Summary.

A Superstring spectrum. (closed / open).

- Introduction to GS.
- Quantization on light-cone gauge.
- Super Yang-Mills multiplet in 10d. / Anomalies.
- First massive state of type I open.
- Closed string spectrum.
- Anomalies in type IIB / type I ⊕ open.
- GS mechanism.
- Descent equations.
- Explicit form of anomalies.

B Superstring spectrum (heterotic).

- Introduction to Het.
- $SO(32)/\mathbb{Z}_2$, $E_8 \times E_8$
- Anomalies.

DIGRESSION

Compactification on CY. Why?

C Complex Manifolds

- Introduction
- Almost complex structure.
- Hermitian Inner product.
- Holomorphic vector Bundles.
- Connections (Levi-Civita, Riemann).
- Kähler manifolds

D Characteristic Classes

- Complexes (Elliptic diff. operators).
- Hodge theory / decomposition.
- Ind(D, E).
- Chern class, (total).
- Chern character.
- [Atiyah-Singer theorem.

- Hodge manifold.
- $\mathbb{CP}^n$ / Computation of $C_{tot}(\mathbb{CP}^n)$.
- Computation of $C_{tot}(M_n)$ when $M_n \subseteq \mathbb{CP}^N$.

E <u>Calabi – Yau</u>

- Definition
- properties.
- $SU(n)$ holonomy and Ricci-flatness.
- Harmonic forms – Spinors.
- <u>Hodge theory</u> – decomposition – Hodge diamond.
- Kähler identities.
- Susy and Kähler. mfld.
- CY as complete intersections examples.

F <u>Compactification on CY</u>.

Superstring spectra

We review the sup. St. spectra before discussing compactifications

Instead of using RNS we adopt a light cone GS. model.

→ Coordinates on W.S.

$$(X^\mu, \theta^\alpha, \theta^{\hat\alpha}) \qquad \begin{array}{l}\mu = 0 \dots 9 \\ \alpha = 1 \dots 16 \\ \hat\alpha = 1 \dots 16\end{array}$$

$\theta^\alpha, \theta^{\hat\alpha}$ Majorana-Weyl spinors

→ Notation: $\gamma^\mu_{\alpha\beta}$: $(\Gamma^\mu)_A{}^B$, $A, B = 1 \dots 32$, $\mu = 0 \dots 9$

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\,\eta^{\mu\nu}, \quad C \text{ - conjugation charge.}$$

$$\begin{cases} (C\Gamma^\mu)_{AB} = (C\Gamma^\mu)_{BA}. \\ (C\Gamma^\mu)_{AB} = \begin{pmatrix} 0 & \gamma^\mu_{\alpha\beta} \\ \gamma^{\mu\,\alpha\beta} & 0 \end{pmatrix} \end{cases} \qquad \gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha}$$

and
$$\begin{cases} \gamma^\mu_{\alpha\beta}\gamma_{\mu\,\gamma\delta} + \gamma^\mu_{\alpha\gamma}\gamma_{\mu\,\delta\beta} + \gamma^\mu_{\alpha\delta}\gamma_{\mu\,\beta\gamma} = 0. \\ \gamma^\mu_{\alpha\beta}\gamma^{\nu\,\beta\delta} + \gamma^\nu_{\alpha\beta}\gamma^{\mu\,\beta\delta} = 2\,\eta^{\mu\nu} \end{cases}$$

Introduce:

$$\Pi^\mu = dX^\mu - \frac{i}{2}\left(\theta^\alpha\gamma^\mu_{\alpha\beta}\,d\theta^\beta + \theta^{\hat\alpha}\gamma^\mu_{\hat\alpha\hat\beta}\,d\theta^{\hat\beta}\right).$$

Notation: θ^α : Weyl (Majorana)

$\theta^{\hat\alpha}$: Weyl (if $\hat\alpha = \beta$) – Anti Weyl if $\theta^{\hat\beta} = \theta_\beta$

If $(\theta_1^\alpha, \theta_2^\beta) \Rightarrow$ IIB $\quad (\theta_1^\alpha, \theta_{2\beta}) \Rightarrow$ IIA., $\quad \theta_1^\alpha = \theta_2^\beta$ (on the b.c.) ⇒ Open, Type I

Green–Schwarz action (GSW chap 5)

$$S = S_1 + S_2'$$

$$S_1 = -\frac{1}{2\pi}\int d^2x \sqrt{-g}\, g^{ij}\, \Pi_i^{\mu}\, \Pi_j^{\nu}\, \eta_{\mu\nu}$$

$g =:$ worldsheet metric.
$\eta_{\mu\nu} =:$ flat 10d. Minkowski space
$x_0, x_1 =:$ worldsheet coords. (x_i)

(1) $\Rightarrow$ Invariance under local worldsheet diffeos.
(they are needed to avoid unphysical degrees of freedom of the type: $[P_\mu, X_\nu] = 2i\eta_{\mu\nu}$ if $\mu,\nu = 0,0$).

(2) It is invariant under susy.

$$\delta\theta^{\alpha} = \epsilon^{\alpha}, \quad \delta\hat{\theta}^{\hat\alpha} = \epsilon^{\hat\alpha} \qquad \epsilon^{\alpha}, \epsilon^{\hat\alpha} \text{ (susy parameters)}$$

$$\delta X^{\mu} = \frac{i}{2}\left(\epsilon\gamma^{\mu}\theta + \hat{\epsilon}\gamma^{\mu}\hat{\theta}\right)$$

and

$$\begin{cases} \delta\Pi^{\mu} = d\left(\frac{i}{2}\epsilon\gamma^{\mu}\theta + \frac{i}{2}\hat{\epsilon}\gamma^{\mu}\hat{\theta}\right) + \\ \qquad - \frac{i}{2}\left(\epsilon\gamma^{\mu}d\theta + \hat{\epsilon}\gamma^{\mu}d\hat{\theta}\right) = 0. \\ \delta\, d\theta^{\alpha} = \delta\, d\hat{\theta}^{\hat\alpha} = 0. \end{cases}$$

(3) Too many degrees of freedom.

$$\begin{cases} 8 \text{ bosons } (10\ X\text{'s} - \text{diff. invariance}) \\ 16+16 \text{ fermions} \end{cases}$$

(the spectrum cannot be susy invariant)

the term: (W-Z term)

$$S_2 = \frac{1}{\pi}\int_{\mathcal{M}} \Pi^\mu \left(d\theta^\alpha \gamma_{\mu\alpha\beta} d\theta^\beta - d\theta^{\hat\alpha}\gamma_{\mu\hat\alpha\hat\beta} d\theta^{\hat\beta}\right) = \frac{1}{\pi}\int_{\mathcal{M}} \omega_{(3)}$$

$\partial\mathcal{M} = \Sigma_1$ (Riemann surface)

$\omega_{(3)}$ is a 3-form which is closed.

$$d\omega_{(3)} = 0 \underset{\text{(Poincaré lemma)}}{\Longrightarrow} \omega_{(3)} = d\mathcal{L}_{(2)}.$$

(but $\mathcal{L}_{(2)}$ cannot be written in terms of susy invariant function Π^μ, $d\theta^\alpha$, $d\hat\theta^{\hat\alpha}$)

(1) It is a topological term (it does not depend upon the w.s. metric g_{ij}).

(2) It is susy invariant

(3) Together: $S = S_1 + S_2$
it acquires a new local symmetry: kappa-symmetry

$$\delta_k x^\mu = i\left(\theta^\alpha \gamma^\mu_{\alpha\beta}\delta\theta^\beta + \theta^{\hat\alpha}\gamma^\mu_{\hat\alpha\hat\beta}\delta\theta^{\hat\beta}\right).$$

$$\delta_k \theta^\alpha = 2i\, \Pi_{\mu i}(\gamma^\mu)^{\alpha\beta} k_\beta{}^i \qquad \left(\begin{matrix} k^i_\beta \\ k^i_{\hat\beta}\end{matrix} \text{ are w.s. vectors}\right)$$

$$\delta_k \theta^{\hat\alpha} = 2i\, \Pi_{\mu i}(\gamma^\mu)^{\hat\alpha\hat\beta} k_{\hat\beta}{}^i$$

in the property:

$$\begin{cases} k^i{}_\beta = \frac{1}{2}\left(g^{ij} + \frac{\epsilon^{ij}}{\sqrt{-g}}\right) k^j{}_\beta & \text{(self-dual vector)} \\ k^i{}_{\hat\beta} = \frac{1}{2}\left(g^{ij} - \frac{\epsilon^{ij}}{\sqrt{-g}}\right) k^j{}_{\hat\beta} & \text{(anti-self dual).} \end{cases}$$

$$\delta_\kappa\left(\sqrt{-g}\,g^{ij}\right) = -16\sqrt{-g}\,\Big(P_{(+)}^{ik}\,\kappa_\alpha^j\,\partial_k\theta^\alpha + P_{(-)}^{ik}\,\kappa_{\hat\alpha}^j\,\partial_k\theta^{\hat\alpha}\Big)$$

① the action is invariant under diffeos ⊕ kappa symmetry. (local).

⇒ <u>Bosons</u>: X^μ ⊕ diffeos → 8 dof. ($8_L + 8_R$).

<u>Fermions</u>: θ^α ⊕ kappa sym (+ eq. of motion) → 8 dof (left-mover)

$\theta^{\hat\alpha}$ ⊕ half sym (+ eq. of motion) → 8 dof (right-movers)

$$\begin{cases} \gamma^\mu_{\alpha\beta}\,\Pi_i^\mu\,P_{(+)}^{ij}\,\partial_j\theta^\beta = 0 \\ \gamma^\mu_{\hat\alpha\hat\beta}\,\Pi_j^\mu\,P_{(-)}^{ij}\,\partial_j\theta^{\hat\beta} = 0. \end{cases}$$

$8_B + 8_F$ left movers
$8_B + 8_F$ right movers.

susy spectrum.

② the theory is interacting on the w.s. (trilinear and quadratic couplings).

