx_3 + i dx_4) =$$

$$= (dx_1 dx_3 - dx_2 dx_4) + i\,(dx_1 dx_4 + dx_2 dx_3)$$

$\Rightarrow \boxed{F_{13} = -F_{24}}$ $\quad \boxed{F_{14} = F_{23}}$

$$dz\,d\bar z + dw\,d\bar w = (dx_1 + i dx_2)(dx_1 - i dx_2) + (dx_3 + i dx_4)(dx_3 - i dx_4)$$

$$= -i2(dx_1 dx_2 + dx_3 dx_4)$$

$$\boxed{F_{12} = F_{34}}$$

$\Rightarrow$ anti self-dual connection $\Leftrightarrow$ set to zero the self-dual part

$$F^{(2,0)} = 0$$

$$F^{(1,1)} \wedge \omega = 0$$

$\uparrow$ Kähler (1,1) form

(*) $$dz\,d\bar w = (dx_1 dx_3 + dx_2 dx_4) + i\,(dx_1 dx_4 - dx_2 dx_3)$$

$\boxed{F_{13} = F_{24}}$ $\quad \boxed{F_{14} = -F_{23}}$ $\Big\}$ anti self-dual

$$dz\,d\bar z - dw\,d\bar w = -2i\,(dx_1 dx_2 - dx_3 dx_4) \quad \boxed{F_{12} = -F_{34}}$$

dim. reduction on $T^2 \times \Sigma_g$

take only zero modes $(w, \bar{w})$ $(z, \bar{z})$

$$A = (A_z, A_w) \to (a_z, \varphi)$$

$$F^{(2,0)} = (\partial_z A_w + \cancel{\partial_w A_z} + [A_z, A_w] = \partial_a \varphi = 0$$

$$0 = F^{(1,1)} \wedge \omega \Rightarrow F_{z\bar{z}} + F_{w,\bar{w}} = F_{z\bar{z}} + \cancel{\partial_w} A_{\bar{w}} - \cancel{\partial_{\bar{w}}} A_w + [A_w, A_{\bar{w}}]$$

$$= f_{z\bar{z}} + [\varphi, \bar{\varphi}] = 0$$

$$\begin{cases} f_{z\bar{z}} + [\varphi, \bar{\varphi}] = 0 \\ \partial_a \varphi = 0 \end{cases}$$ Hitchin system

Instanton & SUSY

SUSY sol's of eom:

1) set fermions to zero $\Rightarrow$ $\delta_{SUSY} B = 0$ automatic

$\delta_{SUSY} F = B = 0$ NON trivial!

$N=1$ vector multiplet: $(A_\mu, \lambda_\alpha, \bar{\lambda}^{\dot{\alpha}})$

$$\delta_{SUSY} \lambda_\alpha \approx F_{\mu\nu} \sigma^{\mu\nu}_{\alpha\beta} \epsilon^\beta$$

$$\delta_{SUSY} \bar{\lambda}^{\dot{\alpha}} \approx F_{\mu\nu} \bar{\sigma}^{\mu\nu\dot{\alpha}\dot{\beta}} \epsilon_{\dot{\beta}}$$

$\sigma^{\mu\nu}$ is self-dual

$\bar{\sigma}^{\mu\nu}$ anti self-dual

$\Rightarrow$ instanton preserves $\frac{1}{2}$ SUSY!

BPS (1st order) eq can generically be obtained in this way

LECTURE TWO

- Solitons & instantons in Quantum Theory -

We will use the formalism of path integral - the well adapted to generalize WKB to q. field theory.

The keyword is semiclassical limit :

$\exists$ physical parameter ?

consider eg. YM Lagrangian

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \qquad (1)$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - (\mu \leftrightarrow \nu) + g [A_\mu, A_\nu]^a$$

$\rightarrow$ (1) contains Kinetic terms $O(g^0)$, and three gluon $g [A_\mu, A_\nu] \partial_\mu A_\nu$ four gluon $g^2 [A_\mu, A_\nu]^2$ vertices. Coupling g CAN BE RESCALED AWAY !

$A_\mu \rightarrow \frac{1}{g} A_\mu \Rightarrow (1) \rightarrow \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - (\mu \leftrightarrow \nu) + [A_\mu, A$

in (euclidean) p.i.

$$Z_{YM} = \int [DA \ldots] \, e^{-\frac{1}{\hbar} \frac{1}{g^2} \int \frac{1}{4} F^2 + \cdots}$$

the true parameter is $g^2 \hbar$. Semiclassical limit $\hbar \rightarrow 0 \Leftrightarrow g^2 \rightarrow 0$

Notice that soliton mass & charge $\sim \frac{1}{g}$ / instanton action $\sim \frac{1}{g}$

$\Rightarrow$ amplitudes $\sim e^{-\frac{1}{\hbar g^2}}$ + corr. not obtainable from pert. theory !

- General remarks on semiclassical limit -

$\hbar \to 0$ fields indicated collectively as φ

$$\varphi = \underset{\substack{\uparrow \\ \text{classical slt.}}}{\varphi_{cl}} + \sqrt{\hbar}\ \underset{\substack{\uparrow \\ \text{quantum fluct.}}}{\tilde{\varphi}}$$

$$S[\varphi] = S[\varphi_{cl}] + \cancelto{0}{\frac{\delta S}{\delta \varphi}\Big|_{[\varphi_{cl}]}} \tilde{\varphi}\, \hbar^{1/2} + \hbar \frac{\delta^2 S}{\delta \varphi^2}\Big|_{[\varphi_{cl}]} \tilde{\varphi}^2 + O(\hbar^{3/2})$$

$$Z = \int [D\tilde{\varphi}]\, e^{-\frac{1}{\hbar} S_{cl} - \frac{\delta^2 S}{\delta \varphi^2}\Big|_{[\varphi_{cl}]} \tilde{\varphi}^2 - \cdots}$$

$$\sim e^{-\frac{1}{\hbar} S_{cl}} \int [D\varphi]\, e^{-\frac{\delta^2 S}{\delta \varphi^2}\Big|_{[\varphi_{cl}]} \tilde{\varphi}^2} + \cdots$$

gaussian integral!

<u>Measure</u>: $\frac{\delta^2 S}{\delta \varphi^2}\Big|_{[\varphi_{cl}]}$ provides the kinetic operator in the soliton/instanton bckgd $\Rightarrow$ the functional measure for the fields is organized in mode expansion w.r.t. this kinetic operator

$$K[\varphi_{cl}]\, \tilde{\varphi}_n = \lambda_n \tilde{\varphi}_n$$

NOTICE that $K[\varphi_{cl}]$ may admit zero modes! Namely

$$K[\varphi_{cl}]\, \tilde{\varphi}_0 = 0$$

e.g. if the slt. is peaked around a point x_0 [as the instanton]

$$\left[\frac{\delta S}{\delta x_0} = 0 \quad \text{because of trasl. inv.}\right]$$

$$0 = \frac{\delta}{\delta x_0} \frac{\delta S}{\delta \varphi}\Big|_{\varphi_{cl}} = \frac{\delta^2 S}{\delta \varphi^2}\Big|_{\varphi_{cl}} \frac{\delta \varphi_{cl}}{\delta x_0} = 0$$

generically, if the soliton/instanton breaks a symmetry which is preserved by the Lagrangian $\Rightarrow$ zero modes

These are related to collective coordinates or in more modern language

moduli $m^\alpha \in \underset{\text{manifold!}}{\mathcal{M}}$ $\quad \tilde{\varphi} = \sum_\alpha \xi^\alpha \tilde{\varphi}_0(m^\alpha) + \sum_n c^n \tilde{\varphi}_n$

Then:

$$Z \sim \int \Big[\underbrace{\sqrt{\det g}}_{\text{metric on } \mathcal{M}} \prod_{\alpha=1}^{m} d\xi^\alpha\Big] \prod_{n\neq 0} dc_n \; e^{-\frac{1}{\hbar} S_{cl} - \frac{\delta^2 S}{\delta\varphi^2}\Big|_{\varphi_{cl}} \tilde{\varphi}_n^2} + \dots$$

after a suitable change of variables I can treat the integration over the zero modes as an integration over the moduli. Jacobian one in $\hbar\to 0$ limit!

Moreover, the Gaussian integral over the q. fluctuations gives rise to (regularized) determinants: then eventually:

$$Z \sim \int_{\mathcal{M}} \sqrt{\det g} \prod_\alpha dm^\alpha e^{-\frac{1}{\hbar} S_{cl}} \left(\det{}' K[\varphi_{cl}]\right)^{-\frac{1}{2}}$$

if there are obs: $\quad \langle \mathcal{O} \rangle \sim Z^{-1} \int_{\mathcal{M}} \sqrt{\det g} \prod_\alpha dm^\alpha \; \mathcal{O}(m_\alpha)\, e^{-\frac{1}{\hbar} S_{cl}} \left(\det{}' K\right)^{-\frac{1}{2}}$

SIMPLE EXAMPLE

* Kink in scalar F.T. $\Rightarrow M = V(\phi_{cl}) + \sum_n \left(n_n + \frac{1}{2}\right)\hbar\omega_n + \text{c.t.}$ λ^0
  zero mode affects $O(\lambda)$
* "trick" for comp. of det'. Gel'fand Yaglom thm

particle in 1d
$$H = \frac{p^2}{2} + V(x) \qquad (m=1) \qquad ; \; Z = N\int [Dx]\, e^{-S/\hbar}$$

Euclidean action
$$S = \int_{-T/2}^{T/2} dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V\right]$$

$[Dx]$: all paths / $x(-T/2) = x_i \qquad x(T/2) = x_f$

$$\Rightarrow \quad x(t) = \bar{x}(t) + \sum_n c_n x_n(t)$$

$\bar{x}(t)$: stationary point; $\sum_n c_n x_n(t)$: fluctuations

$$\left.\frac{\delta S}{\delta x}\right|_{\bar{x}} = -\frac{d^2}{dt^2}\bar{x} + V'(\bar{x}) = 0$$

real ortonormal basis

$$K = \left.\frac{\delta^2 S}{\delta x^2}\right|_{\bar{x}} = -\frac{d^2}{dt^2} + V''(\bar{x}) \quad \Rightarrow \quad \left(-\frac{d^2}{dt^2} + V''(\bar{x})\right) x_n = \lambda_n x_n$$

$$\int_{-T/2}^{T/2} dt\, x_n x_m = \delta_{mn}$$

$$x_n(-T/2) = x_n(T/2) = 0$$

$$Z = N\, e^{-S(\bar{x})/\hbar} \prod_n \lambda_n^{-1/2} \left(1 + O(\hbar)\right)$$

$$= N\, e^{-S(\bar{x})/\hbar} \left[\det{}'\left(-\partial_t^2 + V''(\bar{x})\right)\right]^{-1/2} \left(1 + O(\hbar)\right)$$

assuming:
- $\bar{x}(t)$ isolated s/t. (i.e. no zero modes)
- all $\lambda_n > 0$

trick to compute the determinant: $K = -\partial_t^2 + U$ acting on ψ's vanishing at $-T/2$ and $T/2$

$$(-\partial_t^2 + U)\psi = \lambda\psi \qquad \text{on} \quad t \in [-T/2, T/2]$$

