Geometrical methods for string compactifications

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These notes were prepared for my lectures at LACES 2009. It should be noted that they are a **preliminary version** that I initially prepared for my own reference; as such, they are bound to contain mistakes and inaccuracies. If you think something in it looks fishy, please let me know.

They do contain some extra material that I did not explain in class, and some that I will explain next time. On the other hand, as of today (December 9, 2009), they are **unfinished**: they do not contain the last (and most important) part, about the application of generalized complex geometry to flux compactifications with RR fields.

1 Fluxes and supersymmetry

We will deal in these lectures with type II supergravity. Of course we are interested in this only because it is the low–energy limit of the type II string, but for most of these lectures the stringy effects will not be taken into account, and pure supergravity will be enough.

1.1 Fields

Type II supergravity comes in two flavors, IIA and IIB. Let us begin by recalling the fields of each of them. (In this subsection and the next, we will follow the conventions and language of [1], except for a few changes noted in appendix A.) The matter fields are given by the fermions[1]

$$\psi^a_M \text{ (gravitino)}, \qquad \lambda^a \text{ (dilatino)}. \tag{1.1}$$

where $a = 1, 2$. In IIA, ψ^1_M and λ^1 has chirality $+$, and ψ^2_M, λ^2 have chirality $-$. In IIB, all have chirality $+$. All of them are Majorana. The forces are mediated by the bosons

$$g_{MN} \text{ (metric)}, \qquad B_{MN} \ , \tag{1.2}$$

and by a collection of fields of the form

$$C_{M_1 \ldots M_p} \ , \qquad p = \begin{cases} = 1,\ 3,\ 5,\ 7,\ 9 & \text{(IIA)} \\ = 0,\ 2,\ 4,\ 6,\ 8 & \text{(IIB)} \end{cases} \quad \text{RR} - \text{fields.} \tag{1.3}$$

[1] For notations, see appendix A.

The metric g_{MN} is obviously symmetric, as a metric should be. Both B_{MN} and the $C_{M1\ldots M_p}$ are totally antisymmetric, and they should be understood as gauge potentials, generalizing the potential A_M of electro–magnetism. We will often use for them the form notation

$$B \equiv \frac{1}{2} B_{MN} dx^M \wedge dx^N \ , \qquad C_p \equiv \frac{1}{p!} C_{M_1\ldots M_p} dx^{M_1} \wedge \ldots \wedge dx^{M_p} \ . \tag{1.4}$$

Using also the exterior differential $d \equiv dx^M \frac{\partial}{\partial x^M}$, we can introduce the field–strengths[2] (or "fluxes")

$$H = dB \ , \qquad F_p = dC_{p-1} - H \wedge C_p \ . \tag{1.5}$$

These can be thought of as similar to the F_{MN} of electro–magnetism. For example, there is a gauge transformation $B \to B' = B + d\lambda_1$, with λ_1 a one–form, that leaves H invariant (because $H' = dB' = d(B + d\lambda_1) = dB = H$). The gauge transformations for the C_p potentials are more complicated, and they read

$$C_p \to C'_p = C_p + d\lambda_{p-1} - H \wedge \lambda_{p-3} \tag{1.6}$$

where obviously λ_k is a k–form. Another complication about the C_p is that their field strengths are related by

$$F_p = (-1)^{\lfloor \frac{p}{2} \rfloor} * F_{10-p} \ . \tag{1.7}$$

In most other reviews of supergravity, the constraint (1.7) is solved by keeping as fundamental fields only the field–strengths of the fields with the fewest indices. For example, [2, Vol. 2] keeps only F_0, F_2 and F_4 in IIA, and F_1, F_3 and F_5 in IIB. (Notice that, for IIB, part of the constraint is still with us: we still have $F_5 = *F_5$.) This choice of which field–strengths to keep is called "electric basis". In these lectures, as we will see later, we will choose another electric basis, one which is better adapted to the problem of compactifying to four dimensions.

Equations (1.6) might look clumsy. It is a good idea to collect them into a single object, a differential form of mixed degree. We can think of this as a formal sum:

$$C = \begin{cases} C_1 + C_3 + C_5 + C_7 + C_9 & \text{(IIA)} \\ C_0 + C_2 + C_4 + C_6 + C_8 & \text{(IIB)} \end{cases} \tag{1.8}$$

In the same way, we can define

$$F = \begin{cases} F_0 + F_2 + F_4 + F_6 + F_8 + F_{10} & \text{(IIA)} \\ F_1 + F_3 + F_5 + F_7 + F_9 & \text{(IIB)} \end{cases} \tag{1.9}$$

[2] Actually, this formula is more complicated if $F_0 \neq 0$, because that is a field–strength without a potential, but for the time being we can simply ignore this. F_0 is also called "Romans mass".

This allows to write the RR field strength in (1.5) as

$$F = dC - H \wedge C = (d - H\wedge)C \equiv d_H C \ . \tag{1.10}$$

Notice that

$$d_H^2 = \frac{1}{2}\{d_H, d_H\} = \frac{1}{2}\{d, d\} + \{d, H\wedge\} + \frac{1}{2}\{H\wedge, H\wedge\} = (dH)\wedge \ . \tag{1.11}$$

The idea of these equalities is the same as when we compute the commutator of the derivative operator and the multiplication operator in quantum mechanics: we can imagine acting on a test object (for quantum mechanics, a wave function; in our case, a differential form), compute the (anti)commutator, and then eliminate the test object at the end.

Equation (1.7) does not seem to simplify too much by the introduction of the formal sum (1.9). But the signs we have in (1.7) appear often enough in string theory that it pays off to introduce a symbol to avoid writing them out every time. We introduce an operator λ, defined by its action on a differential form α_k of degree k:

$$\lambda\alpha_k = (-1)^{\lfloor \frac{p}{2} \rfloor}\alpha_k \ . \tag{1.12}$$

Then we can write (1.7) as[3]

$$F = *\lambda F = \lambda * F \ . \tag{1.13}$$

1.2 Supersymmetry transformations

We can now introduce the supersymmetry transformations of type II supergravity. They contain infinitesimal parameters ϵ^a, $a = 1, 2$; these are fermionic, so that the supersymmetry transformations mix bosons and fermions (just like $- \times + = -$, $- \times - = +$). In IIA, ϵ^1 has chirality $+$, ϵ^2 has chirality $-$. In IIB, both have chirality $+$. Both are Majorana.

Before we write down the supersymmetry transformations, we need a bit of notation. The fluxes appear in these equations multiplied by an appropriate number of gamma matrices:

$$\not{H}_M \equiv \frac{1}{2}H_{MNP}\Gamma^{NP} \ , \qquad \not{H} \equiv \frac{1}{6}H_{MNP}\Gamma^{MNP} \ , \qquad \not{F}_k \equiv \frac{1}{k!}F_{M_1\ldots M_k}\Gamma^{M_1}\ldots\Gamma^{M_k} \ . \tag{1.14}$$

Notice that the definition of $\not{F}_k$ is very similar to the form notation $F_k = \frac{1}{k!}F_{M_1\ldots M_k}dx^{M_1}\wedge \ldots \wedge dx^{M_k}$. One can think of $\not{F}_k$ as being obtained from the k–form F_k via a map that

[3]The operators $*$ and λ commute in dimension ten, but not in dimension six.

sends $dx^M \mapsto \Gamma^M$:

$$\text{Clifford map} \ : \ \alpha \equiv \sum_k \frac{1}{k!} \alpha_{i_1 \ldots i_k} dx^{i_i} \wedge \ldots \wedge dx^{i_k} \quad \longleftrightarrow \quad \not{\alpha} \equiv \sum_k \frac{1}{k!} \alpha_{i_1 \ldots i_k} \gamma^{i_i \ldots i_k}_{\alpha\beta} \ . \tag{1.15}$$

Notice also that $\not{F}_k$ has two spinorial indices (because each of the Γ^M does), so it is a "bispinor". The use of slashes to distinguish a form from its corresponding bispinor is more precise, and it is indeed essential to keep track of certain subtle signs, but it can will quickly get out of hand (and make unreadable the equations in which they appear) as soon as one applies it to more complicated forms. In what follows, we will drop the slash whenever it should not lead to confusion.

They can be written as [1]

$$\begin{aligned} \delta\psi^1_M &= \left(D_M + \frac{1}{4} H_M\right)\epsilon^1 + \frac{e^\phi}{16} F\, \Gamma_M \Gamma \epsilon^2 \ , \\ \delta\psi^2_M &= \left(D_M - \frac{1}{4} H_M\right)\epsilon^2 - \frac{e^\phi}{16} \lambda(F) \Gamma_M \Gamma \epsilon^1 \ ; \\ \Gamma^M \delta\psi^1_M - \delta\lambda^1 &= \left(D - \partial\phi - \frac{1}{4} H\right)\epsilon^1 \ , \\ \Gamma^M \delta\psi^2_M - \delta\lambda^2 &= \left(D - \partial\phi + \frac{1}{4} H\right)\epsilon^2 \ . \end{aligned} \tag{1.16}$$

In the third and fourth equations, $D = \Gamma^M D_M$ and $\partial\phi = \Gamma^M \partial_M \phi$; these definitions are in the spirit of (1.14), and, just as for those definitions, we decided to drop the slashes.

Fortunately, in (1.16) we were able to assemble here all the RR fluxes in the combination F from (1.9), without any extra factors. Notice also that (1.16) are valid both in IIA and IIB.

If we are looking for supersymmetric solutions of the equations of motion, we can follow a standard strategy: set to zero the expectation values of the fermions ψ^a_M and λ^a. Invariance under supersymmetry then means that all the variations in (1.16) should be set to zero.

This gives rise to four equations. As we will now explain, these are almost all we need. Because of the supersymmetry algebra $\{Q, Q\} = \not{P}$, one expects that the Hamiltonian can be written as some kind of square. In supersymmetric field theories, this leads to the BPS bound on the energy; in particular, one finds that supersymmetric configurations also solve the equations of motion.

For example, in four–dimensional $\mathcal{N} = 1$ super–Yang–Mills, invariance under supersymmetry gives the self–duality equations $F = *F$. If one recalls the Bianchi identity $dF = 0$ (in absence of monopoles), the equations of motion $d{*}F = 0$ follow automatically.

The situation in type II supergravity is similar. The Bianchi identities that have to be imposed are

$$(d - H\wedge)F = 0 \ , \qquad dH = 0 \qquad \text{(almost everywhere).} \tag{1.17}$$

The reason to specify "almost everywhere" is that there could be sources. We will see later that only some kind of sources are allowed by supersymmetry.

Once one imposes (1.17), almost all the equations of motion follow automatically. Those that do not (the ones for g_{01}) will not play any role in the rest of our lectures. In fact, we will see that, for compactifications, half of the equations in (1.17) are also a consequence of supersymmetry. That is because half of the equations for F should be thought of as true Bianchi identities; the other half are actually the equations of motion for F. Just which half is which depends on one's choice of electric basis.

To summarize, to find supersymmetric solutions of type II supergravity, we need to

- Set to zero the right hand sides of (1.16);
- Impose (1.17);

up to some subtleties that play no role for compactifications.

1.3 Compactifications

We will now specialize the transformations (1.16) to the case we want to focus on in these lectures: solutions that are also "vacua" arising from compactifications. First of all, "compactification" means that the spacetime M_{10} is fibred over a four–dimensional spacetime M_4. Next, we want to define what we mean by the word "vacuum". It should mean a configuration for which that there are no particles in our four–dimensional space–time. One would expect, then, the maximal amount of possible symmetry in four dimensions. In absence of a cosmological constant Λ, that is simply Poincaré symmetry, and $M_4 = \mathrm{Mink}_4$; if $\Lambda < 0$, the Poincaré algebra gets deformed to $\mathrm{SO}(3,2)$, and $M_4 = \mathrm{AdS}_4$. As for dS_4, we do not consider it because it is incompatible with unbroken supersymmetry[4].