⇒ No covariant quantization is available.
(see PURE SPINOR STRING THEORY).

③ $\boxed{\text{We use light-cone gauge.}}$

we set
$$\begin{cases} X^+(\tau,\bar\tau) = X_0^+ + P_0^+\,\tau_0 \\ (\gamma^+\theta)^\alpha = (\gamma^+\theta)^{\hat\alpha} = 0 \end{cases} \quad \left(\text{where } \gamma^\pm = \frac{\gamma^0 \pm \gamma^9}{\sqrt2}\right)$$

Using the light-cone gauge.

$\theta^\alpha = (\theta^a, \theta^{\dot a})$ $\qquad \hat\theta^\alpha = (\hat\theta^a, \hat\theta^{\dot a})$ $\qquad$ type II A/B

$X^{\mu} = (X^\pm, X^I)$

$\begin{cases} I = 1, \dots 8 & \to 8_v \\ a = 1 \dots 8 & \to 8_s \\ \dot a = 1 \dots 8 & \to 8_c \end{cases}$

(Representations of SO(8) group D_4).

$SO(8) \simeq Spin(8)$.

• the action can be rewritten as follow:

$$S_{l.c.} = -\frac{1}{2}\int d^2z \left(\partial X_I \bar\partial X_I - \frac{i}{\pi} \bar S^a_{ab} \rho^i \partial_i S^b \right)$$

where $\boxed{S^a = \sqrt{p^+}(\theta^a + i\hat\theta^a)}$. ← this transforms as a w.s. spinor.

and $\theta^{\dot a}, \hat\theta^{\dot a}, X^\pm$ are set to zero by the gauge fixing

$\rho^i_{ab} \to$ 2d Dirac matrices.

So, the resulting action becomes

$$\bar S^a \rho^i \partial_i S^b \delta_{ab} = S_L \bar\partial S_L + S_R \partial S_R.$$

• For open superstring we use b.c. such that

$\boxed{S = i\bar S}$ at $z = \bar z$ (or $\begin{matrix}\sigma = 0 \\ \sigma = \pi\end{matrix}$).

$\Rightarrow$ type I superstring.

• Quantisation: (open superstring)

ψ^a (S at the boundary).

$$\{\psi^a, \psi^b\} = \delta^{ab} \qquad [X^I, P^J] = i\,\delta^{IJ}$$

We define the ground state.

(in the present formulation the GSO projection is not needed since $\theta^a, \theta^{\dot a}, \cdots$ are already the GSO-projected variables.

- There are other model with different GSO projections (which are not susy invariant) which cannot be described by the same techniques).

$$|\psi\rangle = \{ \underset{\substack{\uparrow \\ \text{bosonic} \\ 8_v}}{|I\rangle}, \underset{\substack{\uparrow \\ \text{fermionic.} \\ 8_c}}{|\dot a\rangle} \}$$

These are obtained as follows:

$$\psi^a = \xi^A + i\,\xi^{\bar A} \qquad A = 1\cdots 4. \qquad SO(8) \to SU(4)$$

subgroup.

$$\{\xi^A, \xi^{\bar A}\} = \delta^{A\bar A} \qquad \{\xi^A, \xi^B\} = 0 \qquad \{\xi^{\bar A}, \xi^{\bar B}\} = 0.$$

$\to$ ξ^A : Annihilation operators

$\xi^{\bar A}$: creation operator.

$|0\rangle \to$ boson $\underline{1}$

$\xi^A|0\rangle \to$ fermion $\underline{4}$ $SU(4)$

$\xi^A\xi^B|0\rangle \to$ boson $\underline{6}$ of $SU(4)$ (antisym.)

$\xi^A\xi^B\xi^C|0\rangle$ fermion $\underline{4}^*$

$\xi^A\xi^B\xi^C\xi^D \epsilon_{ABCD}|0\rangle$

1 - boson

$$\left(|0\rangle,\ \xi^A \xi^B |0\rangle,\ \epsilon_{ABCD}\, \xi^A \xi^B \xi^C \xi^D |0\rangle \right) = |I\rangle$$

(8_V - bosons)

$$\left(\xi^A |0\rangle,\ \epsilon_{ABCD}\, \xi^B \xi^C \xi^D |0\rangle \right) = |\dot{a}\rangle$$

(8_C - fermions).

$$\begin{cases} \psi^a |I\rangle = \frac{1}{\sqrt{2}} \gamma^I_{a\dot{a}} |\dot{a}\rangle \\ \psi^a |\dot{a}\rangle = \frac{1}{\sqrt{2}} \gamma^I_{a\dot{a}} |I\rangle \end{cases}$$

$(|I\rangle, |\dot{a}\rangle)$ are a representation of the Clifford algebra $\{\psi^a, \psi^b\} = \delta^{ab}$.

$\Downarrow$

$\boxed{\text{Super-Yang-Mills multiplet in 10d.}}$

Wave functions:

$$\underbrace{\phi^I(x^I)\, |I\rangle}_{\substack{\text{Gauge bosons} \\ \text{(on light-cone)}}},\quad \underbrace{\lambda^{\dot{a}}(x^I)\, |\dot{a}\rangle}_{\substack{\text{Gluinos} \\ \text{(on light-cone)}}}.$$

This index represent the 8- transverse physical degrees of freedom of a 10-d gauge boson and the dependence upon $\phi(x^I)$ implies that it is on-shell.

So, we have ON-shell SYM in $d=10$.

Action for SYM. in 10d

$$S_{SYM} = \int d^{10}x \, \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\gamma^{\mu}D_{\mu}\lambda\right)$$ for $U(n)$ gauge group.

$$\Rightarrow \begin{cases} D^{\mu}F_{\mu\nu} = \bar{\lambda}\gamma_{\nu}\lambda \\ \not{D}\lambda = 0. \end{cases}$$

Susy: $\begin{cases} \delta A_{\mu} = \frac{i}{2}\bar{\epsilon}\gamma^{\mu}\lambda \\ \delta\lambda = -\frac{1}{4}\epsilon\gamma^{\mu\nu}F_{\mu\nu}. \end{cases}$ ϵ is a MW-parameter

To prove the susy invariance one has to use the Fierz identity $\gamma^{\mu}_{\alpha(\beta}\gamma_{\mu\,\gamma\delta)} = 0$. .

$$\begin{cases} F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + g f^{a}{}_{bc}A^{b}_{\mu}A^{c}_{\nu} \\ (D_{\mu}\psi)^{a} = \partial_{\mu}\psi^{a} + g f^{a}{}_{bc}A^{b}_{\mu}\psi^{c}. \end{cases}$$

(A^{a}_{μ}, ψ^{a}) are in ~~the~~ adjoint representation of the gauge group.

$\boxed{\text{Anomalies for 10d SYM.}}$

→ Remark. 10d SYM is not a renormalizable theory due to dimensionful coupling g. (Therefore, from QFT point of view only the tree level (classical) theory makes sense). However by interpretating SYM as eff. field theory of OPEN SS. one gets that the theory is finite ⇒ but there might be anomalies.

→ They are a breakdown of gauge invariance.

Given the quantum (effective) action

$\Gamma(A_\mu, g_{\mu\nu})$.

$$j_\mu = \frac{\delta \Gamma}{\delta A_\mu}. \quad , \quad \delta A_\mu = D_\mu \Lambda$$

$$\delta_\Lambda \Gamma = \mathrm{tr} \int d^{10}x \, (D_\mu \Lambda) \frac{\delta \Gamma}{\delta A_\mu} =$$

$$= -\mathrm{tr} \left(\int d^{10}x \, \Lambda \, D_\mu \frac{\delta \Gamma}{\delta A_\mu} \right) = 0. \quad \text{if } D_\mu j^\mu = 0.$$

For an anomaly one has:

$$\boxed{\delta_\Lambda \Gamma = \Delta_\Lambda \neq 0}$$

Δ_Λ is linear in the gauge parameter Λ and it is function of $F_{\mu\nu}$, (at linearized level).

It satisfies the WZ consistency-conditions.

$$\boxed{\delta_{\Lambda_1} \Delta_{\Lambda_2} - \delta_{\Lambda_2} \Delta_{\Lambda_1} = \Delta_{[\Lambda_1, \Lambda_2]}}$$

and in the case of 10d SYM:

$$\Delta_\Lambda = \int d^{10}x \, \Big(c_1 \, \mathrm{tr}(\Lambda F_0^5) + c_2 \, \mathrm{tr}(\Lambda F_0) \, \mathrm{tr}(F_0^4) + c_3 \, \mathrm{tr}(\Lambda F_0) \, \mathrm{tr}(F_0^2)^2 \Big).$$

$F_0 = dA$ (only the linear term).

The coefficients c_1, c_2, c_3 depends on the gauge group invariants.

Massive spectrum.

$\alpha^I_{+n}|0\rangle = 0$ $\quad \alpha^I_{-n} \to$ creators.

$S^a_{+n}|0\rangle = 0$ $\quad S^a_{-n} \to$ creators.

First massive state.

NS NS $\to \alpha^I_{-1}|J\rangle$ $\quad \alpha^I_{-1}|\dot{a}\rangle$ — Fermionic states

$S^a_{-1}|I\rangle$ $\quad S^a_{-1}|\dot{b}\rangle$ — RR

$\alpha^I_{-1}|J\rangle$ = 64 states. $\quad \alpha^I_{-1}|\dot{a}\rangle$ = 64 states

$S^a_{-1}|\dot{b}\rangle$ = 64 states $\quad S^a_{-1}|I\rangle$ = 64 "

128 bosons $\quad$ 128 fermions

They corresponds to a massive spin 2 supermultiplet with the fields

$$\boxed{g_{mn},\ C_{mnp},\ \underbrace{\psi_{m\alpha},\ \psi_m^{\hat{\alpha}}}_{\Psi_m}}$$

Ψ_m (Dirac spinor in 10d).

Graviton (massive in 10d) : **44** dof. (traceless, gauge inv.). $\quad \delta g_{mn} = \partial_{(m}\xi_{n)}$, $\quad g_{mn}\eta^{mn} = 0$.

