general Matrix Model literature:

[1] M.L. Mehta, "Random Matrices", 3rd Edition, Elsevier 2004
(1st from 1967 !)

[2] T. Guhr, A. Müller-Groeling, H.A. Weidenmüller: "RMT in Quantum Physics: Common concepts"
Phys. Rep. 299 (1998) 189-425
arXiv: cond-mat/9707301

[3] G. Akemann, J. Baik, P. Di Francesco, "The Oxford Handbook of RMT"
(Editors)
Oxford Univ. Press 2011

[4] P. Forrester, "Log-Gases and Random Matrices",
Princeton Univ Press 2010

[5] G.W. Anderson, A. Guionnet, O. Zeitouni, "An Introduction to Random Matrices", Cambridge Univ. Press 2010

* [2] and [3] give a list of most modern applications in physics and beyond
* [4] and [5] give a rigorous mathematical treatment of the subject

specific literature/reviews:

String theory : Di Francesco, Ginsparg, Zinn-Justin hep-th/9306153
M. Mariño 1206.6272

QCD : Verbaarschot hep-th/0502029

Introduction to Matrix Models and Applications

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What is a random matrix?

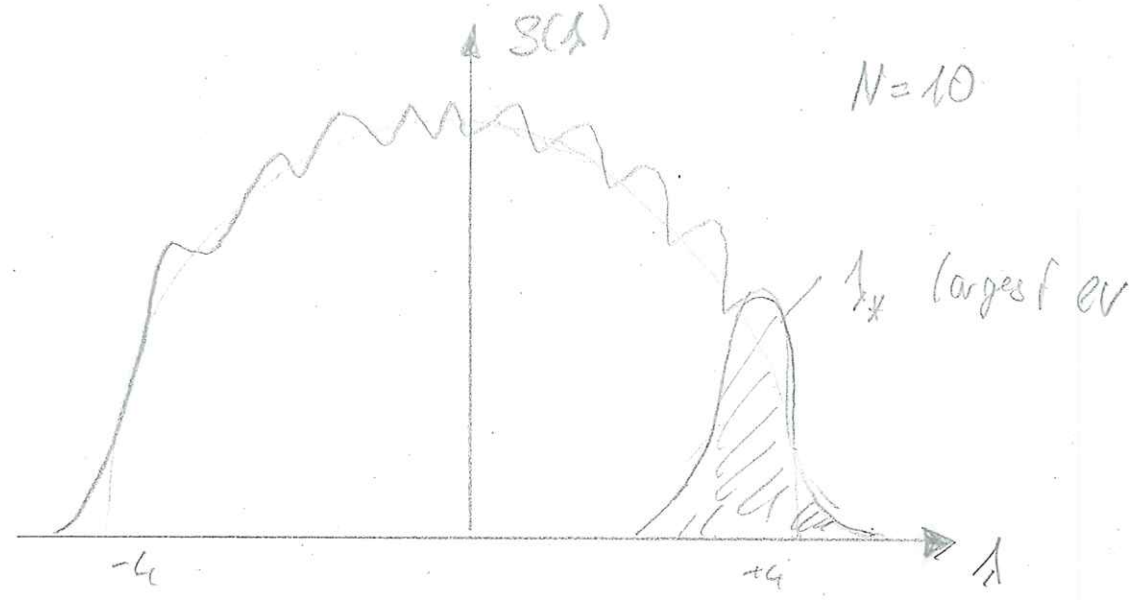
- take H to be an $N \times N$ matrix, with $H_{ij} \in \mathbb{C}$ and $H = H^\dagger$ complex Hermitian (= selfadjoint)
- choose every matrix element (Re- and Im-part) with a Gaussian distribution

$$P(H) = \prod_{i,j=1}^{N} e^{-|H_{ij}|^2} = e^{-\mathrm{Tr}\, H^2}$$

probability measure on $N \times N$ Hermitian matrices (unnormalised)
= product of independent normal random variables

- in this course we will be mainly interested in spectral properties of H: e.g. – distribution of individual eigenvalues (ev)
  or spacing between consecutive ev
  – density, density-density correlation function, etc.

→ make a numerical experiment (average)
choose $N = 10$ say, generate such a random matrix by picking H_{ij} from such a distribution and diagonalise H
repeating this many times you may plot histograms for the obtained eigenvalues:



$S(\lambda)$ spectral density = probability to find an eigenvalue.

$N \gg$ $\quad S(\lambda) \to \frac{1}{\pi N}\sqrt{2N-\lambda^2}$ $\qquad$ Wigner semi-circle

Q: Why should we care about such a simple model, without dynamics, of a "random Hamiltonian"?

A: Random Matrix Theory (RMT) (= Matrix Models MM) has many applications in physics, mathematics, biology, finance, engineering $\to$ literature

- How does this come about?
  - the simple starting point yields a surprisingly rich mathematical structure, e.g.

  scaled distribution of the largest eigenvalue λ_*: Tracy-Widom 1994

$$F_2(t) = \lim_{N\to\infty} \mathrm{Prob}\left(N^{\frac{1}{6}}\left(\lambda_* - 2\sqrt{N}\right) \le t\right) = \exp\left[-\int_t^\infty ds\,(t-s)\,q^2(s)\right]$$

$q''(s) = s\,q(s) + 2q^3(s)$ solution to Painlevé II

$\to$ relation to integrable hierarchy of non-lin differential eqs.