To summarize this preliminary discussion:

[4]There are several very general arguments to show this, but perhaps the most direct is to notice that the potential of $\mathcal{N} = 1$ supergravity in four dimensions is schematiacally of the form $V = e^K(|DW|^2 - 3|W|^2) + D^2$, where W is the superpotential and D are the D–terms; for supersymmetric vacua, $DW = 0$ and $D = 0$, which gives a negative cosmological constant

Definition 1.1. A *vacuum* of type II supergravity is a solution of its equations of motion and Bianchi identities, such that M_{10} is fibered over a spacetime M_4, and such that the whole solution (and not just M_4) enjoys maximal symmetry in four dimensions (that is, Poincaré for $M_4 = \text{Mink}_4$, and SO(3, 2) for $M_4 = \text{AdS}_4$.)

Let us see what else follows (other than the geometry of M_4) from this definition. Let us first look at the metric of M_{10}. We said that, a priori, it is fibred over M_4. Such a fibration, however, would involve a connection A_μ. This connection would be a vector in M_4. But the choice of any such vector would break maximal symmetry in four dimensions. (Alternatively, just think about the off–diagonal components $g_{\mu m}$ as a vector in four dimensions.) Hence, we should choose $A_\mu = 0$.

Since $g_{\mu m} = 0$, the metric is a product. The internal metric g_{mn} is unconstrained by the four–dimensional maximal symmetry. The external metric $g_{\mu\nu}$ is fully constrained by that symmetry, were it not for a possible dependence on the internal coordinates y^m. (Such a dependence is actually important in Randall–Sundrum models.) We can take this into account by introducing a function $A(y)$, called *warping*, and by writing $g_{\mu\nu} = e^{2A} g^{(4)}_{\mu\nu}$.

If we call ds^2_4 the volume element of Mink_4 or AdS_4, we can then summarize this discussion on the metric by writing

$$ds^2_{10} = e^{2A} ds^2_4 + ds^2_{M_6} \ . \tag{1.18}$$

Let us now look at the other fields. First of all, all of them should only depend on the internal coordinates y^m, or else they would break translational symmetry.

This is all that needs to be said about the dilaton ϕ. As for the three–form flux H, we also have to look at its allowed indices. They can only be purely internal: any other choice would select one or more directions in M_4, and thus break maximal symmetry. So $H_{\mu\nu\rho} = H_{\mu\nu p} = H_{\mu n p} = 0$.

The situation is slightly more involved for the RR fluxes F_p. If $p < 4$, the considerations we just saw for H apply verbatim, and the flux is purely internal. For $p > 6$, some of the four–dimensional indices will have necessarily to be on. We can still preserve maximal symmetry, however, with components of the type $F_{0123 m_1 \ldots m_{6-p}}$: these do not select any particular direction in M_4. For $4 \le p \le 6$, both possibilities are allowed.

We can summarize this discussion by writing

$$F = f + \text{vol}_4 \wedge \tilde{f} \ . \tag{1.19}$$

Here, both f and $\tilde{f}$ are forms on M_6. We will call the first term in (1.19) “internal”, and the second “external”. Notice that self–duality (1.7) becomes, in this case,

$$\tilde{f} = \lambda(*_6 f) \ . \tag{1.20}$$

We now turn to ask what happens when one imposes that the vacuum should also be supersymmetric. As we reviewed in the previous subsection, supersymmetric solutions are found by setting to zero the right hand sides of equation (1.16) and the Bianchi identities (1.17). In the case of a vacuum, we have to ask whether the supersymmetry parameters ϵ^a that appear in (1.16) will break maximal symmetry in four dimensions.

To answer this question, it is a good idea to use the fact that $M_{10} = M_4 \times M_6$ and that the metric is a (warped) product. This implies that the gamma matrices can be represented as tensor products:

$$\Gamma_\mu = e^A \gamma_\mu \otimes 1 \ , \quad \Gamma_m = \gamma_5 \otimes \gamma_m \ , \tag{1.21}$$

where Γ’s are with respect to ds^2_{10}, γ_μ with respect to ds^2_4, γ^m with respect to ds^2_6. In the same way, the spinors on which these gamma matrices act should belong to the tensor of the space of the four–dimensional spinors and the space of six–dimensional spinors. In other words, any ten–dimensional spinor ϵ can be written as a *linear combination* of spinors of the form $\zeta \otimes \eta$. If ϵ is Majorana–Weyl, it is convenient to write

$$\epsilon_\pm = \zeta_+ \otimes \eta_\pm + \zeta_- \otimes \eta_\mp \ . \tag{1.22}$$

This is automatically Weyl (if $\zeta_\pm$ and $\eta_\pm$ have chirality $\pm$), and it satisfies the Majorana condition $\epsilon = \epsilon^*$ if[5]

$$\zeta_- = \zeta_+^* \ , \qquad \eta_- = \eta_+^* \ . \tag{1.23}$$

The decomposition (1.22), supplemented by (1.23), can be applied to any of the ϵ^a in (1.16). If we found only one solution to (1.16), that would certainly mean that maximal symmetry is broken: choosing a ζ is a bit like choosing a direction (and in fact, one can connect the two by defining a vector $v^\mu = \zeta^t \gamma^\mu \zeta$). In other words, maximal symmetry acting on a ζ will yield another four–dimensional spinor, ζ'. This suggests a way of preserving maximal symmetry: impose that there (1.16) is solved not just by one choice

[5]In general, the Majorana condition would read $\epsilon = B\epsilon^*$, with B an appropriate intertwiner between the representations Γ^M and $(\Gamma^M)^*$ of the Clifford algebra. For simplicity, however, we will take particular bases in which γ_μ are real, γ_m are purely imaginary (it is non–trivial that such bases exist in four Lorentzian and six Euclidean dimensions). From (1.21) one can then see that the Γ^M are real, and one can choose $B = 1$. (As usual, γ^0 is anti–hermitian, all the others are hermitian. We will also define the chiral gamma’s as $\gamma_5 = i\gamma^{0123}$, $\gamma = -i\gamma^{456789}$, $\Gamma = \Gamma^{0123456789}$).

of ϵ^a, but by all those connected to it by maximal symmetry. In other words, one can look for a solution to (1.16) of the form

$$\begin{aligned} \epsilon^1 &= \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_- \ , \\ \epsilon^2 &= \zeta_+ \otimes \eta^2_\mp + \zeta_- \otimes \eta^2_\pm \ , \end{aligned} \tag{1.24}$$

but one has to impose that the solution works for *any* four–dimensional spinor $\zeta_\pm$ (related by (1.23)). (The upper sign is for IIA, the lower for IIB.)

The reason we have written the same ζ in ϵ^1 and ϵ^2 is that, at least for the time being, we are trying to discuss a "minimal" possibility: the smallest amount of supersymmetry we can preserve without breaking maximal spacetime symmetry. This is obtained by having only one ζ_+: since this is a Weyl spinor in four dimensions, this amount is called $\mathcal{N}=1$. Had we taken two different ζ's, we would have had $\mathcal{N}=2$. The choice (1.24) is actually the most general one can make if one wants $\mathcal{N}=1$ supersymmetry.

Coming back to (1.24), if we plug it in (1.16) and we impose that it should give a solution for any ζ, we see that we can actually factor out ζ altogether. For example, if we take the first of (1.16) in IIB and choose M to be an internal index, $M=m$, we get, using also (1.21) and (1.19):

$$\zeta_+ \otimes \left[\left(D_m - \frac{1}{4} H_m \right) \eta^1_+ + \frac{e^\phi}{8} f\, \gamma_m \eta^2_+ \right] + \zeta_- \otimes \left[\left(D_m - \frac{1}{4} H_m \right) \eta^1_- + \frac{e^\phi}{8} f\, \gamma_m \eta^2_- \right] = 0 \ . \tag{1.25}$$

Since this equation should be satisfied by any ζ, we can simply conclude that $\left(D_m - \frac{1}{4}H_m\right)\eta^1_+ + \frac{e^\phi}{8} f\, \gamma_m \eta^2_+ = 0$; notice that the bracket multiplied by ζ_- is the complex conjugate of this equation, and thus need not be imposed again.

We can continue this process for the other equations in (1.16). The only subtlety worth mentioning appears in the terms $D_\mu \zeta_\pm$. For $M_4 =$Mink$_4$, these vanish. For $M_4 = \mathrm{AdS}_4$, they do not; but we can take for the ζ's a basis with the property $D_\mu \zeta_- = \frac{1}{2}\mu\gamma_\mu\zeta_+$; μ is a number such that $\Lambda = -|\mu|^2$. (That such a basis exists can be seen explicitly, see for example [3].)

If we apply this to (1.16), the first two split in two cases each, $M=m$ and $M=\mu$;

this gives a total of six equations:

$$\left(D_m - \frac{1}{4}H_m\right)\eta^1_+ \mp \frac{e^\phi}{8} f\,\gamma_m \eta^2_\mp = 0 \;, \tag{1.26a}$$

$$\left(D_m + \frac{1}{4}H_m\right)\eta^2_\mp - \frac{e^\phi}{8}\lambda(f)\,\gamma_m \eta^1_+ = 0 \;; \tag{1.26b}$$

$$\mu e^{-A}\eta^1_+ + \partial A \eta^1_+ - \frac{e^\phi}{4} f\,\eta^2_\mp = 0 \;, \tag{1.26c}$$

$$\mu e^{-A}\eta^2_\pm + \partial A \eta^2_\mp - \frac{e^\phi}{4}\lambda(f)\,\eta^1_+ = 0 \;; \tag{1.26d}$$

$$2\mu e^{-A}\eta^1_- + D\eta^1_+ + \left(\partial(2A-\phi) + \frac{1}{4}H\right)\eta^1_+ = 0 \;, \tag{1.26e}$$

$$2\mu e^{-A}\eta^2_\pm + D\eta^1_\mp + \left(\partial(2A-\phi) - \frac{1}{4}H\right)\eta^1_\mp = 0 \;. \tag{1.26f}$$

These equations certainly look scary at this point; the aim of these lectures is to reformulate them in terms of exterior calculus only (that is, in terms of differential forms and their wedges and differentials). As we will see, they will look much more intelligible then.

For completeness, we can also decompose here (1.17) using (1.19):

$$dH = 0 \;, \qquad (d - H\wedge)f = 0 \;, \qquad (d + H\wedge)(e^{4A} *_6 f) = 0 \qquad \text{(almost everywhere)}; \tag{1.27}$$

once again, the "almost everywhere" is there to allow for sources.

To summarize this lecture, we have found that a supersymmetric vacuum satisfies the following properties:

- all fields only depend on the internal coordinates y;
- the metric is given by (1.18), with ds^2_4 the volume element of either Mink_4 or AdS_4, and ds^2_6 a metric on an arbitrary six–manifold M_6;
- the three–form flux H is a closed three–form on M_6;
- the RR flux obeys (1.19), (1.20), and f obeys (1.27);
- the six–dimensional spinors $\eta^{1,2}$ satisfy (1.26).

2 $\mathrm{SU}(3)$ structures

During this lecture we will start learning how to reformulate the equations (1.26) in terms of exterior calculus, by analyzing some easy particular case.

2.1 Spinors and forms

Equations (1.26) contain two internal spinors of definite chirality, $\eta_+^{1,2}$ (and their complex conjugates $\eta_-^{1,2}$, see (1.23)). We will start by considering one such a spinor, and reformulating its data in terms of differential forms.

The existence of a six–dimensional spinor of positive chirality η_+ tells us, first of all, that M_6 has to be *spin*: its frame bundle (the principal SO(6) bundle associated to its tangent bundle) will have a lift to a Spin(6) bundle Σ. We can say more if we use the fact (that we will show only **later**, as a consequence of (1.26)), that $\eta^{1,2}$ vanish nowhere on M_6. In that case, the transition functions of M_6 can be taken to be in SU(3)$\subset$ SO(6). To see why, consider an atlas of M_6, and a spinor η_α on each chart U_α. On each chart, η_α can be made to have a certain fixed form, for example $\eta_\alpha = (1,0,0,0)^t$. Now, the transition functions should obey

$$\eta_\alpha = g_{\alpha\beta}\eta_\beta \; ; \tag{2.1}$$

but this means that $g_{\alpha\beta}$ all keep invariant a certain spinor (which we took to be $(1,0,0,0)^t$). In other words, $g_{\alpha\beta} \in$ SU(3).