U bounded funct. of t. Define $\psi_\lambda(t)$ s.t. of (*) obeying initial data

$$\psi_\lambda(-T/2) = 0 \qquad \partial_t \psi_\lambda(-T/2) = 1$$

eigenvalue $\lambda_n \iff \psi_{\lambda_n}(+T/2) = 0$

check sign

$\det' K := \prod_n \lambda_n$

$\boxed{\text{Gelfand - Yaglom thm}}$

we have:

$$\boxed{\begin{array}{l} \det' K = c\, \psi_0(T/2) \quad \text{were } \psi_0 \text{ solves the initial value problem} \\ K\psi_0 = 0 \qquad \psi_0(-T/2) = 0 \quad \partial_t\psi_0(-T/2) = 1 \end{array}}$$

<u>proof</u>

take two functions $U^{(1)}$ & $U^{(2)}$

$$\det'\left[\frac{-\partial_t^2 + U^{(1)} - \lambda}{-\partial_t^2 + U^{(2)} - \lambda}\right] = \frac{\psi_\lambda^{(1)}(T/2)}{\psi_\lambda^{(2)}(T/2)} \qquad (**)$$

in fact l.h.s. is a meromorphic funct w. zero's at $\lambda_n^{(1)}$ & poles at $\lambda_n^{(2)}$

$\Rightarrow \underset{\lambda\to\infty}{\to} 1$ except $\lambda \in \mathbb{R}$

r.h.s. merom. funct w. EXACTLY same poles & zeros same beh. at ∞ $\Rightarrow$ ratio is an analytic function going to one at ∞ $\Rightarrow$ is $\equiv 1$.

(*) is proven taking

$$\frac{\det'(-\partial_t^2 + U)}{\psi_0(T/2)} = c$$

independent on U!

in fact from (**)

$$\frac{\det'(-\partial_t^2 + U^{(1)} - \lambda)}{\psi_\lambda^{(1)}(T/2)} = \frac{\det'(-\partial_t^2 + U^{(2)} - \lambda)}{\psi_\lambda^{(2)}(T/2)}$$

<u>EXERCISE</u>:

solve for $U = \frac{1}{2}\omega^2 x^2$ and find the spectrum of harm. oscill.

Semiclassical approximation on instanton background

* conventions:

$$A_\mu = A_\mu^a T^a \qquad a = 1, \dots, \dim \mathfrak{g} \leftarrow \text{Lie algebra of } G$$

$$g = e^{-\omega^a T^a}$$

anti herm.

$$(T^a)^\dagger = -T^a$$

$$[T^a, T^b] = f^{abc} T^c$$

structure constants (real & antisymm.)

it is always possible to choose T^a's s.t. $\mathrm{Tr}(T^a T^b) \propto \delta^{ab}$

Cartan inner product $(T^a, T^b) = \delta^{ab} = -\frac{1}{C(r)} \mathrm{Tr}(T^a_r T^b_r)$

$$\boxed{\mathrm{Tr}(T^a T^b) = -C(r)\,\delta^{ab}}$$

$C(r)$ Dynkin index of repr. r ; for fund $SU(N)$ $\underline{N} \Rightarrow C(N) = \frac{1}{2}$

eg. $SU(2)$ $\quad T^a = \frac{\sigma^a}{2i} \qquad \mathrm{Tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}$

$$F_{\mu\nu} = F_{\mu\nu}^a T_f^a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

gauge transf:

$$A_\mu \to A_\mu^g = g(A_\mu + \partial_\mu) g^{-1}$$

$$\Rightarrow F_{\mu\nu}^g = g F_{\mu\nu} g^{-1}$$

behaviour at ∞ plus rotation invariance $\Rightarrow$ $A_\mu = f(|x|^2)\, g \partial_\mu g^{-1}$

$$g^{(1)} = \frac{x_4 + i\,\vec{x}\cdot\vec{\sigma}}{|x|} \qquad k = 1$$

plugging in $F = \pm\tilde{F}$ we get

$$f(|x|^2) = \frac{|x|^2}{|x|^2 + \rho^2}$$

$$\boxed{A_\mu = \frac{|x|^2}{|x|^2 + \rho^2}\, g^{(1)} \partial_\mu g^{(1)^{-1}} = -\sigma_{\mu\nu} \frac{x_\nu}{x^2 + \rho^2}}$$

This is the so-called regular instanton.

Consider a spherically symm. inst around x_0: $x \to x - x_0$ in formulae above. $\Rightarrow$ 5 param. sol. 4 positions $\oplus$ scale

Moreover 3 extra parameters for constant gauge rotations

$\Rightarrow$ 8 par. moduli space

Notice the slow fall off at ∞. To improve, one can go to "singular gauge":

$$A_\mu^{S.} = [A_\mu^{NS}]^{g^{(1)-1}} = g^{(1)^{-1}} A_\mu^{NS} g^{(1)} =$$

$$= \frac{\rho^2}{x^2+\rho^2} g^{(1)^{-1}} \partial_\mu g^{(1)} = -\bar{\sigma}_{\mu\nu} \frac{x_\nu}{x^2} \frac{\rho^2}{x^2+\rho^2}$$

good falloff at ∞ but singularity at $x = x_0$

In any case

$$F_{\mu\nu}(A^{NS}) = 2\sigma_{\mu\nu} \frac{\rho^2}{(x^2+\rho^2)^2}$$

gauge components: use 't Hooft symbols

$$\bar{\sigma}_{\mu\nu} = i \bar{\eta}^a_{\mu\nu} \sigma^a \qquad \sigma_{\mu\nu} = i \eta^a_{\mu\nu} \sigma^a$$

"Clebsch Gordan" coeff. to express $SU(2)_{L,R}$ generators in terms of $SO(4)$.

$$\boxed{A_\mu^{a\,(NS)} = 2 \eta^a_{\mu\nu} \frac{x_\nu}{x^2+\rho^2}}$$

$\boxed{\text{Moduli space } K=1}$

Is a manifold:

For $K=1$ $\qquad \mathcal{M} := $ group of symmetries of action / g. symm. sol.

For YM conformal group $SO(1,5)$ $\left(\begin{array}{l} \text{special conformal } K^\mu: \\ \delta x_\mu = 2 x_\mu (x \cdot \epsilon) - \epsilon_\mu |x|^2 \\ \text{dilatations } \delta x_\mu = \alpha x_\mu \end{array} \right)$

symmetries of inst: $\underbrace{SO(4) \text{ rotations} \otimes (\text{conformal boost} + \text{trasl}) \quad R^\mu = K^\mu + \rho^2 P^\mu}_{SO(5)}$

$$\mathcal{M} = SO(1,5) / SO(5) \qquad \text{Euclidean } AdS_5 \text{ space } (*)$$

Moreover $SU(2)$ global rotations [framed instantons in math. lit.]

simplest repr. for $SO(5,1)$:

$$x_{\alpha\dot\alpha} \equiv x_\mu \sigma^\mu_{\alpha\dot\alpha} = x_4 + i\, \vec{x} \cdot \vec{\sigma} \qquad \text{quaternion}$$

$$SO(5,1): \qquad x \to x' = (Ax + B)(Cx + D)^{-1} \qquad \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1$$

$$A = a_4 + i\, \vec{a} \cdot \vec{\sigma} \qquad \text{all quaternions}$$

remark: non compact manifold (instantons of zero size $\oplus$ $x_0 \to \infty$)

can be partially compactified to add singular points (point like inst. $\rho = 0$)

(*)

$$ds^2_{AdS_5} = \frac{\rho^2}{R^2} dx_m^2 + \frac{1}{\rho^2} d\rho^2 \qquad \sqrt{g} = \rho^3$$

$R^2 = 1$

zero modes

$$A_\mu = \bar{A}_\mu + \delta A_\mu$$

as usual we have to split into zero & non zero modes of Kinetic op. eval. at $\bar{A}$

$$\delta A_\mu = \Sigma_i \, \xi^i \delta_i A_\mu + \delta \tilde{A}_\mu$$

$\delta_i A_\mu$ are slts of the linearized instanton eq:

$$F(\bar{A} + \delta_i A) = \pm \tilde{F}(\bar{A} + \delta_i A)$$

namely

$$D_\mu^{(\bar{A})} \delta_i A_\nu - D_\nu \delta_i A_\mu = \pm \epsilon_{\mu\nu\rho\sigma} D_\rho \delta_i A_\sigma$$

trasversality $\Bigg\} (*)$

$$D_\mu(\bar{A}) \, \delta_i A_\mu = 0$$

normalizable $\boxed{-\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}(\delta_i A_\mu \delta_j A_\mu) \equiv g_{ij}}$ metrics on moduli space

Atiyah-Singer index thm determines the # of slts of the above system $(*)|_{\text{at fixed } k}$ for $G = SU(N)$:

$$\dim \mathcal{M}_{K,N} = 4KN$$

as already anticipated $\partial_i \bar{A}_\mu(x; m^i)$ is a zero mode:

$$0 = \frac{\partial}{\partial m^i} \left.\frac{\delta S}{\delta A}\right|_{\bar{A}} = \left.\frac{\delta^2 S}{\delta A^2} \frac{\partial A}{\partial m^i}\right|_{A=\bar{A}} = 0$$

In order to ensure transverality, we have to add a term

$$\delta_i A_\mu = \partial_i A_\mu + D_\mu \Lambda_i$$

e.g. $k=1$

$$x_0 : \quad \frac{\partial}{\partial x_0^\nu} A_\mu = -\partial_\nu A_\mu \Rightarrow \delta_\nu A_\mu = -\partial_\nu A_\mu + D_\mu A_\nu = F_{\mu\nu}$$

$$0 = \partial_\nu\left(-\partial_\mu A_\nu + D_\nu \Lambda_\mu\right) \Rightarrow \Lambda_\mu = A_\mu \ ;$$

norm of zero modes as obviously S_{inst}!

$$\|\delta_{(\nu)} A_\mu\|^2 = \frac{8\pi^2}{g^2}$$

• dilatations:

$\frac{\partial A_\mu}{\partial \rho}$ is already transversal, as can be checked by expl. cp on $A_\mu^{S.}$ moreover since $\partial_\rho A_\mu \sim A_\mu$ one easily gets also the norm $\| a_\rho \|^2 = 2 S_{inst}$

• rotations:

$\delta_a A_\mu = D_\mu \Lambda_a$ $\quad \Lambda_a \neq 0$ at ∞ $\quad$ solving $D^2 \Lambda^a = 0$

$$\Lambda_a = \frac{x^2}{x^2+\rho^2}\left(-\frac{\sigma^a}{g}\right) \qquad \lim_{x \to \infty} \Lambda^a = -\frac{T^a}{g}$$