Cmnn - 3 form : 84 dof $\quad$, Ψ_m : 128 gravitinos (massive) in 10d.

Closed string spectrum

- Since we have a left-right factorization from w.s. point of view, we can construct the physical spectrum by tensoring the open string states:

Open: $8_v + 8_c$ $(|I\rangle, |\dot{a}\rangle)$.

Closed: $(8_v + 8_c) \otimes (8_v + 8_c) =$

$= \left(|IJ\rangle + |I\dot{b}\rangle + |\dot{a}J\rangle + |\dot{a}\dot{b}\rangle\right)$.

Bosons: $|IJ\rangle, |\dot{a}\dot{b}\rangle = 64 + 64 = 128$ bosons

$|I\dot{b}\rangle, |\dot{a}J\rangle = 64 + 64 =$ fermions 128

In 10d we have:

g_{mn} : 35 dof $\left(\frac{10 \cdot 11}{2} - 2 \cdot 10 = 55 - 20 = 35\right)$.

NS-NS

g_{mn}, δg_{mn}, + gauss const

b_{mn} : 28 dof $\left(\frac{10 \cdot 9}{2} - \underbrace{20}_{\text{gauge}} + 3 = 45 - 20 + 3 = 28 \text{ dof}\right)$.

NS-NS

$$\begin{cases} \delta B_{mn} = \partial_{[m} \Lambda_{n]} \\ \delta \Lambda_m = \partial_m \Lambda \end{cases}$$

ϕ : dilaton 1 dof.

$28 + 35 + 1 = 64$ bosonic dof

In 10d. we have two types of spinors:
<u>Weyl and Anti Weyl.</u>

$F^{\alpha\beta}$ (Weyl – Weyl bispinor)

$F^{\alpha}{}_{\beta}$ (Weyl – Anti Weyl bispinor).

$$F^{\alpha\beta} = \underset{[1]}{(\gamma^{\mu})^{\alpha\beta} F_{\mu}} + \underset{[3]}{(\gamma^{\mu\nu p})^{\alpha\beta} F_{\mu\nu p}} + \underset{[5]^{+}}{(\gamma^{\mu\nu pqr})^{\alpha\beta} F_{\mu\nu pqr}}$$

$$F^{\alpha}{}_{\beta} = \underset{[0]}{\delta^{\alpha}{}_{\beta} F_{(0)}} + \underset{[2]}{(\gamma^{\mu\nu})^{\alpha}{}_{\beta} F_{\mu\nu}} + \underset{[4]}{(\gamma^{\mu\nu pq})^{\alpha}{}_{\beta} F_{\mu\nu pq}}.$$

$$\left\{\begin{array}{llll} \underset{[5+]}{F_{\mu\nu pqr}} \to F_5 = dC_4 & \to 35 \text{ dof.} & & \\ \underset{[3]}{F_{\mu\nu p}} \to F_3 = dC_2 & \to 28 \text{ dof.} & & \\ \underset{[1]}{F_{\mu}} \to F_1 = dC_0 & \to 1 \text{ dof} & (\text{axion}). & \end{array}\right\} |\dot{a}\dot{b}\rangle$$

$\boxed{\text{type IIB}}$

$$\left.\begin{array}{l} F_{\mu\nu pq} \to F_4 = dC_3 \to 56 \text{ dof} \\ F_{\mu\nu} \to F_2 = dC_1 \to 8 \text{ dof} \end{array}\right\} |\dot{a}\dot{b}\rangle.$$

$\boxed{\text{type IIA}}$

Type IIA

(g_{mn}, b_{mn}, ϕ)	NS-NS.	$\longleftrightarrow$	$\lvert I J \rangle$
(C_1, C_3)	R-R	$\longleftrightarrow$	$\lvert \dot a \dot b \rangle$
$(\psi_{m\alpha}, \psi_m{}^{\alpha})$	NS-R R-NS (gravitinos)		$\lvert \dot a J \rangle$ $\lvert I \dot b \rangle$

Type IIB

(g_{mn}, b_{mn}, ϕ)	NS-NS	$\longleftrightarrow$	$\lvert I J \rangle$	64
(C_0, C_2, C_4)	R-R	$\longleftrightarrow$	$\lvert \dot a \dot b \rangle$	64
$(\psi_{m\alpha}, \hat\psi_{m\alpha})$	NS-R R-NS	$\longleftrightarrow$	$\lvert \dot a J \rangle$ $\lvert I \dot b \rangle$	128.

Type I (which is anomalous as it stands $\Rightarrow$ add open strings)

(g_{mn}, ϕ)	NS-NS	$35 + 1$
C_2	R-R.	28
$\psi_{m\alpha}$		64.

Anomalies for Type IIB / Type I + open

Gravitational anomalies

$\Gamma(A_\mu, g_{\mu\nu})$:

$$\delta_\Lambda \Gamma = \int d^{10}x \, \Big(d_1 \operatorname{tr} \hat{\Lambda} R_0^5 + d_2 \operatorname{tr} \hat{\Lambda} R_0 \operatorname{tr} R_0^4 + d_3 \operatorname{tr}(\hat{\Lambda} R_0)\big(\operatorname{tr}(R_0^2)\big)^2 \Big).$$

where $\hat{\Lambda}$ is $so(10)$ gauge parameter. (the [illegible] anomaly for diffeomorphisms can be moved to an anomaly of Lorentz symmetry).

tr: is the $so(10)$ trace.

Mixed anomalies

$$\delta_\Lambda \Gamma = \int d^{10}x \, \Big(e_1 \operatorname{tr} \Lambda F_0 \operatorname{tr} R_0^4 + e_2 \operatorname{tr} \hat{\Lambda} R_0 \operatorname{tr} F_0^4 + e_3 \operatorname{tr} \Lambda F_0 (\operatorname{tr} R_0^2)^2 + e_4 \operatorname{tr} \hat{\Lambda} R_0 (\operatorname{tr} F_0^2)^2 \Big)$$

A term of the form: $\int d^{10}x \operatorname{tr} \Lambda F_0 \operatorname{tr} F_0^2 \operatorname{tr} R_0^2$ is omitted once we can add a local counter term of the form:

$$\Gamma_{C.T.} = \int d^{10}x \operatorname{tr} A dA \operatorname{tr} F_0^2 \operatorname{tr} \omega d\omega .$$

such that:

$$\delta \Gamma_{C.T.} = \int d^{10}x \, \delta(\operatorname{tr} A dA) \operatorname{tr} F_0^2 \operatorname{tr} \omega d\omega =$$

$$+ \operatorname{tr}(AdA)\, \operatorname{tr} F_0^2\, d(\operatorname{tr}(\omega d\omega))\Big) =$$

$$= \int d^{10}x\, d\big(\operatorname{tr}\Lambda F_0\, \operatorname{tr} F_0^2\, \operatorname{tr}\omega d\omega\big) +$$

$$+ \operatorname{tr}(AdA)\, \operatorname{tr} F_0^2\, d(\operatorname{tr}\hat{\Lambda} R_0\big) =$$

$$= -\int d^{10}x\, \big(\operatorname{tr}\Lambda F_0\, \operatorname{tr} F_0^2\, \operatorname{tr} R_0^2 + (\operatorname{tr} F_0^2)^2 \operatorname{tr}\hat{\Lambda} R_0\big)$$

and therefore we can change the form of the anomaly.

Green - Schwarz mechanism

In super-gravity theory we have to consider the field $b_{\mu\nu}$ (in 10d). Its field strength is

$$\boxed{H = db - \operatorname{tr}\left(AdA + \tfrac{2}{3}A^3\right)}$$

(where A is a gauge field).

$$\Rightarrow \quad S_{KT.} = \int d^{10}x \sqrt{-g}\, H_{\mu\nu\rho} H^{\mu\nu\rho} \longrightarrow$$

$$S_1 \Rightarrow \int d^{10}x \sqrt{-g}\, H^0_{\mu\nu\rho}\, \operatorname{tr} A^\mu \partial^\nu A^\rho \qquad H^0_{\mu\nu\rho} = \partial_{[\mu} b_{\nu\rho]}.$$

By a gauge transformation $\delta A = d\Lambda$ we have

$$\delta S_1 = -\int d^{10}x \sqrt{-g}\, \operatorname{tr}(\Lambda F^{\mu\nu}_{(0)})\, \nabla^\rho H^{(0)}_{\mu\nu\rho}$$

(Here B is invariant under $\delta A = d\Lambda$)

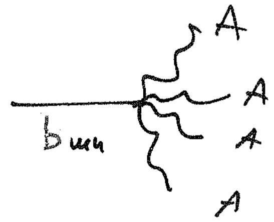
then if there are no other interactions, the eq. of motions would imply! $\delta S_1 = 0$.

($\nabla^m H_{mnp} = 0$)

but if there is another interaction such as:

$$S_2 = \int d^{10}x \, \epsilon \left(b \wedge \mathrm{tr}\, F^4 \right)$$

!



we have!

$$\nabla^m H_{mnp} = \epsilon_{np r_1 \dots r_8} \, \mathrm{tr} \left(F^{r_1 r_2} \wedge \dots \wedge F^{r_7 r_8} \right).$$

then!

$$\delta S_1 = - \int d^{10}x \sqrt{g} \, \mathrm{tr}(\Lambda F_0) \wedge \mathrm{tr}\, F_0^4$$

which is of the form needed to cancel a given term in the anomaly expression. So, adding the coupling S_2, we can cancel the anomaly. (Notice that $S_1 + S_2$ cannot be gauge invariant).