One says that η gives a *reduction* of the structure group to SU(3), or that it defines an *SU(3) structure*. If we look back at (2.1), we see that the crucial point is that the stabilizer in SO(6) of any given spinor η is SU(3): there was nothing special to the choice $\eta = (1,0,0,0)^t$.

We can try to define an SU(3) structure by using tensors instead. To that end, let us define

Definition 2.1. An *almost complex structure* (ACS) is a tensor $I^m{}_n$ such that $I^2 = -1$ (in indices, $I^m{}_p I^p{}_n = -\delta^m{}_n$). We will also always assume that it is *hermitian*, namely that $J \equiv gI$ is antisymmetric. A (hermitian) ACS defines an U(3) structure. As we saw above, the way to see this is to show that the stabilizer of a given I is U(3). This is best seen at the level of Lie algebras: pick $I = \left(\begin{smallmatrix} 0 & 1_3 \\ -1_3 & 0 \end{smallmatrix}\right)$. If we take $m = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with a, b, c, d four 3×3 matrices, and impose that it commutes with I and that it is antisymmetric (as an element of so(6) should be), we get a matrix of the form $m = \left(\begin{smallmatrix} a & s \\ s & a \end{smallmatrix}\right)$, with a antisymmetric and s symmetric. This is the way a hermitian 3×3 matrix $u = s + ia$ (an element of u(3)) is embedded in so(6).

A U(3) structure is thus given by the data I, g, J; two of these three can always be used to derive the third. For example, $g = -JI$. The existence of an U(3) structure on M_6 is equivalent to the vanishing of a certain topological class, $W_3 = 0$, but this will not play any role for us.

Notice that at this point there are no "complex coordinates" z^i, because I is an *almost* complex structure, and not a complex structure. Still, we can define $(1,0)$ forms e^α by

$$I^t e^\alpha = i e^\alpha \ ; \tag{2.2}$$

in components, $I^m{}_n e^\alpha_n = i e^\alpha_m$. Since we have learned how to define holomorphic indices without having holomorphic coordinates, we can also define (p,q)–forms in similar ways. Another thing we can do is to talk about the g_{ij} components of a metric. Notice, then, that an ACS is hermitian if and only if

$$g_{ij} = 0 \ . \tag{2.3}$$

If we now want to reduce further to SU(3), we have to select a nowhere–vanishing "holomorphic volume form" Ω: a $(3,0)$–form. This is because Ω would transform under action by $U \in \mathrm{U}(3)$ as $\Omega \to \det(U)\Omega$. Imposing that Ω be invariant gives $\det(U) = 1$, which means $U \in \mathrm{SU}(3)$. The existence of such an Ω imposes a new topological condition on the space. One can introduce the "canonical bundle" K_I of $(3,0)$–forms, which has $\mathbb{C}$ as a fibre, which makes it a "line bundle". The existence of a nowhere–vanishing Ω says that the canonical bundle is trivial topologically. This is expressed by saying that

$$c_1(I) = 0 \ . \tag{2.4}$$

(You might have already heard that a Calabi–Yau manifold is a *Kähler* manifold with $c_1 = 0$; the condition $c_1 = 0$ is a topological one, and there are many more manifolds which have $c_1 = 0$ than there are Calabi–Yau's.)

We have seen now how to introduce an SU(3) structure using (g, J, Ω). In fact, with some more work one can show that a pair (J, Ω) with an appropriate list of properties is enough to define an SU(3) structure *by itself*, without even specifying a metric g, and that in fact a g is defined by such a pair. Let us see how this is possible.

Definition 2.2. A three–form Ω is said to be *decomposable* if it is locally of the form $e^1 \wedge e^2 \wedge e^3$, for e^i some one–forms, and *nondegenerate* if $\Omega \wedge \bar{\Omega}$ is never–vanishing. Notice that a nondegenerate three–form is necessarily complex. We have a

Theorem 2.1. *A nondegenerate decomposable three–form Ω defines an ACS I_Ω.*

The idea is that the three one–forms in the definition of decomposability are defined to be the $(1,0)$–forms of I_Ω, as defined by (2.2) (there should be three of them). Conversely, given an almost complex structure I, the three–form Ω is determined uniquely (when it exists) up to rescalings

$$\Omega \to \alpha \Omega \ , \tag{2.5}$$

with $\alpha \in \mathbb{C}$.

Once one has an almost complex structure I_Ω, one can add a two–form J and define a metric $g = -JI_\Omega$ as before. That g given by this formula is nondegenerate and symmetric is not obvious: it is a condition of compatibility between J and Ω. We have seen before how to formulate compatibility between g and I, for example in (2.3). One can see that this can be reformulated as a condition between J and I: namely, that J is $(1,1)$ with respect to I. This motivates the following

Definition 2.3. An SU(3) structure is a pair (J, Ω), where J is a real two–form and Ω a decomposable, non–degenerate complex three–form, such that

$$J \wedge \Omega = 0 \ , \qquad J^3 = \frac{3}{4} i \Omega \wedge \bar{\Omega} \tag{2.6}$$

and such that the metric $g = -JI_\Omega$ is positive definite. The first condition in (2.6) is nothing but the condition that J be $(1,1)$ with respect to I. The second is meant to fix the freedom of rescaling Ω as in (2.5) without changing I_Ω.

Notice that we had already called SU(3) structure a nowhere–vanishing spinor η (in addition, this time, to a given metric g; without a metric it does not even make sense to talk of spinors, since the gamma matrices are defined with a metric). For consistency of notation there should be a bijection

$$(g, \eta) \longleftrightarrow (J, \Omega) \ . \tag{2.7}$$

The map from (g, η) to (J, Ω) is easy to describe:

$$J_{mn} = \eta_+^\dagger \gamma_{mn} \eta_+ \ , \qquad \Omega_{mnp} = \eta_+^t \gamma_{mnp} \eta_+ \ . \tag{2.8}$$

At this point it is not obvious why these should satisfy (2.6), but it follows from certain properties of the gamma matrices, the so–called "Fierz identities" that we will learn in detail **later**. The map from (J, Ω) is as follows: once one identifies I_Ω, one has three complex gamma matrices γ^i, and one can define η_+ by imposing

$$\gamma^i \eta_+ = 0 \ . \tag{2.9}$$

J then gives us g, as explained above.

We have seen that an SU(3) exists on M_6 if and only if $W_3 = 0$ (for the existence of an ACS I) and $c_1 = 0$ (for the existence of Ω). Once one exists, how many are there? since we have not imposed any differential properties on (J, Ω), we can modify them locally more or less as we want. In fact, it makes sense to count the dimension of the space of

SU(3) structures at every point. There dimension of the space of metrics is $\frac{6 \cdot 7}{2} = 21$. The space of U(3) structures compatible with a given metric (which is known as *twistor space*) is given by SO(6)/U(3), which is isomorphic to $\mathbb{CP}^3$ and thus has dimension 6. Finally, for every U(3) structure the space of $(3,0)$–forms is real one–dimensional (the volume form is fixed already by the volume condition in (2.6)). That gives us $21 + 6 + 1 = 28$ for the dimension of the space of SU(3) structures at every point. A more direct way of counting this dimension is by the dimension of the quotient $\mathrm{Gl}(6, \mathbb{R})/\mathrm{SU}(3)$, which gives $36 - 8 = 28$.

One can also obtain the same number using only J and Ω, without using the induced ACS. The dimension of the space of real two–forms is $\frac{6 \cdot 5}{2} = 15$; the requirement that it should be non–degenerate is an inequality, so it does not change the dimension. As for Ω, the space of complex three–forms has complex dimension $\frac{6 \cdot 5 \cdot 4}{3!} = 20$, or real dimension 40. Non–degeneracy is again an inequality; decomposability, however, reduces this dimension significantly. It turns out [4] that a decomposable three–form is completely determined by its real part, which belongs to the space Λ^3 of three–forms. Moreover, the space of real three–forms which can be the real part of a decomposable form is an open set in Λ^3. So the *real* dimension of the space of decomposable three–forms is 20. We still have the compatibility constraints (2.6). $J \wedge \Omega$ is a priori a $(3,2)$ form; imposing that it vanishes then gives 3 complex (or 6 real) constraints. Both J^3 and $i\Omega \wedge \bar{\Omega}$ are real forms, so this is 1 real constraint. Summing up, we get a space of real dimension $15 + 20 - 6 - 1 = 28$.

2.2 Intermezzo: operators on differential forms

To go on, we would like to start imposing some differential conditions on the "flabby" geometrical structures we have seen so far. Before we do that, I want to pause to review some differential geometry. In the previous lecture I assumed knowledge of exterior calculus (differential forms, their wedges $\wedge$ and their differential d). We now need to know one thing or two about the Lie bracket. There is the usual definition as commutator:

$$[v, w]_{\mathrm{Lie}}(f) \equiv vwf - wvf \ , \tag{2.10}$$

but I want to promote here another point of view, which generalizes this definition nicely.

First of all, notice that one can think of wedging by one–form $\alpha\wedge$ as an operator acting on the space of differential forms: $\omega \to \alpha \wedge \omega$. Likewise, we can also think of the contractions

$$v \llcorner \equiv \iota_v = v^m \iota_m \tag{2.11}$$

(whose action is defined in (A.2)) as operators on the space of differential forms: $\omega \to v\llcorner\omega$.

Notice, moreover, that these operators obey a nice algebra among themselves:

$$\{dx^m\wedge, dx^n\wedge\} = 0 \ , \qquad \{dx^m\wedge, \iota_n\} = \delta^m{}_n \ , \qquad \{\iota_m, \iota_n\} = 0 \ . \tag{2.12}$$

This has the form of a Clifford algebra with a $2d \times 2d$ "metric"

$$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{2.13}$$

We will see later that this fact is quite important for generalized complex geometry.

Once we take this point of view, many formulas in differential geometry can be shown quite naturally using the operator formalism we know from quantum mechanics. For example, we can check the famous Cartan formula for the Lie derivative on a differential form:

$$L_v = \{d, v\llcorner\} = \{dx^m\wedge\partial_m, v^n\iota_n\} = dx^m\wedge\{\partial_m, v^n\}\iota_n + v^n\{dx^m, \iota_n\}\partial_m = dx^m\partial_m v^n\iota_n + v^m\partial_m \tag{2.14}$$

which can be seen to coincide with the usual expression of L_v in components, using (A.2).

Another formula that can be checked in this way is

$$[\{d, v\llcorner\}, w\llcorner] = [v, w]_{\mathrm{Lie}}\llcorner \ . \tag{2.15}$$

This equation will be useful later: in a sense, it shows that the Lie bracket is not a "new" operation, but that it can be "derived" from the exterior differential d and the contraction operators. (It is indeed an example of a so–called "derived bracket", of which we will see another example later on.)

2.3 Differential conditions

In section 2.1, we have not assumed much of an SU(3) structure, because we did not impose any of the equations (1.26). In this section, we will have a look at a few natural conditions that one can impose on an SU(3) structure mathematically. We are not imposing yet (1.26); this subsection should be seen as a purely mathematical intermezzo – one whose use will become apparent only later on.

The first condition one might want to impose is the *integrability* of I.