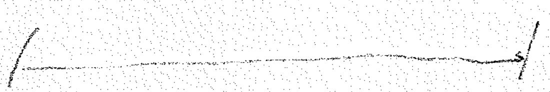
norm: $2 S_{inst} \rho^2$

$$ds^2 = dx_0^2 + \rho^2 d\Omega + d\rho^2 \qquad \text{metric: } \sqrt{g} \propto \frac{\rho^3}{g^8} d^4x_0 \, d\rho \, d^3\vartheta$$

$\mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2$ $\quad$ integrating over $\vartheta \Rightarrow AdS_5$ metric

since A_μ is in the adjoint, center of SU(2) $\mathbb{Z}_2$ leave it invariant!

singular space.



path integral:

$$A_\mu = \bar{A}_\mu + \delta A_\mu$$

$$\delta A_\mu = \sum_i \xi^i \delta_i A_\mu + \tilde{A}_\mu$$

functional measure [$\tilde{A}_\mu = \sum_n c_n \tilde{A}_\mu^{(n)}$ non zero modes]

$$\sqrt{g} \prod_{i,n} d\xi^i \, dc_n \longrightarrow \int \sqrt{g} \prod_i dm_i \, dc_n$$

Jacobian?

$$1 = \int \prod_i dm^i \left|\det \frac{\partial f_j}{\partial m^i}\right| \prod_j \delta(f_j(m))$$

$$f_j(m) \equiv -\frac{1}{2g^2}\int d^4x\, \mathrm{tr}\, \underset{\substack{\uparrow \\ \text{total fluct.}}}{\delta A_\mu}\, \delta_j A_\mu \underset{\substack{\uparrow \\ \text{zero \& non zero mutually orthogonal}}}{=} \sum_i \xi^i g_{ij}(m)$$

$$\frac{\partial f_j}{\partial m^i} = -\frac{1}{2g^2}\int d^4x\, \mathrm{Tr}\left(\frac{\partial}{\partial m^i}\delta A_\mu\, \delta_j A_\mu + \delta A_\mu \frac{\partial}{\partial m^i}\delta_j A_\mu\right)$$

since $A_\mu = \bar{A}_\mu + \delta A_\mu$ doesn't depend on m^i $\Rightarrow$

$$\frac{\partial}{\partial m^i}\delta A_\mu = -\frac{\partial}{\partial m^i}\bar{A}_\mu$$

moreover since $\int d^4x\, \mathrm{tr}\, D_\mu \Lambda\, \delta_j A_\mu = -\int d^4x\, \mathrm{Tr}\, \Lambda_j D_\mu \delta_j A_\mu = 0$

then the first term is precisely $\delta_i A_\mu$

$\Rightarrow$ $$1 = \int \prod_i dm^i \left|\det\left(g_{ij}(m) - \cancel{\frac{1}{2g^2}\int d^4x\, \mathrm{Tr}\, \delta A_\mu \frac{\partial}{\partial m^i}\delta_j A_\mu}\right)\right| \prod_j \delta(\xi^i g_{ij})$$

$\int d\xi^i\, \delta(\xi^i g_{ij})$ sets $\xi^i = 0$ $\Rightarrow$ $\delta A_\mu \equiv \tilde{A}_\mu$ higher order in pert. th.

I get $\int \prod_i dm^i$

Jacobian is one!

example: $\overset{1\text{-inst}}{Z(\vartheta)} = \langle \vartheta | e^{-HT} | \vartheta \rangle \underset{\text{one inst}}{\sim} K e^{-S_{inst}} e^{i\vartheta} VT$

$\nwarrow [\int d\rho \, d\vartheta]$ $\nwarrow [d\dot{x}_0]$

$$\langle \vartheta | e^{-HT} | \vartheta \rangle \approx \sum_{n,\bar{n}} \left(K e^{-S_{inst}} \right)^{n+\bar{n}} (VT)^{n+\bar{n}} e^{i(n-\bar{n})\vartheta} / n! \bar{n}!$$

dilute gas approx

$$= \exp\Big(\overbrace{+ 2K\, VT e^{-S_{inst}} \cos\vartheta}^{-E(\vartheta)\, VT} \Big)$$

$$\frac{E(\vartheta)}{V} \sim -2K \cos\vartheta \, e^{-S_{inst}}$$

$$e^{-S_{inst}} K \sim e^{-S_{inst}} g^{-8} \int_0^\infty \frac{d\rho}{\rho^5} f(\rho M) \underset{RGE}{=} g^{-8} \int_0^\infty \frac{d\rho}{\rho^5} \underbrace{(\rho M)^{8\pi^2 \beta_1} e^{-\frac{8\pi^2}{g^2}}}_{e^{-\frac{8\pi^2}{g^2(\rho)}}} (1 + O(g^2))$$

8 zero modes

energy density

$$\frac{1}{g^2(\rho)} = \frac{1}{g^2} - \beta_1 \ln M + O(g^2)$$

$\beta_1 = 11/12\pi^2$ in $SU(2)$

remarks:

2) infrared embarassment! $\rho^{\frac{22}{3}-5} = \rho^{7/3}$ divergent for $\rho \to \infty$: large scales g^2 blows up!

1) $\langle \vartheta | F\tilde{F} | \vartheta \rangle = \frac{d}{d\vartheta} \ln Z(\vartheta) \sim i K e^{-S_{inst}} \sin\vartheta$ axion potential

Fermions

› integral for fermions : anticommuting variables ξ $\xi^2 = 0$

$$\int d\xi \, f(\xi)$$

such that : 1) linearity $\int (a f(\xi) + b g(\xi)) d\xi = a \int f(\xi) d\xi + b \int g(\xi) d\xi$

2) trasl. $\int f(\xi + b) d\xi = \int f(\xi) d\xi$

notice : only two linearly indep. functs : 1, ξ

Choose normaliz. $\int d\xi \, \xi = 1$

from 2) $\boxed{\int d\xi \, 1 = 0}$

For many variables

$$\int d\psi \, d\bar{\psi} \; e^{-\bar{\psi} A \psi} = \det A$$

easy to show by diagonalizing

Functional integral

$$\int D\psi \, D\bar{\psi} \; e^{-\bar{\psi} K \psi} = \det{}' K$$

remark : Dirac fermions $\psi, \bar{\psi}$ are INDEPENDENT in Euclidean space $\Rightarrow$ fermionic action not real. No form for Green functs. after Wick rotation. Pseudo reality for extended SUSY.

Weyl spinor ψ repr. r

$$S = \int d^4x \, \bar{\Psi} \, i \bar{\sigma}^\mu \underbrace{(\partial_\mu + A_\mu)} \psi$$

$$A_\mu = A_\mu^a \, T_r^a$$

most important novelty: chiral anomaly $\Rightarrow$ selection rule

chiral symm: $\psi \to e^{i\alpha}\psi \quad \bar{\Psi} \to e^{-i\alpha}\bar{\Psi}$; current:

$$J_\mu = \bar{\Psi}\,\bar{\sigma}_\mu\,\psi \qquad \partial_\mu J_\mu = 0 \quad \text{classically}$$

but

$$\partial_\mu J_\mu = -\frac{i}{32\pi^2}\left(F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a\right) 2\,C(r)$$

$C(r)$: Dynkin index of r representation

Invariance of functional integral implies

$$\int D\psi\, D\bar{\Psi}\; e^{-S}\, \mathcal{O}(x_1 \ldots x_n) =$$

$$= \int D\psi\, D\bar{\Psi}\; e^{-S}\left[1 - i\,\partial_\mu\alpha\;\bar{\Psi}\,\bar{\sigma}^\mu\psi - \frac{i}{32\pi^2}\,2C(r)\int d^4x\,\alpha(x)\,F\tilde{F} + O(\alpha^2)\right]$$

$$\left[\mathcal{O}(x_1 \ldots x_n) + \delta_\alpha\,\mathcal{O}(x_1 \ldots x_n) + O(\alpha^2)\right]$$

which implies by integrating by parts

$$-\int D\psi\, D\bar{\Psi}\; e^{-S}\int d^4x\,\alpha(x)\left[\partial_\mu J_\mu(x) + \frac{i}{32\pi^2}\,2C(r)\,F\tilde{F}\right]\mathcal{O}(x_1 \ldots x_n) +$$

$$+ \sum_{i=1}^{n}\int D\psi\, D\bar{\Psi}\; e^{-S}\,\alpha(x_i)\,\frac{\partial\mathcal{O}}{\partial\alpha(x_i)}(x_1 \ldots x_n) = 0$$

i.e.

$$\langle \partial_\mu J_\mu(y)\,\mathcal{O}(x_1 \ldots x_n)\rangle - \sum_{i=1}^{n}\left\langle \frac{\partial\mathcal{O}}{\partial\alpha(x_i)}(x_1 \ldots x_n)\right\rangle \delta(x_i - y) =$$

$$= -\frac{i}{32\pi^2}\,2C(r)\,F_{\mu\nu}^a\tilde{F}_{\mu\nu}^a(y)\,\langle\mathcal{O}(x_1 \ldots x_n)\rangle$$

integrating over y, and noticing that $\int d^4y\, \partial_\mu J_\mu(y) = 0$ [i.e. no surface terms e.g. turn on a small mass for ψ's]
we have finally:

$$\boxed{i 2C(r)\, K \, \langle \mathcal{O}(x_1 \dots x_n) \rangle = \sum_{i=1}^{n} \left\langle \frac{\partial \mathcal{O}}{\partial \alpha(x_i)} (x_1 \dots x_n) \right\rangle}$$

i.e. also the inserted operator has to be anomalous wrt this $U(1)_A$
This is a very important result in that there are correlators which are zero perturbatively ($F\tilde{F} = 0$ no axial anomaly) but now vanishing in the instanton backgd! CHIRAL CONDENSATES