In general we can add: a term of the form!

$$S_2' = \int d^{10}x \left(b \wedge \left[v_1 \mathrm{tr}\, F_0^4 + w_2 \mathrm{tr}\, R_0^4 + \right. \right.$$

$$\left. \left. + v_3 (\mathrm{tr}\, F_0^2)^2 + v_4 \, \mathrm{tr}\, F_0^2 \, \mathrm{tr}\, R_0^2 + v_5 (\mathrm{tr}\, R_0^2)^2 \right] \right).$$

and we can modify the b-field strength as follows

$$H \to H + \mathrm{tr}(A dA) + u \, \mathrm{tr}(\omega d\omega) + \dots$$

(non linear terms)

Descent equations.

$$I_{D+2} = \mathrm{tr}\left(F^{\frac{D+1}{2}}\right) \quad \text{or} \quad \mathrm{tr}\left(R^{\frac{D+1}{2}}\right), \quad \boxed{\delta I_{D+2} = 0}$$

$$dI_{D+2} = 0 \quad \text{(by using the Bianchi id.} \quad dF = [A, F]. \text{)}$$

$$\Rightarrow I_{D+2} = d\, I_{D+1}.$$

$$\delta I_{D+1} = ? \quad : \quad \delta I_{D+2} = 0 = \boxed{d\, \delta I_{D+1} = 0}$$

$$\delta_\Lambda I_{D+1} = d I_D^{(1)}$$

↑ this is linear in Λ.

$$\Delta_\Lambda = \int_M d^D x\, I_D^{(1)} = \int_{\Sigma,\ \partial\Sigma = M} d^{D+1}x\, dI_D^{(1)} = \int_\Sigma d^{D+1}x\, \delta I_{D+1} =$$

δA_Λ

$$= \delta_\Lambda \int_\Sigma d^{D+1}x\, I_{D+1}$$

which automatically satisfy

$$\delta_\Lambda \Delta_{\Lambda'} - \delta_{\Lambda'} \Delta_\Lambda = \Delta_{[\Lambda, \Lambda']}.$$

since $[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{[\Lambda, \Lambda']}$.

In type II B supergravity there are three types of chiral fields: $3/2$, $\frac{1}{2}$, self-dual 5-form. $\cancel{18}$

1) $$\hat{I}_{1/2} = \prod_{i=1}^{2k+1} \frac{\frac{1}{2}x_i}{\operatorname{sh}\frac{1}{2}x_i} \qquad R = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_{2k+1} \\ & & & -x_{2k+1} & 0 \end{pmatrix}$$

$$\operatorname{tr}\left(R^{2m}\right) = 2(-1)^m \sum_{i=1}^{2k+1} x_i^{2m}.$$

2) $$\hat{I}_{3/2} = \hat{I}_{1/2}\left(-1 + 2\sum_{i=1}^{2k+1} \operatorname{ch} x_i\right).$$

$$\left(\hat{I}_{1/2}\right)_{D+2=12} = -\frac{1}{2835}Y_6 - \frac{1}{1080}Y_2Y_4 - \frac{1}{1296}Y_2^{\,3}$$

$$\frac{496}{1296} - \frac{1}{3} - \frac{1}{1296} = \frac{496-432-1}{1296} = \frac{63}{1296} \qquad 64-1$$

$$\left(\hat{I}_{3/2}\right)_{D+2=12} = \left(\hat{I}_{1/2}\right)_{D+2=12} +$$
$$+ \frac{496}{2835}Y_6 + \left(\frac{496}{1080} - \frac{2}{3}\right)Y_2Y_4 + \left(\frac{496}{1296} - \frac{1}{3}\right)Y_2^{\,3}.$$

$$\frac{64}{1296}$$

3) $$\hat{I}_A = -\frac{1}{8}\prod_{i=1}^{2k+1}\frac{x_i}{\operatorname{th} x_i}$$

$$\cancel{64}\ 4$$
$$\left(\hat{I}_A\right)_{D+2=12} = -\frac{496}{2835}Y_6 + \frac{588}{2835}Y_2Y_4 - \frac{\cancel{140}}{2835}Y_2^{\,3}.$$
$$\cancel{864}81$$

To include the dependence of the gauge fields $\rightarrow$

$$\hat{I}_{1/2}(F,R) = \operatorname{tr}\left(e^{iF}\right)\hat{I}_{1/2}(R).$$

In type IIB: $$\boxed{\left(\hat{I}_{3/2}\right)_{D+2=12} - \left(\hat{I}_{1/2}\right)_{D+2=12} + \left(\hat{I}_A\right)_{12} = 0.}$$

21

Spectrum of Heterotic Strings

Flat background superstrings.

$$T = T_L + T_R$$

T_L: Left movers T_R: Right movers.

Now since the two sectors are independent, we can choose to have two different sectors.

We ask Left moving sector to be $N=1$ superstring.

$$(X_L, \theta^\alpha) + \text{ghost} \ldots$$

Right moving sector to be $N=0$ bosonic ⊕ some additional CFT to make the central charge to be 26.

(Notice that in bosonic string theory there is a set of ghost fields b_R, c_R which $C_{(b,c)} = -26$)

So if we start with $d=10$ $X^\mu \to$ and we have X_L^μ, X_R^μ $\mu = 0 \ldots 9$

→ <u>Left</u> $(X_L^\mu, \theta_L^\alpha) + \text{ghosts.} \Rightarrow c_L = 0.$

<u>Right</u> $(X_R^\mu, \widetilde{CFT}) + \text{ghost} \Rightarrow c_R = 10 - 26 + c_{\widetilde{CFT}} = 0$

$$\boxed{c_{\widetilde{CFT}} = 16}$$

this can be realized in several way (for example by a WZW-model, or any $\widetilde{CFT}$).

a way is to introduce 32 w.s. fermions ψ^A with the action:

$$S_\psi = -\frac{1}{2\pi}\int_\Sigma d^2z\,(-2i)\sum_{A=1}^{M}\left(\psi_R^A\,\partial\,\psi_R^A\right)$$

they contribute to the c. charge as $c_{\widetilde{CFT}} = \frac{M}{2}$

and therefore if $M=32$ $\Rightarrow$ $\boxed{c_{\widetilde{CFT}} = 16}$.

If they obey the same boundary conditions.

$\Rightarrow$ $SO(32)$ symmetry (which becomes a gauge symm)

if they obey mixed b.c.

$E_8 \times E_8$.

The $SO(32)$ theory

ψ^A could satisfy. periodic / antiperiod b.c.

P: $\psi^A = \sum_{n\in\mathbb{Z}} \psi_n^A z^{-n}$

A: $\psi^A = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^A z^{-r}$

Physical states: $L_n^{(LEFT)}|\psi\rangle = 0$, $L_n^{(RIGHT)}|\psi\rangle = 0$

$n > 0$.

and $\left(L_0^{LEFT} - a\right)|\psi\rangle = 0$ $\left(L_0^{RIGHT} - \tilde{a}\right)|\psi\rangle = 0$.

Left is susy: $a = 0$. ← X^μ (in light cone)

Right is not susy: $\tilde{a}_A = \frac{8}{24} + \frac{32}{48} = 1$

$$\tilde{a}_P = \frac{8}{24} - \frac{32}{24} = -1$$

(contr. to a: $\frac{1}{24}$ (bos coord), $\frac{1}{48}$ (half integ. fermion), $-\frac{1}{24}$ (integ. fm))

$$L_0|\psi\rangle = \left(\frac{p^2}{8} + N\right)|\psi\rangle = 0. \qquad (p^2 = -m^2)$$

since N (is non negative) $\Rightarrow$ there is no tachyon.

Now we consider $L_0^L|\psi\rangle = (L_0^R - \tilde{a}^R)|\psi\rangle = 0$

$$\Rightarrow \left[L_0^L + (L_0^R - \tilde{a})^R\right]|\psi\rangle = \frac{p^2}{8} + N + \left(\frac{p^2}{8} + \tilde{N} - 1\right) =$$

$$= \left(\frac{p^2}{4} + N + \tilde{N} - 1\right)|\psi\rangle = 0$$

$(m^2 = -p^2)$

$$\Rightarrow \frac{m^2}{4} = N_L + \tilde{N}_R - 1$$

in the Periodic case.

and

$$\frac{m^2}{4} = N_L + \tilde{N}_R + 1$$

antiP./periodic case.

and by level matching

$$L_0^L - (L_0^R - a^R) = \begin{cases} N_L - (N_R - 1) = 0 & A. \\ N_L - (N_R + 1) = 0 & P. \end{cases}$$

Massless sector: $m^2 = 0 \Rightarrow \boxed{N^L = 0}$
(no oscillator).

$\Rightarrow \quad N^L = N^R - 1 = 0 \quad \boxed{N^R = 1}$ in A
(there are massless states)

$N^L = N^R + 1 = 0 \quad N^R = -1$ in P.

(no massless states in P sector).

Left sector $\left(|I\rangle_L, \ |\dot{a}\rangle_L \right)$

R/A sector: $\alpha^I_{R-1}|0\rangle, \quad \psi^{[A}_{R,-\frac{1}{2}} \psi^{B]}_{R,-\frac{1}{2}} |0\rangle$

half integer ($\Rightarrow$ Antiperiodic (they are fermions))

SO(8) reps.

$$(8_V \oplus 8_S) \otimes \left[(8_V, 1) \oplus (1, 496) \right] =$$

$\frac{31 \cdot 32}{2} = 16 \cdot 31 = 496.$

representation of $SO(8) \times SO(32)$

$$= \left[(8_V \oplus 8_S) \otimes (8_V, 1) \right] \oplus \left[(8_V + 8_S) \otimes (1, 496) \right] =$$

spectrum massless of $N=1$ heterotic $SO(32)$.

$\Rightarrow (8_V \otimes 8_V) \rightarrow g_{\mu\nu}, b_{\mu\nu}, \phi$
$(8_V \otimes 8_S) \longrightarrow \psi^\alpha_\mu$
} gravity multiplet $N=1$ $d=10$.

$8_V \otimes (1, 496) \longrightarrow A_\mu$
$8_S \otimes (1, 496) \longrightarrow \lambda^\alpha.$
} gauge multiplet in 10d.