Definition 2.4. An ACS I is said to be *integrable*, in which case it is also called more simply a *complex structure* (CS), if the bundle $T_{1,0}$ of $(1,0)$ vectors satisfies $[T_{1,0}, T_{1,0}]_{\mathrm{Lie}} \subset$

$T_{1,0}$. If I is integrable, it is possible to find in every patch complex coordinates z^i such that $e^i = dz^i$. In many references, you will find this definition in a slightly different form: namely, that a certain "Nijenhuis tensor" $N(I)^m{}_{np}$ should vanish. In our language, that tensor can be seen as follows. Introduce the "holomorphic projector" $\Pi = \frac{1}{2}(1 - iI)$: this projects on $(1,0)$ vectors. Its conjugate $\bar{\Pi} = \frac{1}{2}(1 + iI)$ projects on $(0,1)$ vectors. Then, integrability is clearly equivalent to $\bar{\Pi}([\Pi v, \Pi w]_{\rm Lie}) = 0$. It turns out that the real and imaginary parts of this equation are not independent: we can then simply impose, if $\{\partial_m\}$ is the coordinate basis of vectors,

$$N(I)^m{}_{np} = (\bar{\Pi}[\Pi\partial_n, \Pi\partial_p])^m = 0 \ . \tag{2.16}$$

The integrability condition has actually a convenient reformulation when I comes from a decomposable non–degenerate form Ω, namely $I = I_\Omega$. We have:

Theorem 2.2. *I_Ω is integrable if and only if there exists a form W_5 such that*

$$d\Omega = W_5 \wedge \Omega \ . \tag{2.17}$$

Proof. We will first give an easy argument that shows how this condition is necessary for integrability. Then we will see a more complicated argument that also shows that it is sufficient.

If the complex structure is integrable, so that we have complex coordinates z^i, one can define the Dolbeault operator [6]

$$\partial = dz^i \frac{\partial}{\partial z^i} \ ; \tag{2.18}$$

it is a differential (in that $\partial^2 = 0$), and it sends (p,q)–forms to $(p+1,q)$–forms. We also have $d = \partial + \bar{\partial}$. This implies that d of a $(1,0)$–form is the sum of a $(2,0)$–form and of a $(1,1)$–form. Then, if we recall that locally $\Omega = e^1 \wedge e^2 \wedge e^3$ (where e^i are a basis of $(1,0)$–forms), we recover (2.17).

To show that (2.17) is also sufficient, recall that $\Omega = e^1 \wedge e^2 \wedge e^3$. (2.17) can only be true if the d of a $(1,0)$ form never contains a $(0,2)$ part; or, by conjugation, if

$$(de^{\bar{\alpha}})_{(2,0)} = 0 \ . \tag{2.19}$$

consider any two $(1,0)$ vectors E_β, E_γ. We have the following chain of equalities:

$$[E_\beta, E_\gamma]_{\rm Lie} e^{\bar{\alpha}} = [\{d, E_\beta\}, E_\gamma] e^{\bar{\alpha}} = -E_\gamma \{d, E_\beta\} e^{\bar{\alpha}} = -E_\gamma E_\beta d e^{\bar{\alpha}} = 0 \ . \tag{2.20}$$

[6]Even if I is not integrable, one can define $\partial = dx^m \Pi^n{}_m \partial_n$, but its properties are not as nice as its counterpart when I is integrable: for example, it does not square to zero, and it does not send (p,q)–forms to $(p+1,q)$–forms.

This means that the Lie bracket of any two $(1,0)$ vectors is still $(1,0)$, which is the definition of integrability.

Notice that this argument can also be followed backwards, to show sufficiency again. $\square$

We can also ask what happens when $W_5 = 0$ in (2.17). In that case, $\bar\partial\Omega = 0$, and we can see that the canonical bundle K_I not only has a section, but a holomorphic section. So it is trivial not only as a topological bundle, but as a holomorphic bundle. (The two notions are different, because in the holomorphic classification of bundles one only allows holomorphic gauge transformations.) The space of bundles which are trivial topologically but not holomorphically is called the Jacobian: it has dimension $H^{0,1}$. In fact, K_I can be trivialized holomorphically even if $W_5 = \bar\partial f$, because then we have $\bar\partial(e^{-f}\Omega) = 0$. So what counts is the class of W_5 in $H^{0,1}$. For example, if $H^{0,1}(M_6) = 0$, W_5 is $\bar\partial$–exact, and K_I is holomorphically trivial; and this is in agreement with the fact that there should be no Jacobian in this case.

Let us now leave the realm of complex manifolds. There is another particularly notable class of spaces with SU(3) structures that can be considered: that of symplectic manifolds. A symplectic structure in general is given by a non–degenerate two–form (one whose determinant vanishes nowhere) which is closed. We have a non–degenerate two–form: it is J. So for us M_6 is symplectic if

$$dJ = 0\ . \tag{2.21}$$

We will see that this condition is in a sense "mirror" to the condition (2.17). Beware: (2.21) **does not** mean the space is Kähler. A space is called Kähler if it has a U(3) structure I such that I is integrable *and* $dJ = 0$ (where, as usual for a U(3) structure, $J = gI$).

2.4 No fluxes

We now get back to physics, and to (1.26). We want to try to solve it with some simple Ansatz.

The first, brutal try is to set all the fields to zero except the metric. To further simplify the discussion, we can also set $\eta^1 = \eta^2 = \eta$. That gives us

$$D_m \eta = 0\ . \tag{2.22}$$

Since there are no RR terms to mix the two ϵ's, in this case we can actually take the two $\zeta's$ in (1.24) different, which means we have non–minimal supersymmetry, $\mathcal{N} = 2$.

The condition (2.22) has been considered in other lectures here: it says that M_6 is a *Calabi–Yau* manifold. The holonomy of M_6 is contained in SU(3), because η is kept invariant by the parallel transport, and the stabilizer of a non–zero η in SO(6) is SU(3). This argument sounds superficially similar to the SU(3) structures we saw earlier, but the difference is huge. SU(3) structures are given by the mere existence on the manifold of a well–defined, nowhere–vanishing η; this is a topological condition. SU(3) holonomy is there when η is also covariantly constant: this is a quite rigid constraint on the geometry.

From (2.8), we can also see that $D_m J_{np} = D_m \Omega_{npq} = 0$. This also implies, then,

$$dJ = 0 = d\Omega \ . \tag{2.23}$$

Perhaps surprisingly, these are *equivalent* to (2.22). Of course, $dJ = 0$ alone would not be equivalent to $D_m J_{np} = 0$, but together (2.23) are equivalent to $D_m J_{np} = D_m \Omega_{npq} = 0$ and to (2.22).

Notice that a Calabi–Yau can then be thought of as a manifold which is both complex (because of theorem 2.2) and symplectic (see (2.21)), with some compatibility between the two concepts (see (2.6)).

2.5 No RR fluxes

A more interesting case arises if we set to zero the RR flux f only. It is also a good idea to set $\eta^2 = 0$: not doing so would just result in a second copy of the discussion that follows.

In this case, we have

$$D^H_m \eta_+ \equiv \left(D_m - \frac{1}{8} H_{mnp} \Gamma^{np} \right) \eta_+ = 0 \ . \tag{2.24}$$

The spinor η is still covariantly constant, but with respect to the torsionful connection $\omega_{mnp} - \frac{1}{2} H_{mnp}$. It is also interesting, however, to compute dJ and $d\Omega$. To do that, first we can see that

$$D_m J_{np} = (D_m \eta)^\dagger \gamma_{np} \eta + \eta^\dagger \gamma_{np} D_m \eta_+ = \frac{1}{8} H_{mqr} \eta^\dagger [\gamma_{np}, \gamma^{qr}] \eta = -H_{mqr} \delta_{[n}{}^q J_{p]}{}^r = H_{mr[n} J_{p]}{}^r , \tag{2.25}$$

where we have used the gamma matrix algebra $\{\gamma_m, \gamma_n\} = 2 g_{mn}$ to compute $[\gamma_{np}, \gamma^{qr}]$. Several other manipulations of this sort give [5]

$$d(e^{-2\phi} J) = -e^{-2\phi} * H \ , \qquad d(e^{-2\phi} \Omega) = 0 \ , \qquad d(e^{-2\phi} J^2) = 0 \ . \tag{2.26}$$

One can also write the equation for J without the dilaton, using also the other equations:

$$i(\partial - \bar\partial) J = H \ . \tag{2.27}$$

From the equation on Ω in (2.26) and from theorem 2.2 we see that the manifold is complex. However, it is not a Calabi–Yau, nor is it symplectic, since $dJ \neq 0$.

In type II, however, this case is in trouble, due to a simple argument in [5]. If M_6 is compact, we have

$$\int_{M_6} e^{-2\phi} H \wedge *H = -\int_{M_6} H \wedge d(e^{-2\phi} J) = -\int_{M_6} dH \wedge e^{-2\phi} J = 0 \ , \tag{2.28}$$

where we have used the Bianchi identity for H; this implies that $H = 0$. So there are no compact vacua of this type.

In the so–called "heterotic" version of string theory, however, the Bianchi identity is no longer $dH = 0$, and one can indeed find compact solutions [6], although even there there are subtleties due to the importance of stringy corrections; we will not comment on those solutions any further here.

3 $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures

So far we have seen what kind of geometry can be defined by one spinor. In the equations (1.26) we actually have two six–dimensional spinors $\eta^{1,2}$. In this lecture we will see what kind of geometry one can define with two spinors. We will first describe briefly the situation from the point of view of the structure group on M_6. Then we will turn to a review of generalized complex geometry, adapted from [7, Sec. 3].

If you do not particularly like the math in this section, what you need to know is theorem 3.3, that tells us that many of the data of a supergravity vacuum can be reformulated in terms of two differential forms $\Phi_\pm$ obeying a certain set of *algebraic* constraints. These forms define a "so–called $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure". You also need to know the most important example of solution to those constraints, namely (3.17).

3.1 The traditional point of view

One possible approach is trying to use the same logic we followed in the previous section, and asking what kind of reduction of structure they define. The answer is that they give an $\mathrm{SU}(2)$ structure on M_6, as long as they are never parallel, i. e. if there are no points in which one of the two is zero or proportional to the other.

There are several ways to see this, but the best is probably to consider the vector

$$v_m = \eta_-^{1\,\dagger} \gamma_m \eta_+^2 = \eta_+^{1\,t} \gamma_m \eta_+^2 \ . \tag{3.1}$$

At a point where η^1 and η^2 are proportional, $v_m = 0$, because then we have $v_m \propto \eta^t \gamma_m \eta$, and this vanishes because γ_m are antisymmetric. If $\eta^{1,2}$ are nowhere proportional, v_m has no zeros. The presence of a vector without zeros on a manifold has a non–trivial topological consequence: namely, that the Euler characteristic $\chi = 0$. Actually, v_m is a *complex* vector, and in a sense it selects two privileged directions out of six. This reduces the structure group from SU(3) to SU(2).

Treating this SU(2) structure with differential forms is trickier than the SU(3) structure case, however. The reason is that there are many possible generalizations to (2.8), with many constraints among them, even though geometrically it is enough to have a triple (v, J, Ω) to describe an SU(2) structure. The problem becomes even more acute once one tries to use these bilinears together with the supersymmetry equations (1.26): one gets horrible equations without any clear interpretation.

For these reasons, we will now try a different route.

3.2 Generalized complex structures

In the rest of this section, we will introduce a different point of view, one in which the two spinors will be understood from the point of view generalized complex geometry. This geometry was introduced by Hitchin [8] and developed by Gualtieri [9]. We will review it in $d = 6$ for ease of reading, but most (though not all) of it can be generalized to arbitrary dimensions.

Generalized complex geometry is the generalization of complex geometry to $T \oplus T^*$, the sum of the tangent and cotangent bundle of a manifold. One starts by introducing on $T \oplus T^*$ the analogues of the concepts we have already seen for ordinary complex structures.

Definition 3.1. A generalized almost complex structure (GACS) is a map $\mathcal{J}\colon T \oplus T^* \to T \oplus T^*$ such that $\mathcal{J}^2 = -1_{6+6}$. We will always assume that it is also *hermitian*, namely it obeys $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$ where $\mathcal{I}$ is the natural metric on $T \oplus T^*$ given in (2.13). The metric (2.13) is just the pairing (,) between vectors and one–forms:

$$(v_1 + \xi_1, v_2 + \xi_2) = v_1 \llcorner \xi_2 + v_2 \llcorner \xi_1 \ ; \tag{3.2}$$

also, $\mathcal{I}$ it has signature $(6, 6)$. It reduces the structure group of $T \oplus T^*$ to $\mathrm{O}(6, 6)$, just like an ordinary metric reduces the structure group of T to $O(6)$.