To see this let's expand in eigenvalues of kinetic operator:
notice that

$$\bar{\sigma}^\mu (\partial_\mu + A_\mu) : \quad S_- \to S_+$$

maps to DIFFERENT SPACES. So we have to consider Dirac fields:

$$i \not{D} \psi_n = \lambda_n \psi_n \qquad \lambda_n \in \mathbb{R} \quad \text{since } i\not{D} \text{ is hermitian}$$

notice that, since $\{\gamma_\mu, \gamma_5\} = 0$

$$i \not{D} \gamma_5 \psi_n = - \lambda_n \gamma_5 \psi_n$$

non zero modes always appear in pairs $\lambda_n, -\lambda_n$!
Instead zero modes can be always be chosen as eigenfuncts of γ_5 :

$$\gamma_5 \psi_n = \epsilon_n \psi_n \qquad (\gamma^5)^2 = 1 \Rightarrow \epsilon_n = \pm 1$$

n_R # eigenv. of $\left(\frac{1+\gamma_5}{2}\right)\Psi$ $\quad S_+$ right handed Weyl $\bar{\lambda}^{\dot{\alpha}}$

n_L $\qquad \left(\frac{1-\gamma_5}{2}\right)\Psi$ $\quad S_-$ left " " ψ_α

Atiyah-Singer index thm:

$$\boxed{n_L - n_R = 2C(r)\,K}$$

VANISHING THM

- It is easy to show that for $K>0$ the only normalizable s/t for

$$\gamma_\mu D_\mu \psi = 0$$

$$\gamma_5 \psi = \psi$$

$\Downarrow$ $n_R = 0$

is $\psi = 0$, i.e. there are no Righthanded zero modes on INST. BCKGD

proof

$$D_\nu \gamma_\nu D_\mu \gamma_\mu \psi = D_\mu D_\mu \psi + \frac{1}{2} F_{\mu\nu} \gamma_\mu \gamma_\nu \psi = 0$$

moreover

$$F_{\mu\nu} \gamma_\mu \gamma_\nu \gamma_5 = -\tilde{F}_{\mu\nu} \gamma_\mu \gamma_\nu$$

thus

$$D^2 \psi + \frac{1}{4} \underbrace{\cancel{(F_{\mu\nu} - \tilde{F}_{\mu\nu})}}_{\text{inst. bckgd}} \gamma_\mu \gamma_\nu \psi = D^2 \psi$$

$$\int d^4x \; D_\mu \psi^+ D_\mu \psi = 0 \iff D_\mu \psi = 0 \qquad \text{e.g. } A_3 = 0 \Rightarrow \partial_3 \psi = 0$$

ψ not normalizable unless $\psi \equiv 0$ qed

Then the functional measure reads

$$\int \prod_{1}^{n_L} d\psi_L^{(0)} d\bar{\psi}_L^{(0)} \prod_{n\neq 0} d\Psi_n \, d\bar{\Psi}_n$$

in order this to be not vanishing, one has to insert

$n_L = 2(k) K$ fermionic bilinears $\bar{\Psi} \frac{(1-\gamma_5)}{2} \Psi$

- consequences:

1) in presence of massless spin fields

$$\langle \vartheta | F\tilde{F} | \vartheta \rangle = 0 \, !$$

all ϑ vacua have same energy.
Notice that WI reads

$$\left(\frac{\partial}{\partial \alpha} + 2 \frac{\partial}{\partial \vartheta} \right) \langle \mathcal{O}(x_1 \dots x_n) \rangle = 0$$

namely a variation of ϑ can be undone via an anomalous $U(1)_A$ transf.
$U(1)_A$ connect ϑ vacua.
In other words $U(1)_A$ is spontaneously broken in the instanton vacuum & ϑ-vacua are the vacua associated to this spontaneous breaking.
Relevant phen. applications to the so-called $U(1)$ problem in QCD

see e.g. lecture notes by Coleman.

LECTURE 3

INSTANTONS & SUSY

SUSY $\Rightarrow$ much better control on dynamics

schematically:

$\mathcal{N}=1$

$$\underbrace{\int d^2\vartheta\, d^2\bar\vartheta\; K(\Phi^+, \Phi, V)}_{\text{Kähler potential}} + \underbrace{\int d^2\vartheta\; \overbrace{W(\Phi)}^{\text{superpotential}}}_{\text{F-terms}} + \text{c.c.}$$

$W(\Phi)$ hol.

F terms are under much easier control than non ren. terms. Because of $U(1)_R$ anomaly $\Rightarrow$ chiral selection rule: instanton contributions!

$\mathcal{N}=2$

$$\int d^4\vartheta\; \underbrace{\mathcal{F}(\Psi)}_{\text{prepotential}} \qquad \Psi \quad \mathcal{N}=2 \text{ chiral superfield}$$

the situation is even better in $\mathcal{N}=2$: symmetry between F & D terms $\Rightarrow$ COMPLETE CONTROL ON LOW ENERGY EFFECTIVE ACTION VIA INSTANTONS!

$\mathcal{N}=4$

<u>no</u> $U(1)_R$ anomaly : correlators take contributions both from pert. expansion & instantons

Moreover, in SUSY there are three new important facts, that allow to obtain QUANTITATIVE results:

1 – cancellation of determinants:
- the bosonic & fermionic fluctuations cancel exactly around an inst. bckgrd (for any $\mathcal{N}$!)

2 – integration over the instanton moduli space $\mathcal{M}_{KN}$ is CONVERGENT!

Using the above results one can obtain quantitative results. Moreover

3 – SUSY localisation: by using supersymmetry charges one can localise the p.i. to get contribution only from a set of isolated fixed points in $\mathcal{M}_{KN}$

$\Rightarrow$ completely solve the problem for any K (at least for $G = SU(N)$) [for $\mathcal{N} \geq 2$, also softly broken to $\mathcal{N} = 1$]

~~more principles~~ When can we apply instanton calculus?

It is a semiclassical approach $\Rightarrow$ weak coupling

① superconformal theories ($\beta = 0$) $\Rightarrow$ we can tune g_{YM} to be small. Most important example $N=4$ SYM [AdS/CFT instantons vs D-instantons in IIB]

② asymptotic free theories in Coulomb phase [eg $SU(N) \to U(1)^{N-1}$]

the vev of the scalar $\langle \phi \rangle = a$ can be chosen to be large wrt strong coupling scale $\Lambda/a \ll 1$

$\Rightarrow g_{YM}(a) \ll 1$.

First example in 't Hooft's original paper.

Most celebrated example: Seiberg Witten s/t of $N=2$ SYM.

[remark: valid also with soft (mass) breaking to $N=1$ + $W(\Phi)$.]

③ asymptotic free theories in Higgs phase

[gauge group COMPLETELY BROKEN]

$N=1$ SQCD $N_f = N_c - 1 \Rightarrow$ Affleck-Dine-Seiberg superpotential. $N_f < N_c - 1$ $SU(N_c) \to \Pi_i SU(N_i)$

one can handle the problem from $N_f' = N_c - 1$ by sending the mass of $N_f' - N_f$ chiral multiplet to infty. The results are trustable (w. suitable matching of Λ scale) due to holomorphic decoupling.

REMARK: in all the above instanton is only an APPROXIMATE s/t. of e.o.m. due to the presence of extra scalar fields! But the approx. is OK in semicl. limit. More on this in 4^{th} Lecture.

→ Which are the observables?

To answer this question we have to recall that instantons do preserve a <u>chiral half of SUSY's</u> as can be seen for example from the SUSY variation of the gaugino fields:

$$\delta_\epsilon \lambda_\alpha = [\epsilon Q + \bar{\epsilon}\bar{Q}, \lambda_\alpha] = i\, \sigma_{\mu\nu\,\alpha\beta}\, \epsilon^\beta\, F_{\mu\nu}$$

$$\delta_\epsilon \bar{\lambda}^{\dot{\alpha}} = [\epsilon Q + \bar{\epsilon}\bar{Q}, \bar{\lambda}^{\dot{\alpha}}] = i\, \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}\dot{\beta}}\, \bar{\epsilon}_{\dot{\beta}}\, F_{\mu\nu}$$

and using the (anti) self duality of $\overset{(-)}{\sigma}_{\mu\nu}$.

Then the SUSY observables are the one's preserving the same SUSY's $\Leftrightarrow$ g.i. composites of CHIRAL FIELDS. $\mathcal{O} = f(\Phi^i)$

Since this is closed under multiplication one talk of the CHIRAL RING

For the instantons, the chiral observables are the ones annihilated by $\bar{Q}^{\dot{\alpha}}$:

$$[\bar{Q}^{\dot{\alpha}}, \mathcal{O}] = 0$$

this property, assuming the SUSY invariance of the vacuum, leads to a very important result on the correlators:

$$\partial_i \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle = 0$$

namely the correlator DO NOT DEPEND on the insertion points of the local observables! (Actually one can show that do not depend at all on the metric).(*)

This is a very simple consequence of SUSY algebra:

$$\frac{\partial}{\partial x^{i\mu}} \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle =$$

$$= \langle \mathcal{O}(x_1) \dots \frac{\partial}{\partial x_i^{\mu}} \mathcal{O}(x_i) \dots \mathcal{O}(x_n) \rangle =$$

$$\sim \langle \mathcal{O}(x_1) \dots [\bar{Q}_{\dot\alpha}, \{Q_\alpha, \mathcal{O}(x_i)\}] \dots \mathcal{O}(x_n) \rangle$$

$$= \langle \mathcal{O}(x_1) \dots \{Q_\alpha, \mathcal{O}(x_i)\} \dots \mathcal{O}(x_n) \bar{Q}_{\dot\alpha} | 0 \rangle + \langle 0 | \bar{Q}_{\dot\alpha} \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$$

$$= 0$$

(*) $\Rightarrow$ topological correlators!

Other properties:

- fixed dependence on coupling g_{YM} through $\Lambda = \mu e^{-\frac{8\pi^2}{g^2(\mu)}}$ RGE scale

- analytic mass dependence

remark: for $N=4$ the only topological observable is $\mathcal{O} = \mathbb{1}$: namely the partition fnct Z. However, due to absence of chiral anomaly instantons can contribute to correlators in the perturbative sector.

Semiclassical evaluation

$$S_{YM} = \frac{1}{4g_{YM}^2} F^a_{\mu\nu} F^a_{\mu\nu}$$

$$S_{ferm} = i \bar{\Psi}^a \gamma_\mu D^a_{\mu b} \Psi^b$$

$$\left[S_{scalars} = \bar{\varphi}^a D^2 \varphi^a \right]$$

background method: $A_\mu = \bar{A}_\mu + Q_\mu$

gauge fixing: $D_\mu(\bar{A}) Q_\mu \equiv \bar{D}_\mu Q_\mu = 0$

$$S_{g.f} = b \bar{D}_\mu Q_\mu + \bar{c} \bar{D}_\mu D_\mu c$$

$$S = S_{YM} + S_{ferm} + S_{scalars} + S_{g.f.}$$

at one loop: (b already integrated out)

$$S = S_0 + \int \frac{1}{2} Q_\mu K^{g.f.}_{\mu\nu}(\bar{A}) Q_\nu + i \bar{\Psi} \bar{D}_\mu \gamma_\mu \Psi + \bar{c} \bar{D}^2 c$$

$$\langle \mathcal{O} \rangle = \int_{\mathcal{M}} \frac{\det' i\bar{D\!\!\!/} \; \det \bar{D}^2}{\det'(K_{\mu\nu})^{\frac{1}{2}}} e^{-S_0} \mathcal{O}(m_B, m_F) \sqrt{\mathrm{Sdet}\, g_{\mathcal{M}}} \, dm_B \, dm_F$$

supermanifold !