Indeed at the first massive state level, one needs 2 GSO projections to select correctly the spectrum $\Rightarrow SO(32)/\mathbb{Z}_2$

$E_8 \times E_8$

In this case we assume that 16 ψ^A have periodic boundary conditions and the other 16 anti/periodic.

$$\psi^A(w+2\pi) = \begin{cases} \eta\,\psi^A(w) & A = 1 \dots 16 \\ \eta'\psi^A(w) & A = 17 \dots 32. \end{cases}$$

$$\eta, \eta' \to \pm 1.$$

So we have: on the right-mover sector.

$$\tilde{\alpha}^I_{R,-1}|0\rangle \quad , \quad \psi^A_{R-\frac{1}{2}}\psi^B_{R-\frac{1}{2}}|0\rangle \qquad A \oplus A$$

$$A,B \;:\; \begin{matrix} 1 \dots 16 & \to 120 \\ 17 \dots 32 & \to 120 \end{matrix}$$

but we can have:

$$\begin{cases} a_R^{AA} = \dfrac{8}{24} + \dfrac{M}{48} + \dfrac{32-M}{48} = 1 & \forall m. \\[2ex] a_R^{AP} = \dfrac{8}{24} + \dfrac{M}{48} - \dfrac{32-M}{24} = \dfrac{M}{16} - 1 \\[2ex] a_R^{PA} = \dfrac{8}{24} - \dfrac{M}{24} + \dfrac{32-M}{48} = 1 - \dfrac{M}{16}. \\[2ex] a_R^{PP} = \dfrac{8}{24} - \dfrac{M}{24} - \dfrac{32-M}{24} = -1 & \forall m \end{cases}$$

So we have if $m = 16$ $\quad a_R^{AP} = a_R^{PA} = 0$.

Since the N_L have only integer values, N_R could have integer or half integer, so we have that $m = 8p.\ \forall p \in \mathbb{N}$.

There are 3 cases:

i) $m = 32$ $\qquad$ ii) $m = 16$ $\qquad$ iii) $m = 8$ or 24.

i) $\to$ $\mathcal{S}(32)/\mathbb{Z}_2$.

iii) Anomalies.

ii) $SO(16)\times SO(16)$. Let us study this model:

In the AA sector we have.

$$\psi^A_{-1/2}\,\psi^B_{-1/2}\,|0\rangle_{1\ldots16}\otimes|0\rangle_{17\ldots32} \begin{cases}\to (1,120) & A,B\ 17\ldots32\\ \to (120,1) & A,B\ 1\ldots16.\end{cases}$$

or $\psi^A_{-1/2}|0\rangle_{1\ldots16}\otimes\psi^B_{-1/2}|0\rangle_{17\ldots32} \longrightarrow (16,16)$

$$\begin{array}{r}240+\\256\\\hline 496.\end{array}$$

But then we have to consider also

AP and PA states.

and we have to take into account that the zero modes of $\psi^A\psi^A$ system act like Dirac matrices. Then we have:

$$\begin{rcases}\gamma^{I_1}\ldots\gamma^{I_{16}}|0\rangle_P\otimes|0\rangle_A\\ \gamma^{I_{17}}\ldots\gamma^{I_{32}}|0\rangle_P\otimes|0\rangle_A\end{rcases}\Rightarrow (128,1)\oplus(128',1).$$

and in the same way: on $\underline{AP}$ $\to (1,128)\oplus(1,128')$.

$\Rightarrow$ Too many states, we need a GSO projections.
(no Lie algebra)

$(-1)^{F_1}$ for ψ^A $A=1\ldots16$

$(-1)^{F_2}$ for ψ^A $A=17\ldots32$

Physical states invariant under both $(-1)^{F_1+F_2}$.

So we have:

$$\psi^A_{-\frac{1}{2}}\psi^B_{-\frac{1}{2}} \quad \begin{array}{l} \leftarrow (120, 1) \quad (-)^{F_1+F_2} \text{ invariant.} \\ \rightarrow (1, 120) \quad (-1)^{F_1}(-1)^{F_2} \text{ invariant.} \end{array}$$

Now

$\gamma^{I_1} \dots \gamma^{I_{16}}$, and $\gamma^{I_{17}} \dots \gamma^{I_{32}}$ have opposite behavior under $(-)^F \Rightarrow$

$(128, 1)$ from PA

$(1, 128)$ " AP.

$\Rightarrow$ So in total we have $(248, 1) \oplus (1, 248)$

but these are exactly the generators of E_8

So the final gauge group is $E_8 \oplus E_8$.

<u>Anomalies.</u>

Anomaly cancellation in Het.

The total contribution to the anomaly for Het is

$$\begin{cases} \psi \leftarrow \text{gravitinos} \\ \eta \leftarrow \text{Dilatinos} \\ \lambda \leftarrow \text{Gauginos} \end{cases} \qquad \psi^{\alpha}_{m} \gamma^{m}_{\alpha\beta} = \eta_{\beta}. \qquad \begin{matrix} 3/2 \\ 1/2 \\ 1/2 \end{matrix}$$

$$\hat{I} = \underset{\psi}{\hat{I}_{3/2}(R)} - \underset{\eta}{\hat{I}_{1/2}(R)} - \underset{\lambda}{\hat{I}_{1/2}(R,F)}$$

We compute the $\hat{I}_{12}$ and we get:

$$\hat{I}_{12} = -\frac{1}{720}\mathrm{Tr}\,F^6 + \frac{1}{24\cdot 48}\mathrm{Tr}\,F^4\,\mathrm{tr}R^2 +$$
$$- \frac{1}{256}\mathrm{Tr}\,F^2\left[\frac{1}{45}\mathrm{tr}R^4 + \frac{1}{36}(\mathrm{tr}R^2)^2\right] +$$
number of YM fields $\Rightarrow$ # of generators of GG.
$$+ \frac{n-496}{64}\left[\frac{1}{2\cdot 2835}\mathrm{tr}R^6 + \frac{1}{4\cdot 1080}\mathrm{tr}R^2\,\mathrm{tr}R^4 + \right.$$
$$\left. + \frac{1}{8\cdot 1296}(\mathrm{tr}R^2)^3\right] + \frac{1}{384}\mathrm{tr}R^2\,\mathrm{tr}R^4 + \frac{1}{1536}(\mathrm{tr}R^2)^3$$

Tr is the trace on the adjoint representation.
tr " " " in the fund. repr.

$$\begin{cases} I_{12} = dI_{11} & \delta I_{12} = 0 \Rightarrow d\,\delta I_{11} = 0 \\ \Rightarrow \delta I_{11} + dI_{10} = 0 & d\,\delta I_{10} = 0 \\ \delta I_{10} + dI_9 = 0 & \text{and } G = \int_M I_{10}. \end{cases}$$

No we want to find a way to cancel it.
by introducing a counter term of the form

$$\Delta\Gamma_{10} = \underbrace{\int_M B X_8}_{\text{this is the term has been already discussed.}} - \left(\frac{2}{3}+\alpha\right) \int_M (\omega_{3L} - \omega_{3Y}) X_7$$

α = is a free parameter which can be used to change the form of the anomaly.

by only the variation of B

$$\delta B = \underset{\text{Y.M. transf.}}{\omega^1_{2Y}} - \underset{\text{Lorentz transf.}}{\omega^1_{2L}}.$$

Notice that in order to be factorizable in the form:

$$I_{12} = (\mathrm{tr} R^2 + k \,\mathrm{Tr} F^2) X_8$$

we need to cancel (or to make it vanishing) the term $\mathrm{tr} R^6$ (since $so(10) \to D_5$ has a 6th order Casimir in the fundamental representation)

So, we need $\boxed{n = 496}$

By setting $n = 496$

$$I_{12} \propto \left(-\frac{1}{15} \mathrm{tr} F^6\right) + \frac{1}{24} \mathrm{Tr} F^4 \mathrm{tr} R^2 - \frac{1}{960} \mathrm{Tr} F^2 \left[4 \mathrm{tr} R^4 + 5 (\mathrm{tr} R^2)^2\right] + \frac{1}{8} \mathrm{tr} R^2 \mathrm{tr} R^4 + \frac{1}{32} (\mathrm{tr} R^2)^3$$

this is still to be factorized.

Let us pass from adjoint reps. to fundamental rep.

$$\mathrm{Tr}\,F^2 = (n-2)\,\mathrm{tr}\,F^2$$
$$\mathrm{Tr}\,F^4 = (n-8)\,\mathrm{tr}\,F^4 + 3\,(\mathrm{tr}\,F^2)^2$$
$$\mathrm{Tr}\,F^6 = (n-32)\,\mathrm{tr}\,F^6 + 15\,(\mathrm{tr}\,F^2\,\mathrm{tr}\,F^4).$$

So we have:

$$n = 32$$
$$\mathrm{Tr}\,F^2 = 30\,\mathrm{tr}\,F^2$$
$$\mathrm{Tr}\,F^4 = 24\,\mathrm{tr}\,F^4 + 3(\mathrm{tr}\,F^2)^2$$
$$\mathrm{Tr}\,F^6 = 15\,\mathrm{tr}\,F^2\,\mathrm{tr}\,F^4$$

$$\mathrm{tr}\,F^2 = \frac{1}{30}\mathrm{Tr}\,F^2$$
$$\mathrm{tr}\,F^4 = \frac{1}{24}\left(\mathrm{Tr}\,F^4 - \cancel{8}\frac{1}{900}(\mathrm{Tr}\,F^2)^2\right)$$
$$\mathrm{Tr}\,F^6 = 15\,\frac{1}{30}\,\mathrm{Tr}\,F^2\,\frac{1}{24}\left(\mathrm{Tr}\,F^4 - \frac{1}{300}(\mathrm{Tr}\,F^2)^2\right) =$$

$$\boxed{\mathrm{Tr}\,F^6 = \frac{1}{48}\mathrm{Tr}\,F^2\,\mathrm{Tr}\,F^4 - \frac{1}{14400}(\mathrm{Tr}\,F^2)^3}$$

and finally
$$I_{12} = \left(\mathrm{tr}\,R^2 - \frac{1}{30}\mathrm{Tr}\,F^2\right)X_8$$

$$X_8 = \frac{1}{24}\mathrm{Tr}\,F^4 - \frac{1}{7200}(\mathrm{Tr}\,F^2)^2 - \frac{1}{240}\mathrm{Tr}\,F^2\,\mathrm{tr}\,R^2 + \frac{1}{8}\mathrm{tr}\,R^4 + \frac{1}{32}(\mathrm{tr}\,R^2)^2.$$

A group with 496 generators is $SO(32)$ $\quad \frac{32\cdot 31}{2} = 496$
but also $E_8 \times E_8'$ $(248 + 248)$. These are indeed the two groups for Het.