The hermiticity condition implies that a generalized almost complex structure should have the form

$$\mathcal{J} = \begin{pmatrix} I & P \\ L & -I^t \end{pmatrix} , \tag{3.3}$$

with P and L antisymmetric matrices. The condition $\mathcal{J}^2 = -1_6$ imposes further constraints on I, P and L; for example, $I^2 + PL = -1_6$. $\mathcal{J}$ reduces the structure group of $T \oplus T^*$ further, to $\mathrm{U}(3,3)$.

Example 3.1. There are two "motivating" examples of GACS. One is associated to an ACS I and another to a non–degenerate two–form J:

$$\mathcal{J}_I \equiv \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix} \,, \qquad \mathcal{J}_J \equiv \begin{pmatrix} 0 & J \\ -J^{-1} & 0 \end{pmatrix} \,. \tag{3.4}$$

These two examples will be of fundamental importance for us.

It is also possible to give an integrability condition for a GACS. The role of $T_{(1,0)}$ is played by the i–eigenbundle

$$L_{\mathcal{J}} \equiv \{X \in T \oplus T^* | \mathcal{J}X = iX\} \;; \tag{3.5}$$

the projector on $L_{\mathcal{J}}$ reads

$$\Pi = \frac{1}{2}(1_{6+6} - i\mathcal{J}) \,. \tag{3.6}$$

$L_{\mathcal{J}}$ is null (or "isotropic") with respect to the metric $\mathcal{I}$ in (2.13), since, for $A, B \in L_{\mathcal{J}}$,

$$(A, B) = A\mathcal{I}B = A\mathcal{J}^t\mathcal{I}\mathcal{J}B = (iA)\mathcal{I}(iB) = -A\mathcal{I}B = -(A, B) \,. \tag{3.7}$$

Also it has the maximal dimension that a null space can have in signature $(6,6)$ (namely, 6), since $\Pi A \in L_{\mathcal{J}}$ for any real A. One says that $L_{\mathcal{J}}$ is a *maximally isotropic* subbundle of $T \oplus T^*$.

Example 3.2. In the examples (3.4), we have

$$L_{\mathcal{J}_I} = T^{1,0} \oplus (T^*)^{0,1} \,, \qquad L_{\mathcal{J}_J} = \{v^m + iv^m J_{mn}, \; \forall v = v^m \partial_m \in T\} \,. \tag{3.8}$$

Both have dimension six, as they should.

To define a notion of integrability, we also need a bracket that will play the role the Lie bracket had for T. There is no bracket satisfying the Jacobi identity on $T \oplus T^*$, but fortunately there is one that satisfies it when restricted on isotropic subbundles. This is the *Courant bracket*. We define it in analogy to (2.15):

$$\frac{1}{2}\Big([\{A\cdot, d\}, B\cdot] - [\{B\cdot, d\}, A\cdot]\Big) \equiv [A, B]_{\mathrm{Courant}}\cdot \,, \tag{3.9}$$

where A and B are sections of $T \oplus T^*$. Again all variables are considered as operators on differential forms: $A\cdot = v\llcorner + \zeta\wedge$, namely vectors act by contraction, and one–forms act by wedging. From now on we will write $[\ ,\ ]_{\mathrm{Courant}} = [\ ,\ ]_{\mathrm{C}}$. One can compute explicitly

$$[v + \zeta, w + \eta]_{\mathrm{C}} = [v, w] + \mathcal{L}_v\eta - \mathcal{L}_w\zeta - \frac{1}{2}d(\iota_v\eta - \iota_w\zeta) \,. \tag{3.10}$$

but fortunately this complicated formula is seldom needed.

In this formalism the definitions of the Lie (2.15) and Courant (3.9) brackets are very similar (indeed, Courant contains Lie as a particular case, when $A = v$ and $B = w$). The main feature of a derived bracket is that it contains a differential. For both Lie and Courant the differential is d, but one can generalize it to other differentials, as we will do in (3.23) below.

We are now ready to define integrablity for a GACS. As you will notice, this is very similar to the definition of integrability for an ACS given in section 2.3.

Definition 3.2. A generalized almost complex structure $\mathcal{J}$ is *integrable* if its i–eigenbundle $L_{\mathcal{J}}$ is closed under the Courant bracket:

$$[L_{\mathcal{J}}, L_{\mathcal{J}}]_{\rm C} \subset L_{\mathcal{J}} \ . \qquad (3.11)$$

In this case, $\mathcal{J}$ is called a generalized complex structure (GCS). A manifold on which such a tensor exists is called a generalized complex manifold.

In the two examples (3.4), the integrability of $\mathcal{J}$ turns into a condition on the building blocks, $I^m{}_n$ and J_{mn}. Integrability of $\mathcal{J}_I$ forces I to be an integrable almost complex structure on T and hence a complex structure. In other words, M_6 is complex. For $\mathcal{J}_J$, integrability imposes $dJ = 0$, thus making J into a symplectic form, and M_6 a symplectic manifold.

3.3 Pure spinors

We saw in section 2.1 a close relationship between almost complex structures I (that define U(3) structures) and Weyl spinors η (that define SU(3) structures). An analogous property holds on $T \oplus T^*$ between generalized almost complex structures and a new type of object that we will call *pure spinors*.

We noticed in (2.12) that the list of operators

$$\Gamma_\Lambda = \{\partial_1 \llcorner, \partial_2 \llcorner, \ldots, \partial_6 \llcorner, dx^1\wedge, dx^2\wedge, \ldots, dx^6\wedge\} \qquad (3.12)$$

satisfy a Clifford algebra with respect to the metric (2.13). Since the signature of that $\mathcal{I}$ is $(6,6)$, the corresponding Clifford algebra is called Cliff$(6,6)$. The spinor bundle is then nothing but the bundle of differential forms of all degrees, $\Lambda^\bullet T^* = \sum_p \Lambda^p T^*$. In other words,

$$\text{differential forms} = \text{Cliff}(6,6) \text{ spinors} \ ; \qquad (3.13)$$

the gamma matrices are (3.12). Forms of even (odd) degree can then be thought of as spinors of positive (negative) chirality. Chirality in this sense will be denoted by an index, as in $\Phi_\pm$.

One can also define an inner product between two forms A and B:

$$\langle A, B\rangle \equiv (A \wedge \lambda(B))_6 \tag{3.14}$$

where $_6$ means we only keep the six–form part, and λ was defined in (1.12). A top form is proportional to the volume form vol_6, which means that using the volume form one can extract a number from the Mukai pairing (the constant of proportionality). In $d = 6$ this pairing is antisymmetric; it is then convenient to define the norm of a form Φ as

$$\langle \Phi, \bar\Phi\rangle = -i||\Phi||^2 \mathrm{vol} \ . \tag{3.15}$$

One defines the *annihilator* of a $\mathrm{Cl}(6, 6)$ spinor as

$$L_\Phi = \{v + \zeta \in T \oplus T^* \,|\, (v + \zeta) \cdot \Phi = 0\} \ . \tag{3.16}$$

From standard Clifford algebra, $(X\cdot)^2 = (X, X)1$; because of this, the annihilator space L_Φ of any spinor Φ is isotropic. It can have at most dimension 6, in which case it is maximally isotropic.

Definition 3.3. A *pure spinor* is a form such that L_Φ has dimension 6 and which is non–degenerate (namely, $(\Phi, \bar\Phi)$ never vanishes). As for GACS, we have two "motivating" examples of pure spinors:

Example 3.3.

$$\Phi_\Omega = \Omega \ , \qquad \Phi_J = e^{-iJ} \ . \tag{3.17}$$

The first is a decomposable form (as defined in section 2.1); in the second, J is a non–degenerate two–form. The annihilators are given by

$$L_{\Phi_\Omega} = T^{1,0}_{I_\Omega} \oplus (T^*)^{0,1}_{I_\Omega} \ , \qquad L_{\Phi_J} = \{v^m + iv^m J_{mn}, \ \forall v = v^m\partial_m \in T\} \ ; \tag{3.18}$$

they both have dimension 6. It is also easy to check that both Φ in (3.17) are non–degenerate.

L_Φ for a pure spinor Φ is maximal isotropic; so it makes sense to associate

$$\mathcal{J} \ \leftrightarrow \Phi \qquad \text{if} \ \ L_\mathcal{J} = L_\Phi \ \ , \tag{3.19}$$

which means, we recall, that the i–eigenbundle of $\mathcal{J}$ is equal to the annihilator of Φ. An alternative definition of the generalized almost complex structure $\mathcal{J}$ associated to Φ, maybe more suitable for computations, can be given by

$$\mathcal{J}_{\pm}^{\Lambda\Sigma} = \langle \mathrm{Re}(\Phi_\pm)), \Gamma^{\Lambda\Sigma}\mathrm{Re}(\Phi_\pm)\rangle \ , \tag{3.20}$$

where Γ_Λ are the gamma matrices of Cliff$(6,6)$ given in (3.12).

The correspondence (3.19) is not exactly one to one because rescaling Φ does not change its annihilator L_Φ. Hence, it is more convenient to think about a correspondence between a $\mathcal{J}$ and a *line bundle* of pure spinors. This line bundle need not have a global section, in general; when it does, the structure group on $T \oplus T^*$ is further reduced from $\mathrm{U}(d/2)\times\mathrm{U}(d/2)$ (which was already accomplished by $\mathcal{J}$) to $\mathrm{SU}(d/2)\times\mathrm{SU}(d/2)$.

Example 3.4. Going back to our examples (3.17) of pure spinors, by comparing (3.8) with (3.18), we see that their associated GACS are given by (3.4),

What is remarkable about this correspondence is that integrability of $\mathcal{J}$ can be reexpressed in terms of Φ.

Theorem 3.1. *Let $\mathcal{J}_\Phi$ be the GACS associated to a pure spinor Φ via (3.19). Then $\mathcal{J}_\Phi$ is integrable if and only if there exists a $W \in T \oplus T^*$ such that*

$$d\Phi = W \cdot \Phi \ . \tag{3.21}$$

Proof. Let $A, B \in L_{\mathcal{J}}$. By (3.9) we have

$$[A, B]_{\mathrm{C}}\, \Phi = (AB - BA) \cdot d\Phi \ . \tag{3.22}$$

Assume (3.21) holds. Let us now think of Φ as a Clifford vacuum, of L_Φ as annihilators, and $\bar{L}_\Phi$ as creators. We can think of $W \cdot \Phi$ as obtained from Φ by action of one creator; but then the right hand side of (3.22) is the action of two annihilators on it, and so it is zero. This means that $[A, B]_{\mathrm{C}}\Phi = 0$, or in other words that $[A, B]_{\mathrm{C}} \in L_\Phi = L_{\mathcal{J}}$, which means that $\mathcal{J}$ is integrable. We can also follow the argument backwards to show the opposite implication. □

Example 3.5. If we use $\Phi = \Omega$, then theorem 3.1 reduces to 2.2. For $\Phi = e^{-iJ}$, we reproduce the fact that integrability of $\mathcal{J}_J$ in (3.4) is given by $dJ = 0$.

Definition 3.4. M_6 is a (weakly) *generalized Calabi-Yau* (GCY) if there exists on it a pure spinor Φ which is closed. (A more appropriate, but less catchy, name would be "generalized complex manifold with holomorphically trivial canonical bundle".)

Suppose Φ is a closed pure spinor. One can show that $e^B\Phi$ is still pure (it is its "B–transform"). However, in general it is not closed. But it is closed under $d_H = d - H\wedge$, where $H = dB$ is the curvature of B. From (1.11) and $dH = 0$, we see that d_H is a differential (it squares to zero). So we can use it to define a modified Courant bracket, the *twisted Courant* , $[\ ,\ ]_H$

$$[\{A\cdot, (d - H\wedge)\}, B\cdot] \equiv [A, B]_H \cdot \ . \tag{3.23}$$

This new bracket has all the good properties of the Courant bracket. Moreover, we can now retrace all the steps of the correspondence between generalized complex structures and pure spinors, to get an analogue of the theorem (3.1) in which $d \to d_H$ and integrability is replaced by integrability with respect to (3.23). A manifold on which there exists a pure spinor Φ which is closed under d_H is called *twisted generalized Calabi–Yau*.