(bosonic & fermionic moduli)

KINETIC OPERATORS ON INST. BKGD & DET'S CANCELLATION

VECTORS:

We take the gauge field around the instanton bckgd as

$$A_\mu = \bar{A}_\mu + Q_\mu$$

where $\bar{A}_\mu$ is the inst. slt. and Q_μ the quantum fluct.

Moreover we choose that under gauge transf.

$$Q_\mu \longrightarrow A_\mu^g - \bar{A}_\mu = g(\bar{A}_\mu + Q_\mu)g^{-1} + g\partial_\mu g^{+} - \bar{A}_\mu$$

namely Q_μ undertake the complete g.t. of A_μ.

We gauge fix it by imposing

$$D_\mu Q_\mu = \partial_\mu Q_\mu + [A_\mu, Q_\mu] = 0 \quad (\Delta)$$

The Kinetic operator for Q_μ comes from expansion of

$$[F_{\mu\nu}(\bar{A} + Q)]^2 \quad (\ast)$$

whith

$$F_{\mu\nu}(\bar{A} + Q) = \bar{F}_{\mu\nu} + D_\mu Q_\nu - D_\nu Q_\mu + [Q_\mu, Q_\nu]$$

then by developping $(\ast)$ and using e.o.m. we get [exercise!]

$$Q_\mu \underbrace{\left[-D^2 \delta_{\mu\nu} + D_\mu D_\nu - 2\bar{F}_{\mu\nu} \right]} Q_\nu$$

upon g.f. (Δ)

$$\boxed{-D^2 \delta_{\mu\nu} - 2\bar{F}_{\mu\nu}} \quad (g)$$

Fermions

We wish to consider the eigenvalue eq. for the Dirac operator

$$i D_\mu \gamma^\mu \Psi^{(n)} = \lambda_n \Psi^{(n)} \qquad (*)$$

$$\Psi^{(n)} = \begin{pmatrix} \psi_L^{(n)} \\ \psi_R^{(n)} \end{pmatrix}$$

$(*)$ is equivalent to

$$i D_\mu \sigma_\mu \psi_R^{(n)} = \lambda_n \psi_L \qquad (*')$$

$$i D_\mu \bar\sigma_\mu \psi_L^{(n)} = \lambda_n \psi_R^{(n)}$$

by multiplying the first by $i D_\nu \bar\sigma_\nu$ and the second by $i D_\nu \sigma_\nu$ we get [recall that $\bar\sigma_\mu \sigma_\nu = \delta_{\mu\nu} + \bar\sigma_{\mu\nu}$
$\sigma_\mu \bar\sigma_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu}$]

$$\left(D^2 + \tfrac{1}{2}\bar\sigma_{\mu\nu} F_{\mu\nu}\right)\psi_R^{(n)} = -\lambda_n \psi_R^{(n)}$$

$$\left(D^2 + \tfrac{1}{2}\sigma_{\mu\nu} F_{\mu\nu}\right)\psi_L^{(n)} = -\lambda_n \psi_L^{(n)} \qquad (\overset{*}{*})$$

Since $\sigma_{\mu\nu}$ is self dual & $\bar\sigma_{\mu\nu}$ anti-sd, in an instanton bckgr. $(\overset{*}{*})$ reduce to

$$\boxed{\begin{aligned} D^2 \psi_R^{(n)} &= -\lambda_n^2 \psi_R^{(n)} \\ \left(D^2 + \tfrac{1}{2}\sigma_{\mu\nu} \bar F_{\mu\nu}\right)\psi_L^{(n)} &= -\lambda_n^2 \psi_L^{(n)} \end{aligned}}$$

Using the above expressions for the Kinetic operators in the instanton backgd it is easy to see that all eigenfuncts can be expressed in terms of that of the scalar Kinetic operator

$$D^2 \varphi_n = -\lambda_n^2 \varphi_n$$

in fact, for fermionic fields, it is immediate to realize that $\psi_R^{(n)} = \bar{\epsilon} \varphi_n$ with $\bar{\epsilon}^{\dot{\alpha}}$ a constant fermionic parameter. Plugging this in (*') we immediately obtain

$$\psi_L^{(n)} = \lambda_n^{-1} \, i \, D_\mu \sigma_\mu \bar{\epsilon} \varphi_n$$

(RECALL that we are considering $\lambda_n \neq 0$!)

Concerning gauge bosons, it is easy to see that writing

$$\Psi^{(n)} = \begin{pmatrix} \xi^{(n)} \\ \bar{\chi}^{(n)} \end{pmatrix} \quad \text{then} \quad Q_\mu^{(n)} = \bar{\eta} \bar{\sigma} \xi^{(n)}$$

is an eigenfunct of (g) if $\bar{\eta}$ is a constant right-handed Weyl spinor:

$$D^2 Q_\lambda^{(n)} = \bar{\eta} \bar{\sigma}_\lambda D^2 \xi^{(n)}$$

$$\bar{\sigma}_\lambda D^2 \xi^{(n)} = \bar{\sigma}_\lambda D_\mu D_\nu \left(\sigma_\mu \bar{\sigma}_\nu - \sigma_{\mu\nu} \right) \xi^{(n)} =$$

$$= \bar{\sigma}_\lambda D_\mu \sigma_\mu \left(D_\nu \bar{\sigma}_\nu \xi^{(n)} \right) - \frac{1}{2} F_{\mu\nu} \bar{\sigma}_\lambda \sigma_{\mu\nu} \xi^{(n)}$$

using $D_\nu \bar{\sigma}_\nu \xi^{(n)} = -i \lambda_n \bar{\chi}^{(n)}$ and $D_\mu \sigma_\mu \bar{\chi}^{(n)} = -i \lambda_n \xi^{(n)}$

we get that the first term is just

$$-\lambda_n^2 \bar{\sigma}_\lambda \xi^{(n)}$$

Moreover using : $\bar{\sigma}_\lambda \sigma_{\mu\nu} = \delta_{\lambda\mu} \bar{\sigma}_\nu - \delta_{\lambda\nu} \bar{\sigma}_\mu + \epsilon_{\mu\nu\lambda\rho} \bar{\sigma}_\rho$

~~and~~ selfduality of the curvature, we get for the 2^{nd} term

$$\frac{1}{2} \bar{F}_{\mu\nu} \bar{\sigma}_\lambda \sigma_{\mu\nu} \xi^{(n)} = \frac{1}{2} 2 \bar{F}_{\lambda\nu} \bar{\sigma}_\nu \xi^{(n)} + \bar{F}_{\lambda\rho} \bar{\sigma}_\rho \xi^{(n)}$$

then finally

$$D^2 Q_\mu^{(n)} = -\lambda_n^2 Q_\mu^{(n)} - 2 \bar{F}_{\mu\nu} Q_\nu^{(n)}$$

namely

$$(-D^2 \delta_{\mu\nu} - 2 \bar{F}_{\mu\nu}) Q_\nu^{(n)} = \lambda_n Q_\mu^{(n)}$$

RECAP:

the $\lambda_n \neq 0$ eigenfncts of the kinetic operators are

scalars : $-D^2 \varphi_n = \lambda_n^2 \varphi_n$

fermions : $i D_\mu \gamma_\mu \Psi^{(n)} = \lambda_n \Psi^{(n)}$ $\quad \Psi^{(n)} = \begin{pmatrix} \lambda_n^{-1} i D_\mu \varphi_n \sigma_\mu \\ \varphi_n \bar{\epsilon} \end{pmatrix}$

moreover $i D_\mu \gamma_\mu \gamma_5 \Psi^{(n)} = -\lambda_n \gamma_5 \Psi^{(n)}$

multiplicity 2

gauge field : $(-D^2 \delta_{\mu\nu} - 2 \bar{F}_{\mu\nu}) Q_\nu^{(n)} = \lambda_n^2 Q_\mu^{(n)}$

$$Q_\mu^{(n)} = \bar{\eta} \bar{\sigma}_\mu \sigma_\nu \bar{\epsilon} \cdot \lambda_n^{-1} i D_\nu \varphi_n$$

multiplicity 4

At this point we are ready to show the "cancellation" of determinants:

we choose to work in the Pauli-Villars scheme of regularization then we get ($N=1$ SYM)

$$\prod_{n\neq 0}\left(\frac{(\lambda_n^2+\mu^2)^{4/2}}{(\lambda_n^2)^{4/2}} \;\; \frac{\lambda_n^2}{\lambda_n^2+\mu^2} \;\; \left[\frac{(\lambda_n)^2(-\lambda_n)^2}{(\lambda_n+i\mu)^2(-\lambda_n+i\mu)^2}\right]^{1/2}\right) = 1$$

$$\underset{\text{gauge bosons}}{\Uparrow} \qquad \underset{\text{ghosts}}{\Uparrow} \qquad \underset{\text{fermions}}{\Uparrow}$$

The contribution of zero modes is given by

$$\mu^{n_B - \frac{1}{2} n_F} \qquad (*)$$

where n_B : # bosonic zero modes

n_F : # fermionic zero modes (Weyl repr.)

Adding chiral multiplets do not change the cancellation, since bosonic & fermionic non zero modes compensate. Of course it changes the zero mode contr. (only n_F changes)

The factor $(*)$ combines with the classical inst. action to provide the dependence on RGG scale $\Lambda = \mu\, e^{-\frac{8\pi}{\beta g^2}}$

in fact $\underbrace{}_{\text{it can be shown that}}$ $n_B - \frac{1}{2} n_F = \beta \;\Rightarrow\; \mu^{\beta} e^{-\frac{8\pi^2}{g^2}} = \left(\mu\, e^{-\frac{8\pi^2}{\beta g^2}}\right)^{\beta} = \Lambda^{\beta}$

Fermionic zero modes

$$i \gamma_\mu D_\mu \Psi = 0 \qquad \begin{cases} i \sigma_\mu D_\mu \bar{\chi} = 0 \\ i \bar{\sigma}_\mu D_\mu \lambda = 0 \end{cases}$$

$$\Psi = \begin{pmatrix} \lambda \\ \bar{\chi} \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

From index thm. we already know that $\bar{\chi} \equiv 0$ identically.

Then we have to solve

$$i D_\mu \bar{\sigma}_\mu \lambda = 0$$

Use SUSY + SUPERCONF. transf. on $A = \bar{A}$ $\lambda = \bar{\lambda} = 0$

SUSY: $\quad \delta_{SUSY} \lambda = \frac{1}{2} \bar{F}_{\mu\nu} \sigma_{\mu\nu} \eta$

SCONF: $\quad \delta_{SCONF} \lambda = \frac{1}{2} \bar{F}_{\mu\nu} \sigma_{\mu\nu} (x_\rho \sigma^\rho \bar{\epsilon})$

Instead both vanishing on $\bar{I}$ [inst. preserves $\frac{1}{2}$ SUSY's]

* In other words, we already saw that $\mathcal{M}^{bos}_{K=1} = G_{action} / G_{sit.}$

now we have a supergroup acting $\Rightarrow$ supermanifold!