Killing spinors and SU(3) holonomy

Bosonic background (all fermions are set to zero).

→ performing a susy transf ϵ on ψ_μ, λ we should get zero. – This means that the susy parameters are satisfying some equation: killing-spinor equations. and the solution of these are named KILLING SPINORS.

suppose that the background $M_D = M_4^{(mink)} \otimes M_{D-4}$

M_4: d-d. Minkowski space

M_{D-4}: internal space. (compact.)

$M_k = \#$ killing spinors $\Leftrightarrow N$ (susy).

manifested by integrating over M_{D-4}.

$$\boxed{\epsilon^\alpha_A(x,y) = \sum_{I=1}^{N} \epsilon^\alpha_I \, u^i_I(y)}$$

commuting

this is D-dim. spinorial modes

constant 4-d susy parameters (anticommuting).

$I = 1..N$ (# of susy)

Background:
$$\begin{cases} H_{mnp} = 0 \\ \psi_\mu = \lambda = 0 \quad (\chi = 0) \\ T_{mnp} = Z_{mnp} = 0 \end{cases}$$

$$\Rightarrow \delta\psi_\mu = D_\mu[\omega]\epsilon \qquad \delta\chi = 0 \qquad \delta\lambda^a = -\frac{1}{4}(\Gamma^{mn}\epsilon) F^a_{mn}$$

Hence a killing spinor satisfies: $\boxed{D_\mu \epsilon = 0}$ $\boxed{\Gamma^{mn}\epsilon F_{mn} = 0}$

the indices are decoupled as follows

$$\hat{\mu} = (\mu = 0, \dots 3) \oplus (i = 1 \dots 6)$$
$$\hat{a} = (\alpha = 0, \dots, 3) \oplus (A = 1 \dots 6)$$
$$\hat{\alpha} = (\alpha = 1 \dots 4) \oplus (\Lambda = 1 \dots 8)$$

Inserting into the ansatz for the spinor $\epsilon^{\hat{\alpha}} =$

$$\begin{cases} D_\Lambda u_I(y) = 0 \\ F_{ij} (e^{-1})^i_A (e^{-1})^j_B \Gamma^{AB} u_I = 0 \end{cases}$$

Vielbein of the 6-dim internal space.

D_Λ is the cov. derivative ass. to the Levi-Civita connection.

$F_{\Lambda\Sigma}$ gauge field str. on the internal manifold. (only non vanishing components) $\Rightarrow$ For Lorentz cov. in 4d.

Γ^A bosonic Dirac Matrices for $so(6)$ Clifford algebra.

$$[D_\Lambda, D_\Sigma] u_I(y) = -\frac{1}{4} R^{AB}_{\Lambda\Sigma} (\Gamma_{AB} u_I) = 0$$

This implies that

$$Hol(\nabla) = SO(3) \subset SO(6).$$

let us discuss this point.

$$\underbrace{R^{AB}_{\Lambda\Sigma} \Gamma_{AB}}_{\text{generators of } so(6)} u_I = 0 \qquad R^{AB}_{\Lambda\Sigma}(y) \text{ is evolution in } y \text{ where } u^I(y)$$

to have a non-vanishing solution

$u_I \neq 0.$

it is necessary that the irreducible representation $\underline{8}$ of $so(6)$ decompose under $Hol(\nabla)$

If u_I is a singlet $R^{AB}_{\Lambda\Sigma}\Gamma_{AB}\, u_I = 0$

since the action of the generators $so(6.)$

$$so(6) \simeq su(4) \qquad 8 \simeq 4 \oplus 4^*$$

if $Hol(\nabla)$ is $SO(3)$

$$4 = 1 \oplus 3$$

$$8 = 1 \oplus 1^* \oplus 3 \oplus 3^*$$

So there is $1 \oplus 1^*$ complex singlet $\Rightarrow$ we construct the solution. $\Rightarrow N=1$ $D=a$ SUSY.

the second equation $F_{\Lambda\Sigma}(e^{-1})^{\Lambda}_{A}(e^{-1})^{\Sigma}_{B}\Gamma^{AB}u_I = 0.$

$$\Rightarrow \boxed{F_{\Lambda\Sigma} = 0}$$

Indeed it is true!

Hm.

M_{2n} with $\dim_{\mathbb{C}} M_{2n} = n$ and $Hol(M_{2n}) = SU(n)$ is Ricci-flat.

Ricci flat $\Rightarrow T_{\mu\nu} = 0$ which compatible with $F_{\Lambda\Sigma} = 0$

and also

$$dH = -\beta_1\, \mathrm{tr}(F \wedge F)$$

with $H = 0 \Rightarrow \mathrm{tr}(F \wedge F) = 0 \Rightarrow F_{\Lambda\Sigma} = 0$

This situation is not really advantageous:

1) The vacuum we have found is a solution to the WRONG field theory (anomalous $N=1$ SUGRA)

2) No chiral families

$\Rightarrow$ <u>ANOMALY FREE SUGRA</u>

<u>Bianchi id.</u>: $\underbrace{dH - \beta_1 \mathrm{Tr}(F \wedge F) + \gamma_1 \mathrm{tr}(R \wedge R)}_{\text{this modification is due to the anomaly}} = 0$

and usage $H = 0 \Rightarrow$

$$\beta_1 \mathrm{Tr}(F \wedge F) = \gamma_1 \mathrm{tr}(R \wedge R)$$

which is solved by "embedding" the spin connection into gauge connection:

$$\underset{\substack{\uparrow \\ \text{gauge conn.}}}{A^a_\Lambda} = \underset{\substack{\uparrow \\ \text{Embedding tensor}}}{C^a_{AB}} \, \underset{\substack{\uparrow \\ \text{spin-conn.}}}{\omega^{AB}_\Lambda}$$

$SU(3) \hookrightarrow SO(6) \hookrightarrow E_8 \oplus E_8$

$\rightarrow$ Solution of the consistency condition for killing spinors and Einstein's eqs.

$$R^{AB}{}_{CD} = (e^{-1})^\Lambda_C \, (e^{-1})^\Sigma_D \, R_{\Lambda\Sigma}{}^{CD}$$

and

$$(e^{-1})^\Lambda_C \, (e^{-1})^\Sigma_D \, F^a_{\Lambda\Sigma} \, \sigma_a^{AB} \, \Gamma^{CD} = R^{AB}_{CD} \, \Gamma^{CD}.$$

$\Rightarrow$ <u>CY 3-folds</u> w. Ricci-flat metric and spin conn. embedded in the gauge conn. are exact compactified solutions of $N=1$ SOGRA with 1 killing spinor.

The embed of $SU(3) \to SO(6)$

Given $U(3) \ni \mathcal{U} \Rightarrow \mathcal{U}^\dagger \mathcal{U} = I$ $\quad \mathcal{U} = (a_{ij}) \quad a_{ij} \in \mathbb{C}$.

$$a_{ij} = \mathrm{Re}\, a_{ij} + i\, \mathrm{Im}\, a_{ij} \Longrightarrow$$

$$\tilde{a}_{ij} = \begin{pmatrix} \mathrm{Re}\, a_{ij} & \mathrm{Im}\, a_{ij} \\ -\mathrm{Im}\, a_{ij} & \mathrm{Re}\, a_{ij} \end{pmatrix} =$$

$$\tilde{a}_{ij} = \mathrm{Re}\, a_{ij} + J\, \mathrm{Im}\, a_{ij}$$

this is Real matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which has the property $J^2 = -\mathbb{I}$

$$\mathcal{O}(a) = \begin{pmatrix} \mathrm{Re}\,\mathcal{U} & \mathrm{Im}\,\mathcal{U} \\ -\mathrm{Im}\,\mathcal{U} & \mathrm{Re}\,\mathcal{U} \end{pmatrix}.$$

$$\mathcal{O}^T \mathcal{O} = \begin{pmatrix} \mathrm{Re}\,\mathcal{U} & -\mathrm{Im}\,\mathcal{U} \\ \mathrm{Im}\,\mathcal{U} & \mathrm{Re}\,\mathcal{U} \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,\mathcal{U} & \mathrm{Im}\,\mathcal{U} \\ -\mathrm{Im}\,\mathcal{U} & \mathrm{Re}\,\mathcal{U} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}$$

So the matrix $\mathcal{O}(a)$ is a $SO(2n)$ matrix and we found the embedding.

In particular the matrix of $\mathcal{U}(3)$: $\mathcal{U} = i\,\mathbb{1}_3 \quad \mathcal{U}^\dagger \mathcal{U} = \mathbb{1}_n$.

is embedded in $SO(6)$ as follow

$$\mathcal{O}(i) = \begin{pmatrix} J_2 & & \\ & J_2 & \\ & & J_2 \end{pmatrix} = \qquad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J_2^T J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbb{1}$$

$$\boxed{\mathcal{O}(i) = J_2 \otimes \mathbb{1}_3}$$

let me call $\mathcal{O}(i) = \bar{I}$ is the only $U(1)$ matrix which is an invariant matrix of $SO(6)$ in the fund. representation.

Indeed, to commute with $U(3)$ a matrix should be constant in each reducible representation - Since $\underline{6}$ of $SO(6)$ $\to$ $\underline{3} \oplus \underline{3}^*$ then we can chose $\mathbb{1}$ on $\underline{3}$ and $\pm \mathbb{1}$ on $\underline{3}^*$ and therefore we have two choices.

$\Rightarrow$ <u>Complex manifolds</u>.

Complex str. on 2n-d manifold

Complex Manifolds

M is $2n$-dim. manf.