Remarkably, one can actually prove [9] that in any dimension, a pure spinor must have the form

$$\Phi = \Omega_k \wedge e^{B+ij} \tag{3.24}$$

where Ω_k is a complex k–form and B, j are two real two–forms. Hence the most general pure spinor is a hybrid of the two examples in (3.17).

Finally, we can ask how many pure spinors there are on M_6. It turns out that a pure spinor in dimension 6 is determined by its real part [8], which belongs to either $\Lambda^\pm$ (the space of even or odd forms; both have real dimension 32). Moreover, the space of real forms (either odd or even) that can be real parts of a pure spinor is open in $\Lambda^\pm$, so it also has dimension 32. (This is similar to the result in the case of decomposable three–forms, at the end of section 2.1.) Hence the space of pure spinors at a given point has real dimension 32.

3.4 Metric from U(3) × U(3) structures

We have seen how the existence of a generalized almost complex structure reduces the structure group of $T \oplus T^*$ from O(6, 6) to U(3, 3). We will now see how to reduce the structure group further to its maximal compact subgroup, U(3)×U(3)

We will consider two GACS that commute, $[\mathcal{J}_1, \mathcal{J}_2] = 0$. There are two remarks to be made about such a situation. First, since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, they can be diagonalized simultaneously, and one can divide the complexified $T \oplus T^*$ in four sub–bundles of rank 3:

$$\begin{aligned} L_{++} &= L_{\mathcal{J}_1} \cap L_{\mathcal{J}_2} \ , \qquad & L_{+-} &= L_{\mathcal{J}_1} \cap \bar{L}_{\mathcal{J}_2} \\ L_{-+} &= \bar{L}_{\mathcal{J}_1} \cap L_{\mathcal{J}_2} \ , \qquad & L_{--} &= \bar{L}_{\mathcal{J}_1} \cap \bar{L}_{\mathcal{J}_2} \ . \end{aligned} \tag{3.25}$$

Second, the product

$$G = -\mathcal{J}_1\mathcal{J}_2 \tag{3.26}$$

has the properties

$$G^2 = 1_{6+6} \ , \qquad \mathcal{I}G = G^t\mathcal{I} \tag{3.27}$$

where recall again that $\mathcal{I}$ was defined in (2.13). It is easy to see [9, Chap. 6] that these two properties imply that G has the form

$$\begin{aligned} G = -\mathcal{J}_1\mathcal{J}_2 = & \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \\ = & \mathcal{E} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{E}^{-1} \ ; \qquad \mathcal{E} = \begin{pmatrix} 1 & 1 \\ g+B & -g+B \end{pmatrix} \ . \end{aligned}$$

with g symmetric and non–degenerate, and B antisymmetric. So we see that *two commuting GACS determine a metric and a B–field.*

Definition 3.5. Two GACS $\mathcal{J}_1$ and $\mathcal{J}_2$ are said to be *compatible* if they commute and if the metric they determine is positive definite. Two compatible GACS structure group reduce the structure group of $T \oplus T^*$ to U(3)×U(3).

Let me make a few more remarks about (3.28).

- The metric $M = \mathcal{I}G$ appeared in the context of T–duality (see for example [10]), where it was noted that it transforms by conjugation under $\mathrm{Sl}(2,\mathbb{R})$.

- Both $\mathcal{J}_1$ and $\mathcal{J}_2$ in a U(3) × U(3) structure commute with the projector G. From (3.28), it follows that

$$\mathcal{J}_{1,2} = \mathcal{E} \begin{pmatrix} I_1 & 0 \\ 0 & \pm I_2 \end{pmatrix} \mathcal{E}^{-1} \ , \tag{3.28}$$

  where $I_\pm$ are two ACS. Hence $\mathcal{J}_{1,2}$ are determined by g, B and $I_\pm$.

- It is easy to see that the transformation $\Phi \to \exp\left(\begin{smallmatrix} -\omega^t & \beta \\ B & \omega \end{smallmatrix}\right)\Phi$ corresponds under (3.19) to the conjugation $\mathcal{J} \to O\mathcal{J}O^{-1}$, where $O = \exp[(B\wedge + \frac{1}{2}\omega_m{}^n[dx^m\wedge, \iota_{\partial_n}] + \iota_\beta]$. Hence the matrix $\left(\begin{smallmatrix} 1 & 0 \\ B & 1 \end{smallmatrix}\right)$ that appears in (3.28) is induced by the B–transform $\exp[B\wedge]$, whose effect on pure spinors was discussed in the previous subsection.

Example 3.6. One can check that the two GACS in (3.4) are compatible if and only if $g = IJ$ is symmetric. Remember that, in this situation, I is a hermitian ACS, namely a U(3) structure. In this case, the two ACS in (3.28) are $I_1 = I_2 = I$.

We end this subsection with a definition:

Definition 3.6. If two compatible $\mathcal{J}_{1,2}$ are also integrable, they define a generalized Kähler structure, as defined in [9]. This condition is equivalent to the existence of $(2,2)$ σ–models. The two ACS in (3.28) are then integrable, and they play the role of complex structures for the left–movers and right–movers. The world–sheet applications of generalized complex geometry are very interesting, but unfortunately the world–sheet treatment of RR fluxes is not the easiest aspect of string theory. For that reason, we will not discuss it any further in these lectures.

3.5 Compatible pairs

We now get to the most important point of this lecture: we will learn how many of the data of a supergravity vacuum can be summarized by a pair of pure spinors with an appropriate compatibility condition.

Definition 3.7. Two pure spinors $\Phi_{1,2}$ are said to be a *compatible pair* if the corresponding $\mathcal{J}_{1,2}$ are compatible, and if their norms are the same: $||\Phi_1|| = ||\Phi_2||$. Fortunately, in six dimensions the condition that $[\mathcal{J}_1, \mathcal{J}_2] = 0$ can be reformulated directly as a constraint on $\Phi_{1,2}$. First of all, $\Phi_{1,2}$ should have opposite parity. So from now on we will denote them by $\Phi_\pm$.

$$(\Phi_-, X\Phi_+) = 0 = (\bar{\Phi}_-, X\Phi_+) \qquad\qquad \forall X \in T \oplus T^* \ . \tag{3.29}$$

I will not give the proof of the equivalence of this claim here; it can be found in [11, App. A].

Example 3.7. The two examples (3.17) of pure spinors are compatible if and only if (J, Ω) describe an SU(3) structure.

Remember now the Clifford ("slash") map (1.15); as we said there, we will not actually write the slash every time. So we will mix in some equations bispinors with differential forms, such as in (3.32) below.

By now, it should come as no surprise that a compatible pair of pure spinors reduces the structure group of $T \oplus T^*$ to $\mathrm{SU}(3) \times \mathrm{SU}(3)$.

Theorem 3.2. *The Clifford algebra* $Cl(6,6)$ *is isomorphic to two copies of the ordinary Clifford algebra Cl(6).*

Proof. From the Cl(6) algebra $\{\gamma^m, \gamma^n\} = 2g^{mn}$, we get the following action of a γ^m

matrix acts on the left and on the right of the $\gamma^{m_1\ldots m_k}$:

$$\begin{aligned}\gamma_m \gamma^{m_1\ldots m_k} =& \gamma_m{}^{m_1\ldots m_k} + k\delta_m{}^{[m_1}\gamma^{m_2\ldots m_k]}\\ \gamma^{m_1\ldots m_k}\gamma_m =& (-1)^k\left(\gamma_m{}^{m_1\ldots m_k} - k\delta_m{}^{[m_1}\gamma^{m_2\ldots m_k]}\right)\ .\end{aligned} \tag{3.30}$$

In other words, the gamma matrix action on bispinors is mapped to the following action on forms α_k of degree k:

$$\gamma^m\,\alpha_k = [(dx^m\wedge + g^{mn}\iota_n)\alpha_k]\ , \qquad \alpha_k\,\gamma^m = (-)^k[(dx^m\wedge - g^{mn}\iota_n)C_k]\ . \tag{3.31}$$

These equations express the Γ^Λ in (3.12) in terms of gamma matrices acting from the left or on the right. These are two copies of the Clifford algebra Cl(6).

□

Here is now the central result:

Theorem 3.3. *Any compatible pair of pure spinors $\Phi_\pm$ can be written as*

$$\Phi_\pm = e^{B\wedge}\eta^1_+\otimes\eta^{2\,\dagger}_\pm \tag{3.32}$$

for some two–form B and ordinary (Cliff(6)) spinors $\eta^{1,2}_+$ (with, as usual, $\eta^{1,2}_- = (\eta^{1,2}_+)^*$).

Proof. Let us first show that (3.32) are a compatible pair.

First of all, it is easy to see that the operation $\Phi_\pm \to e^{B\wedge}\Phi_\pm$ (the "B–transform" we saw earlier) sends compatible pairs into compatible pairs. For this reason, we lose nothing if we give the proof for $B = 0$.

Next, notice that each of the Cl(6) spinors $\eta^{1,2}_+$ is pure, again in the sense that it is annihilated by half (in this case 3) of the gamma matrices. We have already used this in (2.9); it is true for Cl(d) with $d \le 6$ (but not for $d > 6$). We can call the annihilators γ^{i_1} and γ^{i_2}; so, $\gamma^{i_a}\eta^a_+ = 0$. (The notation here is that the index i_a is holomorphic with respect to the complex structure I_a.)

Now, $\eta^1_+\otimes\eta^{2\,\dagger}_\pm$ are annihilated by γ^{i_1} acting on the left, $\overrightarrow{\gamma}^{i_1}$, and by $\gamma^{\bar\imath_2}$ (γ^{i_2}) acting on the right, $\overleftarrow{\gamma}^{i_2}$ ($\overleftarrow{\gamma}^{\bar\imath_2}$). Thanks to (3.31), we can translate these 3 + 3 annihilators into 6 annihilators in Cl(d,d). This means $\Phi_\pm$ are both pure.

Hence Φ_+ and Φ_- share three annihilators: the three gamma matrices $\overrightarrow{\gamma}^{i_1}$. Call L_{++} this subbundle of $(T\oplus T^*)\otimes\mathbb{C}$. Similarly, Φ_+ and $\overline{\Phi}_- = \eta^1_-\otimes\eta^{2\,\dagger}_+$ are both annihilated by the three gamma matrices $\overleftarrow{\gamma}^{\bar\imath_2}$. Call this bundle L_{+-}. In this way we construct four bundles $L_{\pm\pm}$ of dimension 3 each. Now, $\mathcal{J}_+$ (the GACS associated to Φ_+) is i on $L_{+\pm}$ and $-i$ on $L_{-\pm}$. In the same way, $\mathcal{J}_-$ (the GACS associated to Φ_-) is i on $L_{\pm+}$ and $-i$ on $L_{\pm-}$. This implies that $[\mathcal{J}_+, \mathcal{J}_-] = 0$. The bundles $L_{\pm\pm}$ are simply the ones we had defined in (3.25).

It remains to see that the two norms are equal. This follows easily from the formulas

$$\langle \alpha_k, \beta_{6-k} \rangle = \frac{1}{8}(-)^{k+1}\mathrm{Tr}(\alpha_k * \beta_{6-k})\,\mathrm{vol} \tag{3.33}$$

and

$$\alpha\ \gamma = i\lambda(*\alpha)\ ; \qquad \gamma\ \alpha = -i * \lambda(\alpha)\ . \tag{3.34}$$

This finishes the proof that (3.32) is a compatible pair. The proof that it is the most general is not much more difficult: one uses the bundles $L_{\pm\pm}$ to define four sets of gamma matrices, and one uses those to define the gamma matrices. More details can be found in [7, Sec. 3.4]. □

We conclude once again with a counting argument. We ask how many compatible pairs of pure spinors there are at a given point. We know from the end of section 3.3 that the space of either $\Phi_\pm$ is 32. The compatibility constraint (3.29) can be seen to count as 12 real constraints (rather than 12 complex ones, as one might naively think). Finally, the equal norm constraint is one real constraint. This gives a grand total of $32 + 32 - 12 - 1 = 51$.