Namely, beyond trasl., conf. trasf & rotations we have to consider also SUSY & SUCONF transf which are symmetries of the action but not of the sol.

<u>REMARK</u>: this reasoning is valid ONLY for $K=1$ both in bosonic & fermionic case. For higher K there are zero modes NOT related to any symm. of the action, and one has to resort to a more general and powerful construction (ADHM).

thus we have 4 zero modes

$$\lambda_i^{(0)} = \frac{1}{2} \bar{F}_{\mu\nu} \sigma_{\mu\nu} \xi_i(x)$$

$$\xi_i(x) = \eta_i + (x-x_0)_\mu \sigma_\mu \bar{\epsilon}_{i-2} \qquad i = 1, \dots, 4$$

$$\eta_1 = \bar{\epsilon}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \eta_2 = \bar{\epsilon}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \eta_{3,4} = \bar{\epsilon}_{-1,0} = 0$$

$i = 1, 2$ SUSY $\qquad i = 3, 4$ SUCONF

metric:

$$g_{ij} = \int d^4x \; \lambda_i^{(0)} \lambda_j^{(0)} = \mathrm{diag}\left(\frac{8\pi^2}{g^2}, \frac{8\pi^2}{g^2}, \frac{8^2\pi^2}{g^2}\rho^2, \frac{8^2\pi}{g^2} \right.$$

<u>remark</u>: functional measure $\prod_i \frac{\delta b_{0i}}{\| \lambda_i^{(0)} \|}$

recall $\boxed{\text{Metric on supermanifold}}$

local coordinates (x, ϑ)

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{even} & \text{odd} \end{array}$$

if Kähler supermetric is

$$\gamma = g_{i\bar{j}} \, dx^i dx^{\bar{j}} + \underset{\substack{\uparrow \\ \text{Kähler form}}}{\omega_{i\bar{j}}} \, d\vartheta^i \, d\vartheta^{\bar{j}}$$

volume form $\sqrt{S\det\gamma}$

Instanton calculus in $N=2$ SYM

recall

$$\mathcal{L}_{N=2} = \frac{\mathrm{Im}}{16\pi^2}\int d^4\vartheta\, \mathcal{F}(\Psi)$$

classical action
$$\mathcal{F}(\Psi) = \frac{\tau_{cl}}{4\pi i}\,\mathrm{Tr}\,\Psi^2$$

Ψ $N=2$ chiral superfield w. cps $(A_\mu, \lambda_\alpha^A, \bar\lambda^{\dot\alpha A}, \phi, \vec{D})$

$A =$ fund $SU(2)_R$

Coulomb phase $\langle\phi\rangle = a$
(take $SU(2)$ for simpl.)

low energy Wilsonian effective action abelian

$$\mathcal{A} = a + \vartheta\lambda + \vartheta^2 F^- + \dots$$

$$\mathcal{L}^{eff}_{N=2} = \tau(a)\, F^-\wedge F^- + \bar\tau\, F^+\wedge F^+ + \mathrm{Im}\,\tau\, da\wedge * d\bar a$$

$$\tau = \frac{\partial^2\mathcal{F}}{\partial a^2}$$

for $a \gg \Lambda$, weak coupling exp. of $\mathcal{F}$

$$\mathcal{F} = \frac{i}{2\pi}\, a^2 \ln\frac{a^2}{\Lambda^2} + a^2\sum_{1k}^{\infty} \mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{2Nk}$$

pert. one loop $\qquad\qquad$ instanton contrib.

Quantum moduli space parametrized by g.i. quantity

$$u(a) = \langle \mathrm{Tr}\, \phi^2 \rangle$$

for $a \gg \Lambda$ semiclassical exp.

$$u(a) = -\frac{1}{2} a^2 + a^2 \sum_K u_K \left(\frac{\Lambda}{a}\right)^{4K}$$

classical + instanton corrections

REMARK: corrections from any K! see what follows for explan.

properties:

1) dim analysis + RGE $\Rightarrow$ $\left(\frac{\Lambda}{a}\right)^{2NK}$ for $SU(N)$

in fact recall $\mu^{n_B - \frac{1}{2} n_F}$ in this case is

$$n_B = n_F = 4K \cdot N \quad \text{and} \quad \beta = 2N$$

$$\Rightarrow \quad \mu^{2KN} e^{-\frac{8\pi^2}{g^2}} = \left(\mu e^{-\frac{8\pi^2}{g^2 2N}}\right)^{2KN} \quad \square$$

2) $U(1)_R$ anomaly:

for $N=2$ $\qquad \partial_\mu J^R_\mu = -\frac{i}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \cdot 4N$

i.e. $\qquad U(1)_R \longrightarrow \mathbb{Z}_{4N}$

moreover $\quad u \to e^{2\pi i \frac{m}{N}} u \quad$ under $\mathbb{Z}_{4N}$

since $\phi \xrightarrow[U(1)_R]{} e^{2i\alpha}\phi \quad \Rightarrow \quad m = 0, .., 4N-1$

so $\qquad u \neq 0 \Rightarrow \mathbb{Z}_{4N} \to \mathbb{Z}_4$

and $\mathbb{Z}_N$ connects physically equivalent vacua

e.g. for $N=2$ $\quad u$ & $-u$ are completely equivalent

(same mass spectrum, same singularities)

3) non-renormalization

both $\mathcal{F}$ & u do not get contrib. in pert theory ($\mathcal{F}$ only at 1 loop) and do not receive any pert. corrections around instantons!

For $N=2$ semiclassical limit around instantons is EXACT! [Seiberg PLB 206 ('88) 75]

The coefficients u_k can be calculated

from semiclassical expansion around the instanton bckgd.

We have almost all the ingredients, except the

expression of the observable $u = \mathrm{Tr}\,\phi^2$ in terms

of bosonic & fermionic zero mds. This can be readily

obtained recalling that the $N=2$ SYM Lagrangian

contains the Yukawa interaction term + kin =)

$$+\frac{1}{g^2}\mathrm{Tr}\left(\bar{\Phi}\,[\lambda_{\dot{A}},\lambda_{\dot{B}}]\,\epsilon^{\dot{A}\dot{B}}\right)$$

e.o.m.

$$D^2\phi = [\lambda_{\dot{A}},\lambda_{\dot{B}}]\,\epsilon^{\dot{A}\dot{B}}$$

but on inst. bckgd we know that λ has zero modes =)

$$\phi = (D^2)^{-1}\,[\lambda^{(0)}_{\dot{A}},\lambda^{(0)}_{\dot{B}}]\,\epsilon^{\dot{A}\dot{B}}$$

where

$$\lambda^{(0)}_{\dot{A}} = \frac{1}{2}\bar{F}_{\mu\nu}\,\sigma_{\mu\nu}\,\zeta_{\beta\dot{A}}$$

$$\zeta_{\beta\dot{A}} \equiv \xi + \frac{(x-x_0)_\mu}{\rho}\,\sigma_\mu\,\bar{\eta}$$

$(\xi,\bar{\eta})$ quaternions of Grassmann numbers

Moreover, in presence of vev $\langle\phi\rangle = a$ (16)

the conformal symmetry of the action is broken $\Rightarrow$

the associated moduli APPEAR in the inst. action

$$S_{inst}^{K=1} = \frac{8\pi^2}{g^2} + \underset{\substack{\uparrow \\ \text{'t Hooft}}}{4\pi^2 |a|^2 \rho^2} + \underset{\substack{\uparrow \\ \text{Yukawa}}}{\mathrm{Tr}\left(\bar{a} \left[\lambda^{(0)}_{\dot{A}}, \lambda^{(0)}_{\dot{B}} \right] \epsilon^{\dot{A}\dot{B}} \right)}$$

this is a very important fact since explains why $\mathcal{U}$ receives contributions for any K! in fact instanton zero modes can be absorbed by expanding the term $e^{\mathrm{Tr}\,\bar{a}\,\lambda^{(0)}\lambda^{(0)}}$ in the instanton action.

Finally we can give the dependence on a:

e.g. SU(2)

$$\left.\begin{aligned} \int d\rho &\rightarrow \frac{1}{|a|^2} \\ \int d^4\xi &\rightarrow \bar{a}^2 \end{aligned}\right\} \frac{1}{a^2} \quad \text{holomorphic dependence}$$

in agreement with dim + RGE analysis $a^2 \left(\frac{\Lambda}{a}\right)^4$

The precise coeff can also be computed easily and is in agreement w. SW results.

Non renorm. & topological twist

LECTURE 4

(1)

In the previous lectures we have seen to noticeable properties of instanton-bounded correlators:

(1) space-time independence

(2) non-renormalisation - semiclassical approx. exact

These point to a deep geometrical structure underlying, can we uncover it? The answer is given by a change of variables, "topological twist".

The reason for that is that in the twisted variables we can push the instanton calculation to ANY instanton charge k! using local formulae

Topological Twist

~~Recall field content of $\mathcal{N}=2$ SYM~~

$$\cancel{(A_\mu, \lambda_\alpha^{\dot{A}}, \bar{\lambda}^{\dot{\alpha}\dot{A}}, \phi)}$$

The $SU(2)_A$ R-symmetry group of $\mathcal{N}=2$ SUSY can be used to "twist" the SUSY algebra, namely to redefine the Lorentz group $SU(2)_L \times SU(2)_R$ of Euclidean $\mathcal{N}=2$ in

$$SU(2)_L \times SU(2)_R' \qquad SU(2)_R' = \operatorname{diag}\left(SU(2)_R \times SU(2)_A\right)$$

This changes the repr's of the SUSY algebra

$$\left(Q_{\alpha \dot{A}} \, , \, \bar{Q}^{\dot{\alpha} \dot{A}} \right) \to \left(Q_\mu \, , \, Q \, , \, Q_{\mu\nu} \right)$$

explicitely:

$$Q_{\alpha \dot{A}} = \sigma^\mu_{\alpha \dot{A}} \, Q_\mu$$

$$\bar{Q}^{\dot{\alpha} \dot{A}} = \epsilon^{\dot{\alpha} \dot{A}} \, Q + \bar{\sigma}^{\mu\nu \, \dot{\alpha} \dot{A}} \, Q_{\mu\nu}$$

namely the 8 SUSY charges are reorganized in a scalar, antiselfdual tensor and a vector.

The presence of Q implies that the twisted theory can be formulated on ANY manifold M ~~Γ~~ (w. that breaks Q!). If M has further structure then beyond Q also some other supercharges can be conserved.

For example if M is Kähler (i.e. it admits a globally defined ~~covariantly~~ non degenerate (1,1) form $\omega_{\mu\nu}$, otherwise stated holonomy $U(2)$) then it admits a further scalar $Q' = \omega_{\mu\nu} \, Q^{\mu\nu}$

If M has isometries generated by vector fields $V_a = V_a^\mu \partial_\mu$ then $Q_a = V_a^\mu \, Q_\mu$ and so on. "read Lorentz"?