TM is tangent space

T^*M is cotangent space

M is a manifold (seen as diff. manifold.)

$\{\phi^\alpha\}_{\alpha=1,\dots,2n}$ on a patch $U_\alpha \subset M$.

1) $\vec{t} \in \Gamma(M, TM)$ (vector field)

$$\vec{t} = t^\alpha_{(\phi)} \partial_\alpha$$

2) $\omega \in \Gamma(M, T^*M)$ (diff. 1-form)

$$\omega = d\phi^\alpha \omega_\alpha(\phi)$$

3) Contraction $\forall \vec{t} \in \Gamma(M, TM)$

$$i_{\vec{t}} : T^*M \longrightarrow C^\infty(M)$$
$$\omega \longmapsto i_{\vec{t}}\omega = t^\alpha(\phi)\,\omega_\alpha(\phi)$$

4) if $\omega = df = d\phi^\alpha \partial_\alpha f$

$$i_{\vec{t}} df = t^\alpha \partial_\alpha f = \vec{t}(f)$$

5) $\forall \vec{t} \in \Gamma(M, TM)$ $\begin{cases} i_{\vec{t}} : \Omega^p(M) \to \Omega^{p-1}(M) \\ \omega \longmapsto t^\alpha \omega_{\alpha\beta_1\dots\beta_{p-1}} d\phi^{\beta_1} \wedge \dots \wedge d\phi^{\beta_{p-1}} \end{cases}$

6) $L : \Gamma(M, TM) \to \Gamma(M, TM)$

$$\vec{t} \longmapsto L(\vec{t})$$

(linear operator) $L(\alpha \vec{t}_1 + \beta \vec{t}_2) = \alpha L(\vec{t}_1) + \beta L(\vec{t}_2)$

7) on every $U_\alpha \subset M$: $L \to L_\alpha^{\ \beta}(\phi)$

$$L(\vec{t}) = \underbrace{t^\alpha(\phi)\, L_\alpha^{\ \beta}(\phi)}\, \partial_\beta = \hat{t}^\alpha \partial_\alpha = \hat{\vec{t}}$$

(pull back of L)

8) $L_* : \Gamma(T^*M, M) \to \Gamma(T^*M, M)$

$$\boxed{i_{\vec{t}}\, L_*(\omega) = i_{L(\vec{t})}\,\omega}$$

$$L_*(\omega) = d\phi^\alpha \big(L_*(\omega)\big)_\alpha =$$

$$= d\phi^\alpha (L_*)_\alpha^{\ \beta} \omega_\beta$$

$$i_{\vec{t}}\, L_*(\omega) = t^\alpha (L_*)_\alpha^{\ \beta}(\phi)\, \omega_\beta$$

$$i_{L(\vec{t})}(\omega) = t^\alpha L_\alpha^{\ \beta} \omega_\beta$$

$$\Rightarrow \boxed{L_\alpha^{\ \beta} = (L_*)_\alpha^{\ \beta}}$$

$\boxed{\text{Almost complex structure}}$ $\boxed{(M, J)}$

$$J : \Gamma(M, TM) \to \Gamma(M, TM)$$

$$\vec{t} \longmapsto J(\vec{t}) = t^\alpha J_\alpha{}^\beta \partial_\beta = t^\alpha J_\alpha{}^\beta(\phi) \partial_\beta$$

such that: $\boxed{J^2 = -\mathbb{1}} \to \boxed{J_\alpha{}^\beta(\phi) J_\beta{}^\gamma = -\delta_\alpha{}^\gamma}$

– $\forall p \in M$: $J_\alpha{}^\beta(\phi) \longrightarrow J_\alpha{}^\beta(p) = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$

$\boxed{\mathbb{1}_{n\times n}}$

– A local frame where $J \to \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$ is denoted "well adapted"

– $\vec{e}_\alpha = \partial_\alpha$ we have:

$$\begin{cases} J(\vec{e}_\alpha) = -\vec{e}_{\alpha+n} & \alpha \le n \\ J(\vec{e}_\alpha) = \vec{e}_{\alpha-n} & \alpha > n \end{cases}$$

– Introduce

$$\begin{cases} \vec{E}_i = \vec{e}_i - i\,\vec{e}_{i+n} \\ \vec{E}_{\bar{\imath}} = \vec{e}_i + i\vec{e}_{i+n} \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} J(\vec{E}_i) &= i\,\vec{E}_i \\ J(\vec{E}_{\bar{\imath}}) &= -i\,\vec{E}_{\bar{\imath}} \end{aligned}}$$

$\vec{E}_i = \partial_{z_i}$ $\quad z^i = \phi^i + i\phi^{i+n}$

$\vec{E}_{\bar{\imath}} = \partial_{\bar{z}_{\bar{\imath}}}$ $\quad \bar{z}^{\bar{\imath}} = \phi^i - i\phi^{i+n}$

Coordinate transformation

Two well adapted basis are related by coordinate transformations which is holomorphic function:

$$\phi^\alpha \longrightarrow \phi^\alpha + \zeta^\alpha(\phi) = \phi'^\alpha$$

$$J_\alpha{}^\beta \longrightarrow \boxed{(\mathcal{J})_\alpha{}^\beta J_\beta{}^\gamma (\mathcal{J}^{-1})_\gamma{}^\delta = J_\alpha{}^\delta}$$

$$\Downarrow$$

$$\partial_\alpha \xi^\beta J_\beta{}^\gamma = J_\alpha{}^\beta \partial_\beta \xi^\gamma$$

(eq. di Cauchy-Riemann)

$$z^i \longrightarrow z^i + \xi^i(z)$$

holomorphic function

C-R equations

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_x v = -\partial_y u \end{cases} \qquad \begin{matrix} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{matrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \partial_x v \\ \partial_y v \end{pmatrix}$$

$$\partial_0 \xi^0 J_0{}^1 + \cancel{\partial_0 \xi^1 J_1{}^1} = J_0{}^1 \partial_1 \xi^1 + \cancel{J_0{}^0 \partial_0 \xi^1}$$

$$\begin{cases} \partial_0 \xi^0 J_0{}^1 = J_0{}^1 \partial_1 \xi^1 \Longrightarrow \boxed{\partial_0 \xi^0 = \partial_1 \xi^1} \\ \cancel{\partial_0 \xi^0 J_0{}^0} + \partial_0 \xi^1 J_1{}^0 = \cancel{J_0{}^0 \partial_0 \xi^0} + J_0{}^1 \partial_1 \xi^0 \end{cases}$$

$$\partial_0 \xi^1 J_1{}^0 = J_0{}^1 \partial_1 \xi^0 \qquad \boxed{\partial_0 \xi^1 = -\partial_1 \xi^0}$$

$$\cancel{\partial_1 \xi^0 J_0{}^0} + \partial_1 \xi^1 J_1{}^0 = J_1{}^0 \partial_0 \xi^0 + \cancel{J_1{}^1 \partial_1 \xi^0}$$

$$\partial_1 \xi^0 J_0{}^1 + \cancel{\partial_1 \xi^1 J_1{}^1} = J_1{}^0 \partial_0 \xi^1 + \cancel{J_1{}^1 \partial_1 \xi^1}$$

$\boxed{\text{on } T^*M}$

$x \in V,\ x^* \in V^*$

$\langle J(x), x^* \rangle = \langle x, J(x^*) \rangle$

Hermitian inner product.

$h(J(x), J(y)) = h(x,y) \implies$ see

$$\begin{cases} J_*(dz^i) = i\,dz^i \\ J_*(d\bar{z}^{\bar{\imath}}) = -i\,dz^{\bar{\imath}} \end{cases}$$

if $w(x)$ of generic system of coords $\{x^\alpha\}$

$w(x): M \to \mathbb{C}$ is holomorphic if

$$\boxed{J_*(dw) = i\,dw} \qquad d\phi^\alpha J_\alpha{}^\beta \partial_\beta w = i\,d\phi^\alpha \partial_\alpha w$$

$$\boxed{J_\alpha{}^\beta \partial_\beta w = i\,\partial_\alpha w}$$

if the solution $\exists \Rightarrow z^i = w^i(x)$.

if (*) is integrable $\exists$ holomorphic coord. system

$$\phi = \phi(z, \bar{z})$$

using (**)

$$\begin{cases} d\phi = \partial_i \phi\, dz^i + \bar{\partial}_{\bar{\imath}} \phi\, dz^{\bar{\imath}} \\ J_*(d\phi) = i(dz^i \partial_i \phi - dz^{\bar{\imath}} \partial_{\bar{\imath}} \phi) \end{cases}$$

$$dJ \wedge d\phi = -2i\, \partial_i \partial_{\bar{\imath}} \phi\, dz^i \wedge dz^{\bar{\imath}}$$

$$\begin{cases} (1-J)\, dz^i \wedge dz^{\bar{\imath}} = dz^i \wedge dz^{\bar{\imath}} - J(dz^i) \wedge J(dz^{\bar{\jmath}}) = \\ = dz^i \wedge d\bar{z}^{\bar{\imath}} - i\,dz^i \wedge (-i)\,dz^{\bar{\imath}} = 0 \end{cases}$$

$$\Rightarrow \quad \boxed{(1-J)\, dJ \wedge d\phi = 0} \qquad \text{true in any coordinate system.}$$

$$T^{\alpha}_{\beta\gamma}\, \partial_\alpha \phi \, dx^\beta \wedge dx^\gamma = 0$$

$$\boxed{T^{\alpha}_{\beta\gamma} = \partial_{[\beta} J_{\gamma]}{}^{\alpha} - J_\beta{}^{\delta} J_\gamma{}^{\sigma}\, \partial_{[\delta} J_{\sigma]}{}^{\alpha}}$$

see next page

Hermitian Complex Manifolds

$$\boxed{(M, J, h)}$$

See: $\boxed{h(J(x), J(y)) = h(x,y) = x^\alpha h_{\alpha\beta} y^\beta}$

$J(x) = x^\alpha J_\alpha{}^\beta \partial_\beta$

$$x^\alpha J_\alpha{}^\beta h_{\beta\gamma} \left(x^\delta J_\delta{}^\gamma\right) = x^\alpha \left(J_\alpha{}^\beta h_{\beta\gamma} J^\gamma{}_\delta\right) x^\delta$$

$$\Rightarrow \boxed{J^T h J = h}$$ Hermitian inner product.