We can compare this with the data that this encodes by theorem 3.3, namely $(g, B, \eta_+^{1,2})$. This is $21 + 15 + 8 + 8 = 52$, which is one more than the 51 we got before.

To see why this does not contradict theorem 3.3, let us go back to equation (3.32). It gives a map $(g, B, \eta_+^{1,2}) \to \Phi_\pm$; theorem (3.3) does not say that this map is invertible, but that it is *surjective*. The discrepancy between the dimensions 52 and 51 is then simply due to the fact that this map "forgets" one dimension out 52. It is easy to see why: (3.32) is invariant under

$$\eta^1 \to a\eta^1\ , \qquad \eta^2 \to a^{-1}\eta^2\ . \tag{3.35}$$

4 Compactifications with RR fluxes

In this section we will finally attack the general case with $f \neq 0$. The geometry we have learned so far will help us describe a general result.

Let me repeat here, just in case you decoupled during the previous section, a summary of the previous section. Theorem 3.3 tells us that many of the data of a supergravity vacuum can be reformulated in terms of two differential forms $\Phi_\pm$ obeying a certain set of *algebraic* constraints. These forms define a "so–called $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure". You also need to know the most important example of solution to those constraints, namely (3.17).

4.1 Branes and orientifolds

We also need some preliminary definitions of stringy ingredients.

Definition 4.1. A *supersymmetric cycle* B with respect to the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure $\Phi_{1,2}$ is a submanifold of M_6 such that $(\mathrm{Im}\Phi_2)_{|B} = \mathrm{vol}_B$. (This also implies that $\iota^*\mathrm{Re}\Phi_2 = 0$ and that $\iota^*(X\cdot\Phi_1) = 0$ for any $X \in T\oplus T^*$.) Here vol_B is the *Born–Infeld volume form* computed using $\det(g+b)$, with g and b determined by $\Phi_{1,2}$. We will also say that B is *almost calibrated* by $\mathrm{Im}\Phi_2$. We will denote by δ_B the current supported on B, normalized such that $\langle\delta_B, \mathrm{Im}\Phi_2\rangle = \delta(B)$. Locally, if B is described by a set of equations $\{f_i = 0\,, i = 1,\ldots,d\}$, one can write $\delta_B = \delta(f_1)\ldots\delta(f_d)\, df_1\wedge\ldots\wedge df_d$. Notice that so far we are not assuming that any of the forms that define the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure is closed; if $\mathrm{Im}\Phi_2$ were closed, we would drop the "almost" and we would just say that B is calibrated by it. However, the case of interest for us will be precisely when $\mathrm{Im}\Phi_2$ is not closed.

Let now σ be an involution on M_6 with fixed locus O_σ.

Definition 4.2. The $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure $\Phi_{1,2}$ is *compatible with* σ if

$$\sigma^*\Phi_1 = -(-)^{\mathrm{Int}(\frac{p}{2})}\lambda(\Phi_1)\ , \qquad \sigma^*\Phi_2 = (-)^{\mathrm{Int}(\frac{p-1}{2})}\lambda(\bar\Phi_2) \qquad (p = \dim(O_\sigma)+3)\ . \tag{4.1}$$

We also call σ an *orientifold involution* for the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure $\Phi_{1,2}$. The rules of string theory dictate that there should be a source δ_{O_σ} on O_σ, normalized such that

$$\langle\delta_{O_\sigma}, \mathrm{Im}\Phi_2\rangle = -\delta(O_\sigma)\ ; \tag{4.2}$$

notice the sign difference with respect to the normalization for branes.

4.2 The pure spinor equations

Theorem 3.3 says that some of the data of a vacuum, namely $(g, B, \eta^{1,2})$, can be encoded in a compatible pair of pure spinors $\Phi_\pm$:

$$(g_{\Phi_\pm}, B_{\Phi_\pm}, \eta^{1,2}_{\Phi_\pm}) \to \Phi_\pm \tag{4.3}$$

We will now show that the supersymmetry equations themselves can also be encoded in some elegant differential equations on $\Phi_\pm$.

In this section we will use the notation

$$\mathrm{IIA}: \quad \begin{array}{c}\Phi_1 = \Phi_+ \\ \\ \Phi_2 = \Phi_-\end{array} \qquad\qquad \mathrm{IIB}: \quad \begin{array}{c}\Phi_1 = \Phi_- \\ \\ \Phi_2 = \Phi_+\end{array} \quad . \tag{4.4}$$

We can now get

Theorem 4.1. *The supersymmetry equations (1.26) are equivalent to the system*

$$\boxed{\begin{array}{l} d_H\Phi_1 = 0 \ , \qquad d_H(e^{-A}\mathrm{Re}\Phi_2) = c_- f \ , \\ d_H(e^{A}\mathrm{Im}\Phi_2) = c_+ e^{4A} * \lambda f \end{array}} \qquad (\Lambda = 0) \ , \tag{4.5a}$$

where $c_\pm$ are constants, and $c_+ \neq 0$[7]*, when $M_4 = \mathrm{Mink}_4$, and to the system*

$$\boxed{d_H\Phi_1 = -2\mu\, e^{-A}\mathrm{Re}\Phi_2 \ , \qquad d_H(e^{A}\mathrm{Im}\Phi_2) + 3\mu\,\mathrm{Re}\Phi_1 = e^{4A} * \lambda f} \qquad (\Lambda = -3\mu^2 < 0) \tag{4.5b}$$

when $M_4 = \mathrm{AdS}_4$, where $\Phi_\pm$ is an $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure such that $B_{\Phi_\pm} = 0$; $*$ is the six–dimensional Hodge star determined by the metric $g_{\Phi_\pm}$.

The solutions of (4.5) and (1.26) are related to one another via theorem 3.3 (with $B_{\Phi_\pm} = 0$). The invariance (3.35) is fixed by $||\eta^1||^2 \pm ||\eta^2||^2 = c_\pm e^{\pm A}$ in the Minkowski case; in the AdS case, the norms need to be equal, and the (3.35) is fixed by $||\eta^1||^2 = ||\eta^2||^2 = e^A$. The dilaton is determined by

$$e^{3A-\phi} = ||\Phi_1|| = ||\Phi_2|| \ . \tag{4.6}$$

The proof of this theorem can be found in [7, App. A]. We are not going to give it here, but its spirit is similar to (2.25), although there are several technical differences. There is also a version of (4.5) in which the $*$ does not appear [11], that looks especially nice for $M_4 = \mathrm{Mink}_4$.

If we take (4.5) and act on it by d_H, we see that we find $d_H(e^{4A} * \lambda f) = 0$, which is the third equation in (1.27); it is actually even true everywhere, and not just almost everywhere. In other words, there are no electric sources. This is reasonable: a source for f would be a brane that looks like a point in M_4, and that would break maximal symmetry, which is part of our definition of "vacuum".

[7] As we will discuss, compact solutions require $c_- = 0$; one can then work in conventions where $c_+ = 1$. Sometimes the equations are quoted in this simplified form.

The other two equations in (1.27) still need to be imposed separately. The good news is that, once one does that, all the equations of motion follow. That gives a vacuum. To simplify the discussion, we will assume in what follows that we have no NS sources, so that $dH = 0$. We then have:

Theorem 4.2. *There is a supersymmetric vacuum with internal space M_6, with warping A and internal fluxes H, f if and only if there exists on M_6 an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure $\Phi_{1,2}$ (such that $B_{\Phi_\pm} = 0$) and supersymmetric branes B_i such that*

- *(4.5) holds (namely, (4.5a) if $M_4 = \mathrm{Mink}_4$ and (4.5b) if $M_4 = \mathrm{AdS}_4$);*
- *$dH = 0$ and $df = \delta$,*

where, for δ, one of the two following possibilities is realized:

a) $\delta = \sum_i \delta_{B_i}$, with B_i supersymmetric cycles;

b) there exists an orientifold involution σ for the $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure $\Phi_{1,2}$, and $\delta = -\delta_{O_\sigma} + \sum_i \delta_{B_i}$, again with B_i supersymmetric cycles; as we noticed before, O_σ is automatically supersymmetric. (This case is the only possibility if $M_4 = \mathrm{Mink}_4$ and M_6 is compact).

The proof of this is in several papers. The fact that supersymmetry and the Bianchi identities imply the Einstein equations and the equations of motion for ϕ has been obtained in [12]; the equation of motion for H was shown in [13]. Finally, the last statement in *b)* follows from an older and more general argument in [14–16]; it canalso be rederived [7, Eq. (4.13)] by an integration by parts. Here is how. We compute:

$$\int e^A \langle \delta, \mathrm{Im}\Phi_2 \rangle = \int \langle d_H f,\ e^A \mathrm{Im}\Phi_2 \rangle = \int \langle f,\ d_H(e^A \mathrm{Im}\Phi_2) \rangle = \int e^{4A} \langle f,\ *\lambda(f) \rangle \leq 0\ , \tag{4.7}$$

where we have used the self–adjointness of d_H with respect to $\langle\ ,\ \rangle$ (which can be easily proved), and (4.5a). Looking back at (4.2) and at the definition of brane, we see that this means that there has to be an orientifold plane somewhere.

Notice that, when M_6 is compact, since we need at least an orientifold source, $\delta \neq 0$. If $c_- \neq 0$ in (4.5a), by acting on the second equation with d_H we get $d_H f = 0$, which means $\delta = 0$. Hence, if M_6 is compact, we need to take $c_- = 0$.[8] From now on, we will take

$$c_- = 0\ , \qquad c_+ = 1\ . \tag{4.8}$$

[8] This would not have been true had we not assumed $dH = 0$ earlier; in that case, dH might be non–zero, and $d_H^2 \neq 0$. One can indeed get solutions with $dH \neq 0$ and $c_- \neq 0$ by applying $Sl(2, \mathbb{Z})$ duality to other solutions, but these solutions are simply old solutions in an unfamiliar frame.

Finally, before we go on to analyze some particular cases, we notice a corollary of theorem 4.1. The first equation in (4.5a) says that the internal manifold M_6 must be a weakly generalized Calabi–Yau manifold, as defined in section 3.3.

4.3 Minkowski vacua and SU(3) structures

One might think one does not gain much by reformulating (1.26) as (4.5), since we still have the algebraic constraints on the differential forms $\Phi_\pm$ for them to be an SU(3)×SU(3) structure.

However, the most general solution to those algebraic constraints is now known [17,18]. So really all we would need to do to obtain a complete classification of supersymmetric vacua would be to solve some differential equations.

As one can imagine, that is more easily said than done. In this section, we will content ourselves with using a particular class of SU(3) × SU(3) structure: the "SU(3) structure" case in (3.17). In fact, slightly more generally, we are going to write

$$\Phi_+ = e^{3A-\phi} e^{i\theta} e^{-iJ} \ , \qquad \Phi_- = \Omega \ . \tag{4.9}$$

All we have done here is to rescale Φ_+ by a complex number. This does not change any of the considerations in section 3, and it allows us to reproduce (4.6) automatically. The angle θ is a new piece of data.

We notice right away the following result, from (4.4) and the first equation in (4.5a) gives

$$\begin{aligned} &\mathrm{IIA}: \ dJ = 0 \qquad (\Rightarrow M_6 \text{ is symplectic}); \\ &\mathrm{IIB}: \ d\Omega = 0 \qquad (\Rightarrow M_6 \text{ is complex}). \end{aligned} \tag{4.10}$$

As we remarked (without proof) in theorem 4.2, in the Minkowski case we need an orientifold if we want M_6 to be compact. We will use the type of the orientifold to organize the various possibilities. It should be noted, however, that we will make some assumptions along the way, and thus our discussion will not be exhaustive.

4.3.1 Kähler manifolds from O7/03 planes

This is the most important case: it leads to the type of geometry which is under highest technical control, and it hence has given so far most of the examples of supersymmetric vacua we know.