In particular on hyper-Kähler manifolds since the holonomy group is just $SU(2)$ the twist do not alter ~~by~~ any means the theory.

The true advantage of the twisted formulation is that

we have a SCALAR SUSY charge Q

Moreover it can be shown that

$$S^{\text{twisted } N=2} = \frac{1}{e}\int d^4x \; \mathrm{Tr}\,(F \wedge F) + \{Q, \Psi\} \qquad (*)$$

namely the $N=2$ SUSY action is Q-exact, up to a topological term!

It is easy to find Ψ. First of all we see that as for SUSY charges, also the fermions of $N=2$ multiplet change repr. under twist:

$$\lambda_{\alpha\dot{A}} = \sigma^{\mu}_{\alpha\dot{A}}\,\psi_{\mu}$$

$$\bar{\lambda}^{\dot{\alpha}\dot{A}} = \eta\,\epsilon^{\dot{\alpha}\dot{A}} + \bar{\sigma}^{\mu\nu\dot{\alpha}\dot{A}}\,\chi_{\mu\nu}$$

while the bosons (gauge vector A_μ and scalar ϕ) stay unchanged

The twisted $N=2$ Lagrangian then reads

$$\mathcal{L}^{\text{twisted } N=2} = -\frac{1}{2g^2}\mathrm{Tr}\Big(F_{\mu\nu}F_{\mu\nu} + \eta D_\mu \psi_\mu - \chi_{\mu\nu} D_{[\mu}\psi_{\nu]}$$

$$+ \bar{\Phi}[\psi_\mu,\psi_\mu] - \frac{1}{2}\Phi[\eta,\eta] + \frac{1}{2}\Phi[\chi_{\mu\nu},\chi_{\mu\nu}]$$

$$+\frac{1}{2}[\bar{\Phi},\Phi]^2\Big)$$

[cont.]

the Q-transf. read:

$$\{Q, A_\mu\} = \psi_\mu \qquad \{Q, \psi_\mu\} = D_\mu \phi$$

$$\{Q, \chi_{\mu\nu}\} = B_{\mu\nu} \qquad \{Q, B_{\mu\nu}\} = [\chi_{\mu\nu}, \phi]$$

$$\{Q, \bar\phi\} = \eta \qquad \{Q, \eta\} = [\bar\phi, \phi]$$

$$\{Q, \phi\} = 0$$

notice that $Q^2 = \delta_\phi$ gauge transf. w. parameter ϕ

It's easy now to write

$$\Psi = -\frac{1}{2}\mathrm{Tr}\left\{\chi_{\mu\nu}\left(F_{\mu\nu} - \frac{1}{2}B_{\mu\nu}\right) + \bar\phi D_\mu \psi_\mu + \frac{1}{2}\eta[\bar\phi, \phi]\right\}$$

in fact

$$\{Q, \Psi\} = -\frac{1}{2}\mathrm{Tr}\left\{B_{\mu\nu}\left(F_{\mu\nu} - \frac{1}{2}B_{\mu\nu}\right) + \frac{1}{2}\chi_{\mu\nu}[\chi_{\mu\nu}, \phi] - \chi_{\mu\nu} D_{[\mu}\psi_{\nu]} + \eta D_\mu \psi_\mu \right.$$
$$+ \bar\phi D_\mu D_\mu \phi + \bar\phi[\psi_\mu, \psi_\mu] + \frac{1}{2}[\bar\phi, \phi]^2$$
$$\left. - \frac{1}{2}\eta[\eta, \phi]\right\}$$

□

Notice moreover that the topological term is invariant

$$-\frac{1}{2}\int Q\,\mathrm{Tr}(F\wedge F) = -\int \mathrm{Tr}(F\wedge D\psi) = \int \mathrm{Tr}(\underbrace{DF}_{0\ \text{B.I.}}\wedge\psi)$$

Eq. (8) has two deep consequences:

1 - correlators of Q-invariant observables are INDEPENDENT on the metric on M

2 - semiclassical approximation is exact

The proof of both relies on the assumption that the vacuum is annihilated by Q (no SUSY breaking) and uses W.I. of Q

For 1. we observe first of all that the energy momentum tensor is Q-exact

$$T_{\mu\nu} = \{Q, V_{\mu\nu}\} \qquad (**)$$

in fact

$$\delta_g S = \frac{1}{2}\int d^4x \sqrt{g}\, T_{\mu\nu}\, \delta g^{\mu\nu} = \delta_g \int \{Q, \Psi\} = \int \{Q, \delta_g \Psi\}$$

Q does not depend on g

theories satisfying (**) are called topological theories; their correlators do not depend on metric g but only on global properties of the manifold

$$\delta_g \langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle = \int D[A\ldots]\left(\frac{1}{2}\int d^4x\sqrt{g}\,\{Q, V_{\mu\nu}\}\,\delta g^{\mu\nu}\right)\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\, e^{-S}$$

$$= \langle 0|\{Q, V_{\cdot\cdot}\}\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle = 0$$

assuming $\delta_g \mathcal{O} = \{Q, G\}$ □

Moreover, twisted $N=2$ is a special kind of top. theory because it satisfies the more stringent property (★) from which follows the <u>exactness of semiclassical limit</u>

$$\frac{\delta}{\delta e}\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle = \int D[A\ldots]\int \frac{1}{e^2}\{Q,\Psi\}\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\,e^{-\frac{1}{2}S}$$

$$= \langle \{Q,\Psi\}\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle = 0$$

<u>saddle point</u> eq. ? From the explicit form of Ψ we see that this is a gauge fermion implementing the g.f. cds

$$\begin{cases} F^-_{\mu\nu} = 0 \\ D_\mu \psi_\mu = 0 \end{cases}$$

for the BRST symmetry Q. $\Rightarrow$ instantons ! ($\longrightarrow$ see other page fixed point!)

To summarize, we have found that

- correlators around instanton bckg in (twisted) $N=2$ SYM are non-renormalized (one loop exact)
- they provide topological invariants.

More precisely, these are constructed from Q-invariant observables and from (Δ) we see that the only local ones are g.i. polynomial in ϕ. <u>Donaldson polynomials</u>.

6 bis

Only fixed points of Q contribute to the path integral : Q as a trasl. in superspace w. coord. ϑ

if Q field $\neq 0$ we can write (fermionic analog of Faddeev-Popov)

$$\int_{\text{FIELDS}} = \int d\vartheta \int_{\text{FIELDS}/Q\text{ action}}$$

$\int d\vartheta = 0$! volume of fermionic part supergroup is zero!

notice that in our case

$$Q\,\chi_{\mu\nu} = B_{\mu\nu} \qquad \text{and on-shell} \qquad B_{\mu\nu} = F^-_{\mu\nu}$$

namely

$$Q\,\chi_{\mu\nu} = F^-_{\mu\nu} = 0$$

So the ∞ dim functional integral is localised on a finite dimensional (super)manifold, the instanton (super)moduli space.

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_n) \rangle = \int_{\mathcal{M}_{k,N}} dm_B \, dm_F \, e^{-S(m_B, m_F)} \prod_i \mathcal{O}_i(m) \frac{\det'(S_F)}{\det'(S_B)}$$

How to perform this integral? Localisation formulae!

Using symmetries $G \circlearrowright \mathcal{M}$

$$\int_{\mathcal{M}} = \sum_{\text{fixed points of } G}$$

when the symmetries of $\mathcal{M}$ are enough to get a discrete set of isolated fixed points we reduce just to count these points with suitable weight.

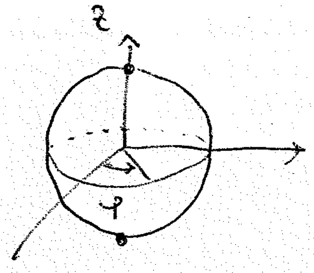
This is luckily the case for $\mathcal{M}_{k,N}$!

Before entering into that, I recall localisation formulae.

LOCALIZATION FORMULAE

Let's start with a simple example

S^2 sphere



$$Z(\mu) = \int_{S^2} e^{\mu z} \, d\varphi \, dz$$

$\mu \in \mathbb{R}$ parameter

$$= 2\pi \, \frac{e^{\mu} - e^{-\mu}}{\mu}$$

this integral is computed in terms of the EXTREMA of the funct. z. These are reached at the fixed points of the rotation around z axis.

The above integral is a simple example of exact saddle point!

A general formula has been found by Duistermaat-Heckman:

Let (M, ω) be a compact symplectic manifold $\dim M = 2\ell$, on which one can define a hamiltonian circle action $dH = i_\xi \omega$

where ξ is the vector field generating the circle action.

If the critical points $\{p\}$ of H are isolated and non degenerated (that is H is a good Morse funct) one has

$$\int_M e^{uH} \frac{\omega^\ell}{\ell!} = (-2\pi)^\ell \sum_{\{p\}} \frac{e^{uH(p)}}{u^\ell (\mathrm{Hess}_p H)^{1/2}}$$

the determination of the square root is chosen from the canonical orientation of $T_p M$.

More precisely

$$(\mathrm{Hess}_p H)^{1/2} = (-)^{\frac{\lambda(p)}{2}} \prod_i w_i(p)$$

where

$\lambda(p)$ is the Morse index (# negative eigenvalues)

$w_i(p)$ are the weights of the circle action

example:

come back to the sphere $\quad M = S^2 \quad \omega = d\cos\vartheta\, d\varphi$

$$z = (1-\cos\vartheta) \qquad \xi = \frac{\partial}{\partial\varphi}$$

$$i_\xi \omega = -d\cos\vartheta = dz \quad \Rightarrow \quad H = z$$

$$\left.\begin{array}{ll} \lambda(p) = 0 & \text{minimum} \\ 2 & \text{maximum} \\ w_i(p) = 1 & \end{array}\right\} \quad (-2\pi)\left[-\frac{e^u}{u} + \frac{e^{-u}}{u}\right]$$

□

D-H can be applied also on non-compact manifolds

(provided the functs are integrable)

e.g.

$$Z(u) = \int_{\mathbb{R}^2} e^{-\frac{u}{2}(x^2+y^2)}\, dx\, dy = \frac{2\pi}{u}$$

$$H = \frac{1}{2}(x^2+y^2) \qquad \xi = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$$

$$i_\xi\, dx\, dy = x\,dx + y\,dy$$

$$Z(u) = (-2\pi)\left[-\frac{1}{u} + \frac{0}{u}\right] = \frac{2\pi}{u}$$

□

REMARK: it is important that the vector field generates a COMPACT action, otherwise the thm. doesn't apply!

Generalities on $\mathcal{M}_{k,N}$

We have already seen that the metric on $\mathcal{M}_{kN}$ is

$$g_{ij} = \frac{1}{g^2_{YM}} \int d^4x \; \delta_i A_\mu \, \delta_j A_\nu$$

where $\delta_i A_\mu$ are zero modes in the instanton backgd.