- Given h an Hermitian inner product in a $\mathbb{R}$ vector space V and a complex structure J. Then h can be uniquely extended to a complex symmetric bilinear form

1) $h(\bar{z}, \bar{w}) = \overline{h(z,w)}$ $\quad z, w \in V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$

2) $h(z, \bar{z}) > 0$ $\quad \forall z \neq 0 \in V^{\mathbb{C}}$

3) $h(z, \bar{w}) = 0$ $\quad$ if $z \in V^{1,0}$ $w \in V^{0,1}$

- $\forall h$ on V and J on V $\exists\, \varphi \in \Lambda^2 V^*$ as follows:

$$\boxed{\varphi(x,y) = h(x, J(y))}$$

It is skew symmetric:

$$\varphi(y,x) = h(y, Jx) = h(Jy, J^2x) =$$
$$= h(Jy, -x) = -h(x, Jy) =$$
$$= -\varphi(x,y).$$

$$y^\alpha \varphi_{\alpha\beta} X^\beta = y^\alpha h_{\alpha\beta'} J^{\beta'}_\beta X^\beta \Rightarrow \varphi_{\alpha\beta} = h_{\alpha\beta'} J^{\beta'}_\beta$$

Using matrix formulation $\left\{ \varphi = hJ \quad \varphi^T = J^T h = J^T J^T h J = (J^2)^T h J = \right.$

$$\boxed{\varphi^T = -hJ = -\varphi.}$$

Torsion of J — Argument on flatness ⇒ Using Riemann tensor. ⟹ we would like to find a global coord system such that $\boxed{\begin{array}{l} J(z) = z \\ J(\bar z) = -\bar z \end{array}}$

$$\forall x, y \in TM, \qquad J(x) = x^\alpha J_\alpha^\beta \partial_\beta$$

$$N(x,y) = 2\{[J(x), J(y)] - [x,y] - J[x, Jy] - J[J(x), y]\}$$

$$N^\alpha_{\beta\gamma}: \quad N(x,y) = N^\alpha_{\beta\gamma} x^\beta y^\gamma \partial_\alpha$$

$$[x,y] = [x^{\alpha'} \partial_{\alpha'} y^{\beta'} - y^\alpha \partial_\alpha x^{\beta'}] \partial_{\beta'}$$

$$[J(x), J(y)] = [x^\alpha J_\alpha^{\alpha'} \partial_{\alpha'} [y^\beta J_\beta^{\beta'}] - y^\alpha J_\alpha^{\alpha'} \partial_{\alpha'} [x^\beta J_\beta^{\beta'}]] \partial_{\beta'} =$$
$$= [\underline{x^\alpha J_\alpha^{\alpha'} \partial_{\alpha'} y^\beta} J_\beta^{\beta'} + x^\alpha J_\alpha^{\alpha'} y^\beta \partial_{\alpha'} J_\beta^{\beta'} +$$
$$- \underline{y^\alpha J_\alpha^{\alpha'} \partial_{\alpha'} x^\beta} J_\beta^{\beta'} - y^\alpha J_\alpha^{\alpha'} x^\beta \partial_{\alpha'} J_\beta^{\beta'}] \partial_{\beta'}$$

$$J[x, Jy] = (x^{\alpha'} \partial_{\alpha'} [y^{\beta'} J_{\beta'}^\beta] - y^{\alpha'} J_{\beta'}^{\alpha'} \partial_{\alpha'} x^\beta) J_\beta^\mu \partial_\mu$$

Hermitian metric h on <u>HVB</u>

$$\langle \xi, \eta \rangle = \bar{\xi}^{\bar{I}} \eta^J h_{\bar{I}J}(z,\bar{z}) = (\xi^\dagger h \eta)$$

$\xi, \eta \in \Gamma(M, E)$

<u>Def</u> An hermitian metric for a complex manifold M is a hermitian fiber metric on TM. The transition functions a the Jacobian transfs.

<u>Def</u> CANONICAL CONNECTION

$$\boxed{\begin{array}{ll} 1) & d\langle \xi, \eta \rangle_h = \langle D\xi, \eta \rangle_h + \langle \xi, D\eta \rangle_h \\ 2) & D^{(0,1)}\xi = \left(\bar{\partial} + \theta^{(0,1)}\right)\xi = 0 \end{array}}$$

$$d\left[\bar{\xi}^{\bar{I}} h_{\bar{I}J} \eta^J\right] = d\bar{\xi}^{\bar{I}} h_{\bar{I}J}\eta^J + \bar{\xi}^{\bar{I}} dh_{\bar{I}J}\eta^J + \bar{\xi}^{\bar{I}} h_{\bar{I}J} d\eta^J$$

$$= \left(d\bar{\xi}^{\bar{I}} + \bar{\theta}^{\bar{I}}_{\bar{J}}\bar{\xi}^{\bar{J}}\right) h_{\bar{I}J}\eta^J +$$

$$+ \bar{\xi}^{\bar{I}} h_{\bar{I}J}\left(d\eta^J + \theta^J_K \eta^K\right)$$

$$\Rightarrow \quad dh_{\bar{I}J} = \bar{\theta}^{\bar{J}}_{\bar{I}} h_{\bar{J}J} + h_{\bar{I}I}\theta^I_J$$

$$\boxed{dh_{\bar{I}J} - \bar{\theta}^{\bar{J}}_{\bar{I}} h_{\bar{J}J} - h_{\bar{I}I}\theta^I_J = 0}$$

this implies that

$$\theta^I_J = dz^k h^{I\bar{I}} \partial_k h_{J\bar{I}}$$

$$\Downarrow$$

$$\boxed{\nabla h_{\bar{I}J} = 0}$$

This implies that holomorphic section are transported into holomorphic sections.

d) trasf. the functions between two local trivilitation (U_α, h_α) and (U_β, h_β).

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \otimes \mathbb{C}^n \to (U_\alpha \cap U_\beta) \otimes \mathbb{C}^n$$

induce:

$$\boxed{g_{\alpha\beta} : (U_\alpha \cap U_\beta) \longrightarrow GL(n, \mathbb{C})}$$

holomorphic.

E (HVB), $U \subset M$.

a FRAME is a set of section $\{s_1 \dots s_r\} : M \to E$ such that:

$\{s_1(z), \dots, s_r(z)\}$ is a basis of $\pi^{-1}(z)$ $\forall z \in U$.

given a frame: $\{e_I(z)\}_{I=1 \dots r}$

$$\Gamma(M, E) \ni \xi(z) = \xi^I(z)\, e_I.$$

$$\bar\partial \xi^I = dz^{\bar J}\, \partial_{\bar J} \xi^I = 0$$

Connections

$$D\xi = d\xi + \theta\xi$$

θ is a $GL(r, \mathbb{C})$ 1-form. $= \theta^I{}_J$

$$\theta = \theta^{(1,0)} + \theta^{(0,1)} = dz^I \theta_I + dz^{\bar I} \theta_{\bar I}$$

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$$= \left(X^{\alpha'}\partial_{\alpha'}Y^{\beta'}J_{\beta'}{}^{\beta} + X^{\alpha'}Y^{\beta'}\partial_{\alpha'}J_{\beta'}{}^{\beta} + - Y^{\beta'}J_{\beta'}{}^{\alpha'}\partial_{\alpha'}X^{\beta}\right)J_{\beta}^{\mu}\partial_{\mu} = - \cancel{X^{\alpha'}\partial_{\alpha'}Y^{\mu}\partial_{\mu}} + X^{\alpha'}Y^{\beta'}\partial_{\alpha'}J_{\beta'}{}^{\rho}J_{\rho}{}^{\mu}\partial_{\mu} - Y^{\beta'}J_{\beta'}{}^{\alpha'}\partial_{\alpha'}X^{\beta}J_{\beta}{}^{\mu}\partial_{\mu}$$

$$J[J(X),Y] = \left(X^{\alpha'}J_{\alpha'}{}^{\alpha}\partial_{\alpha}Y^{\beta} - Y^{\alpha}\partial_{\alpha}X^{\beta'}J_{\beta'}{}^{\beta} - Y^{\alpha}X^{\beta'}\partial_{\alpha}J_{\beta'}{}^{\beta}\right)J_{\beta}^{\mu}\partial_{\mu}$$

$$= \left(X^{\alpha'}J_{\alpha'}{}^{\alpha}\partial_{\alpha}Y^{\beta}J_{\beta}{}^{\mu} + Y^{\alpha}\partial_{\alpha}X^{\mu} - Y^{\alpha}X^{\beta'}\partial_{\alpha}J_{\beta'}{}^{\beta}J_{\beta}{}^{\mu}\right)\partial_{\mu}$$

$$\Rightarrow \boxed{N_{\alpha\beta}^{\mu} = J_{\alpha}{}^{\nu}\partial_{\nu}J_{\beta}{}^{\mu} - J_{\beta}{}^{\nu}\partial_{\nu}J_{\alpha}{}^{\mu} - \partial_{\alpha}J_{\beta}{}^{\nu}J_{\nu}{}^{\mu} + \partial_{\beta}J_{\alpha}{}^{\nu}J_{\nu}{}^{\mu}}$$

An. complex structure has vanishing $N=0$

Holomorphic Vector Bundle

M (complex manifold)
E " "

HVB: $\pi : E \to M$

a) π is holomorphic map of E into M.

b) $\forall p \in M$ fiber over p:

$$E_p = \pi^{-1}(p).$$

is a complex vector space $\dim E_p = r$ (rank of vector bundle)

c) $\forall p \in M$, $\exists U_p$ and an homeo.

$$h: \pi^{-1}(U) \to U \otimes \mathbb{C}^r$$

s.t. $h(\pi^{-1}(U)) = \{p\} \otimes \mathbb{C}^r$