As one can see from (4.1), $\sigma^*\theta = \pi - \theta$ for O7/O3 projections, so $\theta = \frac{\pi}{2}$ on O7 and on O3 planes. In this section we just **assume** that $\theta = \frac{\pi}{2}$ everywhere. This leads to

$$\begin{aligned} & d\Omega = 0 \ , \qquad d\tilde{J} = 0 \ , \qquad H \wedge \Omega = H \wedge \tilde{J} = 0 \\ & \tilde{J} \wedge \Omega = 0 \ , \qquad i\Omega \wedge \bar{\Omega} = \frac{3}{4} e^{\phi} \tilde{J}^3 \\ & * f_1 = -\frac{1}{2} e^{-4A} d(e^{\phi} \tilde{J}^2) \ , \qquad * f_3 = e^{-\phi} H \ , \qquad * f_5 = e^{-4A} d(e^{4A-\phi}) \ . \end{aligned} \tag{4.11}$$

where $\tilde{J} = e^{2A-\phi} J$. The first two lines constrain the geometry alone. We have a complex structure with $c_1 = 0$ from $d\Omega = 0$, a symplectic structure from $d\tilde{J} = 0$, and the two are compatible because $J \wedge \Omega = 0$. This is called Kähler geometry. Our original definition of SU(3) structure requires $J^3 = \frac{4}{3} i\Omega \wedge \bar{\Omega}$, which is different from what we have on (4.11). Because of this, we cannot quite say M_6 is a Calabi–Yau: it would be if $\phi = \text{const}$. Interestingly, however, Yau's theorem can still be used to show that a solution exists.

Among the equations for the flux, especially notable is the one that relates f_3 and H. A combination which is often used in IIB supergravity is

$$G = f_3 - i e^{-\phi} H \ . \tag{4.12}$$

Another way of expressing this is to introduce[9] $f_3^0 = f_3 + HC_0$. When $df_3^0 = 0$, we have $0 = d(f_3^0 + HC_0) = df_3^0 - Hf_1$, which is part of (1.27). Then we can also write $G = f_3^0 - \tau H$, where

$$\tau = C_0 + i e^{-\phi} \tag{4.13}$$

is another useful combination in IIB: the so–called *axio–dilaton*.

Now, from (4.11) we have

$$*G = iG \ , \tag{4.14}$$

that is, G is *imaginary self–dual*. In general, just as a consequence of the definitions of a SU(3) structure, one can show that the action of $*$ on the space of three–forms is given by

$$*\Omega = -i\Omega \ , \qquad *\alpha^0_{2,1} = i\alpha^0_{2,1} \ , \qquad *(\alpha_{1,0} \wedge J) = -i\alpha_{1,0} \wedge J \ , \tag{4.15}$$

plus the complex conjugates of these equations. (The superscript ^{0} means "primitive": namely, $\alpha^0_{2,1} \wedge J = 0$.) Thus, G can only have components that are $(2,1)$ and primitive, or $(0,3)$, or of the form $\alpha_{0,1} \wedge J$. However, $H \wedge J = 0$ and $H \wedge \Omega = 0$ in (4.11) show that the latter two components should be zero. That means

$$G = G^0_{2,1} \ , \tag{4.16}$$

[9] I am afraid most people call $f_{\text{here}} = \tilde{f}_{\text{most people's}}$, and $f^0_{\text{here}} = f_{\text{most people's}}$.

namely, G is $(2,1)$ and primitive.

We can also massage the equation for f_1 a bit further. Using $*\left((\alpha_{1,0}+\alpha_{0,1})\wedge \frac{J^2}{2}\right) = i(\alpha_{1,0}-\alpha_{0,1})$, we get

$$f_1 = -i(\bar\partial - \partial)(e^{-\phi}) \tag{4.17}$$

(which would have been obtained more quickly from the formulation in [11] we mentioned earlier), which in turn means

$$\bar\partial \tau = 0\ . \tag{4.18}$$

The observation that the axio–dilaton is holomorphic is at the core of a non–perturbative completion of IIB called *F–theory*: this essentially promotes τ from holomorphic function to holomorphic section of a certain line bundle. We will not discuss it here, though

A special subcase which is worth commenting about further is $f_1 = 0$. In that case, we get $d(e^{\phi}\tilde J^2) = 0$. This implies $d\phi \wedge \tilde J^2 = 0$, and hence $\phi = \text{const}$. We can simplify then the expression to $f_5 = 4dA$. More importantly, we now have that $d\tilde J = 0 = d\Omega$, and $i\Omega \wedge \bar\Omega = \frac{4}{3}\tilde J^3$. This means that there is a Calabi–Yau metric on M_6! Remember though: this is not the metric determined by the original pure spinors $\Phi_\pm$; thus, it is not really the metric in the supersymmetry equations (1.26). This is because we have rescaled $\tilde J = e^{2A-\phi}J = e^{2A}J$. This means that $e^{2A}ds^2_6$ is a Calabi–Yau. In this case, one says M_6 is "conformally Calabi–Yau".

Finally, the Bianchi identity for f_5 becomes a Laplacian equation once one assumes $*f_5 = d\alpha$ for some function α.

4.3.2 Complex manifolds from O5 planes

In this case, O_σ is a subspace of dimension two. As one can see from (4.1), $\sigma^*\theta = -\theta$ for O5 projections, so $\theta = 0$ on O5 planes. Once again, we will actually just **assume** $\theta = 0$ everywhere. This leads to

$$\begin{aligned} &d\Omega = 0\ , \qquad dJ^2 = 0\ , \qquad \phi = 2A + \text{const} \\ &f_1 = f_5 = 0\ , \qquad f_3 = -e^{-4A} * d(e^{2A}J)\ , \qquad i\Omega\wedge\bar\Omega = \frac{4}{3}e^{2A}J^3\ . \end{aligned} \tag{4.19}$$

Since $d\Omega = 0$, from theorem 2.2 it follows that the space is complex; but, since $dJ \neq 0$ in general, it is not Kähler any more.

It is not easy to find examples of this type (recall we also have to satisfy (1.27)). Some have been found in [7] in a certain approximation.

4.3.3 Symplectic manifolds from O6 planes

In IIA we have

$$\begin{aligned} &dJ = H = 0 \ , \qquad d(e^{-A}\mathrm{Re}\Omega) = 0 \ , \qquad f_0 = f_4 = f_6 = 0 \ , \\ &d(e^{A}\mathrm{Im}\Omega) = -e^{4A} * f_2 \ , \qquad 3A = \phi + \mathrm{const} \ , \qquad \theta = \mathrm{const} \ , \qquad i\Omega \wedge \bar{\Omega} = \frac{4}{3} J^3 \ . \end{aligned} \tag{4.20}$$

The actual value of θ is irrelevant this time, in sharp contrast with the IIB cases we saw earlier. We see that this time $dJ = 0$; this means that M_6 is symplectic. The equation for the flux can be reformulated in terms of a certain "Lifschitz differential" [11].

It can be shown with some work that any solution of this type actually corresponds to the reduction of a (singular) G_2 manifold. This is done by introducing an M_7 which is described as follows. In our case, (1.27) read $df_3 = \delta_3$; this δ_3 gives the locus where an O6 and possibly D6's (all of which are on subspaces of dimension 3 of M_6) are located. On M_6 minus these loci, we can describe M_7 as the total space of the U(1) bundle whose curvature is f_2, with the usual M–theory metric

$$ds_7^2 = e^{\frac{4}{3}\phi}(dz + C_1)^2 + e^{-\frac{2}{3}\phi} ds_6^2 \ . \tag{4.21}$$

where z is the coordinate on the M–theory circle. Close to the loci of the D6's and O6, the metric has the explicit form known from their flat space solutions: in particular, the size of the M–theory circle goes to zero on the D6's and explodes on the O6.

Now, if we introduce the form

$$\phi = e^{-3A}\mathrm{Im}\Omega + (dz + C_1) \wedge J \ , \tag{4.22}$$

one can see that it defines a G_2 structure on M_7. If one imposes

$$d\phi = d *_7 \phi = 0 \ , \tag{4.23}$$

one gets that M_7 has actually G_2 *holonomy* (this is similar to the fact that (2.23) is equivalent to SU(3) holonomy in six dimensions). Now, if one plugs (4.22) into (4.23), one gets exactly the conditions in (4.20).

4.4 AdS vacua and SU(3) structures

In this subsection, we will give a quick bird's eye view on AdS_4 solutions with SU(3) structure, namely when $\Phi_\pm$ is taken as in (4.9). Just like for section (4.3), it should be stressed that this is an Ansatz; solutions of more general type exist.

4.4.1 IIA

In this case, one gets that θ, A and ϕ are constant. We can set $A = 0$ without loss of generality, and define the (constant) string coupling $g_s = e^\phi$ and the real numbers $m = \mu\cos(\theta)$, $\tilde{m} = -\mu\sin(\theta)$, so that the cosmological constant now reads $\Lambda = -3(m^2 + \tilde{m}^2)$. The other equations we get from (4.5b) can then be summarized as follows [7,12]:

$$\begin{gathered} dJ = 2\tilde{m}\mathrm{Re}\Omega \ , \qquad d\Omega = i(W_2^- \wedge J - \tfrac{4}{3}\tilde{m}J^2) \ , \qquad H = 2m\mathrm{Re}\Omega \ ; \\ g_s f_0 = 5m \ , \qquad g_s f_2 = -W_2^- + \tfrac{1}{3}\tilde{m}J \ , \qquad g_s f_4 = \tfrac{3}{2}mJ^2 \ , \qquad g_s f_6 = -\tfrac{1}{2}\tilde{m}J^3 \ . \end{gathered} \tag{4.24}$$

Here, W_2^- is a *primitive* $(1,1)$–form (the strange notation comes from [19,20]; primitive means that $W_2^- \wedge J^2 = 0$).

Notice that M_6 is no longer symplectic, nor is it complex. One easy type of geometry which solves this Ansatz (as well as the Bianchi identities) is the so–called "nearly Kähler" geometry [21]; other solutions can be found in [22,23].

4.4.2 IIB

Let us plug (4.9) into the first equation of (4.5b). The zero–form part and two–form part of that equation read

$$0 = \rho\cos(\theta) \ , \qquad 0 = \rho\sin(\theta)J \ . \tag{4.25}$$

It follows that [23] there are no supersymmetric AdS_4 solutions of SU(3) structure in type IIB.

A Notations

- The ten–dimensional space–time index $M = 0, \ldots, 9$ gets decomposed into a four–dimensional index $\mu = 0, \ldots, 3$, and a six–dimensional internal index $m = 1, \ldots, 6$. Coordinates are denoted x^M.

- In the internal space, m is a spacetime real index, $i = 1, 2, 3$ a holomorphic spacetime index; $a = 1, \ldots, 6$ a frame real index, $\alpha = 1, 2, 3$ a holomorphic frame index. (No confusion should arise with fermionic indices, because we never write those down explicitly.)

- We use for $*$ the same conventions used in [1]: in dimension d, $*_d e^{a_1\ldots a_k} = \frac{1}{(d-k)!}\epsilon_{a_{k+1}\ldots a_d}{}^{a_1\ldots a_k} e^{a_{k+1}\ldots a_d}$, where $e^{a_1\ldots a_k} \equiv e^{a_1} \wedge \ldots \wedge e^{a_k}$.

- Our supersymmetry transformations (1.16) are the same as in [1], except for a redefinition of fluxes in IIA:

$$F_{\text{here}} = -\lambda(F_{\text{there}}) \ , \qquad H_{\text{here}} = -H_{\text{there}} \ . \tag{A.1}$$

  This change allows us to get the same form in IIA and IIB for the pure spinor equations (4.5).

- The contraction $\iota_m \equiv \iota_{\partial_m}$ acts by

$$\iota_m(dx^{m_1} \wedge \ldots \wedge dx^{m_p}) = p\, \delta_m^{[m_1} dx^{m_2} \wedge \ldots \wedge dx^{m_p]} \ . \tag{A.2}$$

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