By using this formula we immediately see that $\mathcal{M}_{kN}$ inherits the isometries from the isometries of $\mathbb{R}^4$ & of $SU(N)$.

In particular both $SO(4)$ rot. group of $\mathbb{R}^4$ & $SU(N)$ gauge action provide isometries of $\mathcal{M}_{kN}$

One can show that the action of the maximal torus

$$U(1)^2 \times U(1)^{N-1} \subset SO(4) \times SU(N)$$

on $\mathcal{M}_{kN}$ has a finite # of isolated fixed points.

$\Rightarrow$ we can apply torus action to reduce $\int_{\mathcal{M}_{kN}} \to \sum_{f.p.}$!

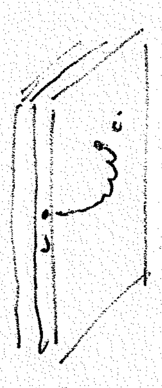
In order to classify the fixed points we need to know something more about the moduli space $\mathcal{M}_{kN}$

ADHM construction

ADHM showed that the moduli space of instantons $\mathcal{M}_{k,N}$ can be described in terms of a set of matrices obeying suitable constraints. From mathematical viewpoint the ADHM constr. is naturally formulated in the twistor space.

From physics viewpoint the simplest way to recover ADHM construction is by branes in type II string theory.

Roughly speaking these are defects of dim $(p+1)$ in space time where open strings can end. The low energy dynamics of open strings ending on a stack of N Dp branes is the dimensional reduction of $\mathcal{N}=1$ SYM from 10 to $(p+1)$ dimensions (*)



N Dp

It is clear from the geometry that open strings ending on N Dp branes have Chan Paton indexes transforming in the symmetric $N \times \bar{N}$ i.e. adjoint repr. of $U(N)$

(*) For example for $p=3$ we have the familiar $\mathcal{N}=4$ SYM in 4 dimensions.

The low theory of the brane contains also coupling to the various RR fields in the bulk. This includes the term

$$\int_{Dp} d^{p+1}x \ \mathrm{Tr}\left(C_{p-3} \wedge F \wedge F \right)$$

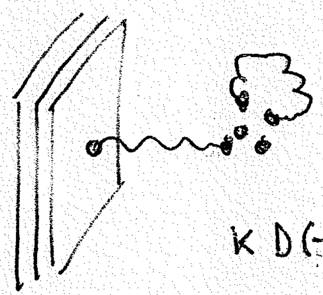
where F is the curvature of $U(N)$ gauge field.

Recall that C_{p-3} RR fields couple to $D(p-4)$ branes $\Rightarrow$

$D(p-4)$ in Dp branes $\Leftrightarrow$ instantons in SYM theory

in fact a non trivial term $F \wedge F$ induces a source term $\frac{8\pi^2}{e^2}\int d^{p-3}x\, C_{p-3}$ for RR, which is precisely the source induced by a $D(p-4)$ brane. One can check indeed that inst has same action & charge as a $D(p-4)$ brane.

The moduli space $\mathcal{M}_{kN}$ can now be constructed by considering a system $D(p-4)/Dp$ $(p=3)$



K $D(-1)$

N $D3$

Consider the wv theory for $D(-1)$ branes: dim reduction to 0 dimension of $N=1$ SYM $(A_\mu, \lambda) \to (X_\mu, \hat{X}_m, \text{ferm.})$

$\mu = 1,..,4$ directions along D3 $\qquad B_1 = X_1 + iX_2$

$m = 5,..,10$ $\perp$ D3 $\qquad B_2 = X_3 + iX_4$

Moreover, we have strings stretched between $D(-1)$ $D3$

$\Rightarrow$ in $(K, \bar{N})$ & $(\bar{K}, N)$ repr. I, $J^\dagger$ + fermions then the bosonic part of the action reads

$$S^{D(-1)} = \mathrm{Tr}_K \Big(\frac{1}{g^2} [\hat{X}_m , \hat{X}_n]^2 + [\hat{X}_m , X_\mu]^2$$

$$+ I \hat{X}_m \hat{X}^m I^+ + J^+ \hat{X}_m \hat{X}^m J \Big) +$$

$$\left.\begin{array}{l} + g^2 \, \mathrm{Tr}_K \left(I I^+ - J^+ J + [B_i , B_i^+] \right) \\ + g^2 \, \mathrm{Tr}_K \left(I J + [B_1 , B_2] \right) \end{array}\right\} \text{"D terms"}$$

We consider D(-1) on top of D3 $\Rightarrow$ $\hat{X}_m = 0$

then the non trivial eq. to be solved are the "D terms"

$$\mathcal{M}_{k,N} = \left\{ \begin{array}{c} (B_1, B_2, I, J^+) \\ \text{satisfying} \\ \text{D-terms} \end{array} \right\} \Big/ \left\{ \begin{array}{l} g B_i g^{-1} \\ g I \\ J g^{-1} \end{array} \quad g \in U(k) \right\}$$

stability cd $\nexists g$ s.t. $(g B_i g^{-1} = B_i , \; g I = I , \; J g^{-1} = J \}$

ADHM construction!

example:

$K = 1$ $N = 2$ BPST instanton

B_1 , B_2 c-numbers $I = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$ $J = (j_1 , j_2) \in \mathbb{C}$

$$\left.\begin{array}{lll} I J = 0 & \Rightarrow & i_1 j_1 + i_2 j_2 = 0 \\ I I^+ - J^+ J = 0 & \Rightarrow & |i_1|^2 + |i_2|^2 - |j_1|^2 - |j_2|^2 = 0 \end{array}\right/ U(1)$$

easier to consider $\mathbb{C}^*$

$$(i_1, i_2, j_1, j_2) \longrightarrow (w i_1, w i_2, w^{-1} j_1, w^{-1} j_2)$$

$w \in \mathbb{C}^*$

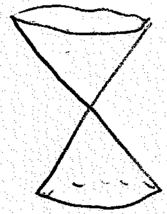
and only cplx constant $i_1 j_1 = -i_2 j_2$

invariant comb.

$$i_1 j_1 = X$$

$$i_1 j_2 = Y \qquad\qquad YZ = -X^2 \quad \text{cone } \mathbb{R}^4/\mathbb{Z}_2$$

$$i_2 j_1 = Z$$



$\Rightarrow \quad (B_1, B_2) \to \mathbb{R}^4 \qquad (I, J^+) \to \mathbb{R}^4/\mathbb{Z}_2$

$$\mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2$$

two non compactness

① x_0 run to infty

② shrink to zero size

regularise ② by setting

$$[B_i, B_i^+] + II^+ - J^+J = \zeta_{\mathbb{R}}$$

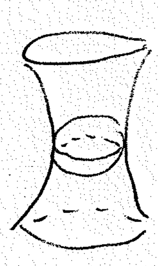
$$[B_1, B_2] + IJ = \zeta_{\mathbb{C}}$$

eg:

$$X^2 + YZ = \zeta_{\mathbb{C}}^2$$

setting $Y = v_1 + i v_2$ $\qquad X^2 + V_1^2 + V_2^2 = \zeta_{\mathbb{C}}^2$ sphere

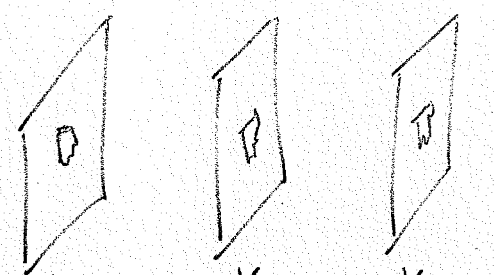
$Z = v_1 - i v_2$



① regularized by torus action $U(1)^2$.

FIXED POINTS

separate D3 branes (Higgs mech. vev to scalars)

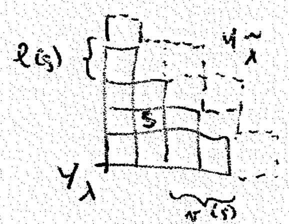


$k_1 \quad k_2 \ \ldots \ k_N \qquad \sum_i k_i = K$

- term $(\pi)^2 a^2 \rho^2$ → inst. shrink to minimal size
inst action

- $U(1)^2 \subset SO(4)$ rotation invariance forces inst to cluster around the origin

fixed points classified in terms of Young tableaux $Y_\lambda \ Y_{\tilde\lambda}$



k_λ boxes

$$Z_k = \sum_{\{Y_\lambda\}} \prod_{\lambda,\tilde\lambda=1}^{N} \prod_{s\in Y_\lambda} \frac{1}{E(s)(E(s)-\epsilon)}$$

$$E(s) = a_{\lambda\tilde\lambda} - \epsilon_1 h(s) + \epsilon_2 (v(s)+1)$$

$$Z^{inst} = \sum_k Z_k(a,\epsilon)\, \Lambda^{2NK} = \exp\left[-\mathcal{F}(a,\epsilon,\Lambda)\right]$$

$$\mathcal{F}(a, \epsilon_1 = -\epsilon_2 = t, \Lambda) = \sum_{g=0}^{\infty} t^{2g-2} \mathcal{F}_g(a,\Lambda)$$ → string expansion!

$g=0$ term

$$\mathcal{F}_0(a;\Lambda) = \sum_k \Lambda^{2kN} f_k(a) \qquad \text{SW prepotential}$$

from quasiclassical limit $k \sim \frac{1}{\epsilon_1\epsilon_2} \to \infty$ one can show that $\left(a_i, \frac{1}{2\pi i}\frac{\partial F_0}{\partial a_i}\right)$ coincides w. the set of A & B periods of the meromorphic differential

$$dS = \frac{1}{2\pi i} z \frac{dw}{w}$$

on the curve

$$\Lambda^N\left(w + \frac{1}{w}\right) = P_N(x) \qquad \text{SW curve !}$$

higher order terms in ϵ are related to gravitational corrections(*) to SW theory and can be also computed in terms of a dual string theory, topological string on A_{N+1} singularities.

(*) $$\int d^4x\, d^4\theta\, \mathcal{F}_g(a)\, \mathcal{W}^{2g} = \int d^4x\, \mathcal{F}_g(a)\, \underset{\substack{\uparrow \\ \text{5-d Riemann tensor}}}{R_+^2}\, \underset{\substack{\uparrow \\ \text{graviphoton field strength}}}{T_+^{2g-2}}$$

S-duality & modular forms

For $\mathcal{N}=4$ theory expect exact S-duality symm.

soft breaking to $\mathcal{N}=1^*$ $\Rightarrow$ S-duality exchanges vacua

e.g.

$$\langle \mathrm{Tr}\,\phi^2 \rangle^{\text{"}U(1)\text{"}}_{\mathcal{N}=1^*} = -\frac{1}{24}\left(m^2 - h^2\right) E_2(q)$$

$\uparrow$

2^{nd} Eisenstein series