* key observation: eigenvalues are correlated (while H_{ij} are not!)

e.g. the probability that 2 consecutive eigenvalues are at distance s

$$\lim_{N\to\infty} P(s) \approx \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi}s^2}$$ Wigner surmise (from N=2 calculation)

normalised, scale of ev fixed by $\int_0^\infty ds\, s P(s) = 1$ 1st moment

vs Poisson $P(s) \approx e^{-s}$

→ in many physical systems (including many-body, strongly interacting) the ev (of Hamiltonian, Dirac operator, ...) repell each other on a certain scale like in MM, or show fluctuations like these ev uncorrelated: $P(s) \sim e^{-s}$ Poisson.

→ pictures / history

J. Wishart (1928) – E. Wigner & F. Dyson ('57–63) – H. Montgomery ('73)
math. statistics — nuclear physics — $\zeta(s) = \sum_{n=1}^\infty n^{-s}$

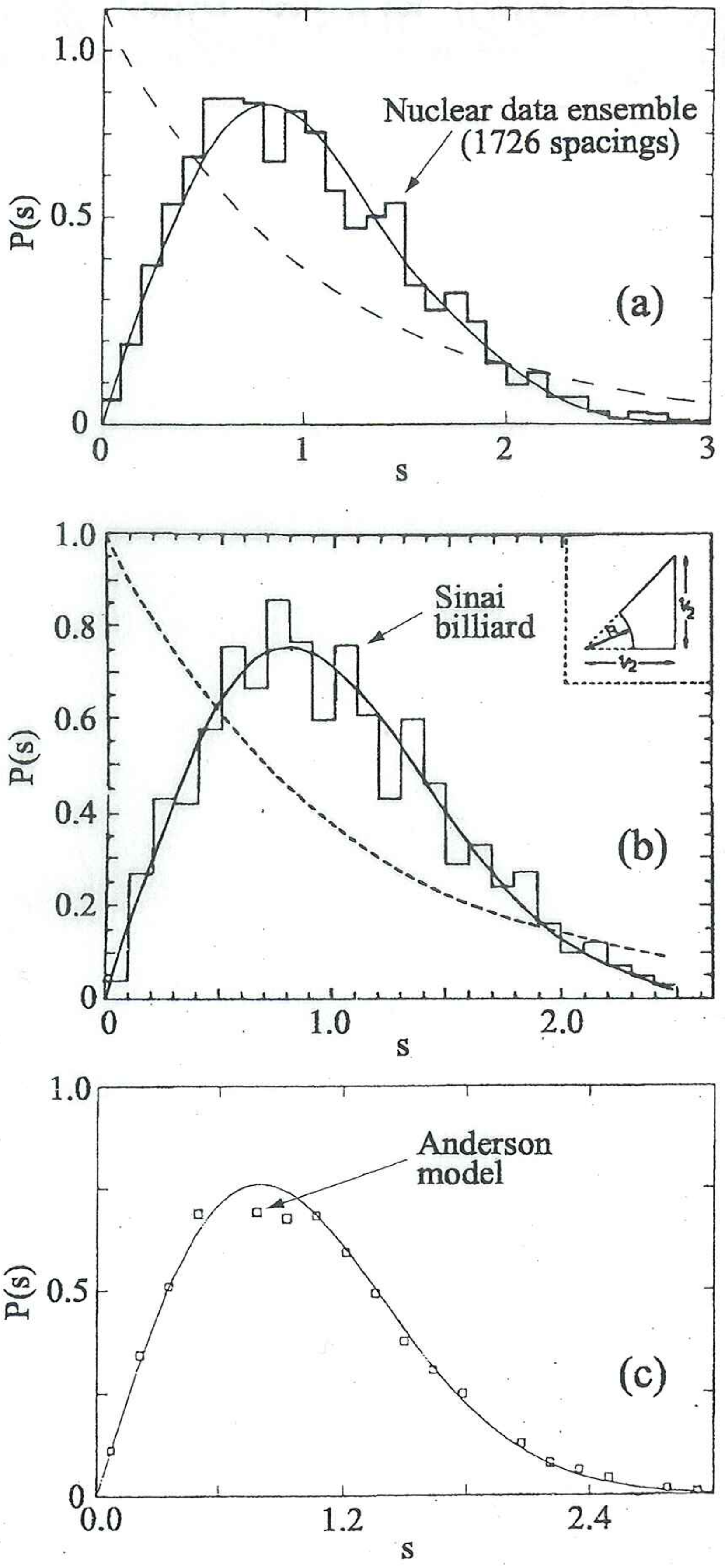
– BGS 85 — t'Hooft, BIPZ 70s — Verbaarschot 92 – ...
quantum chaos — QFT, Strings — QCD

"RMT is a new kind of statistical mechanics" from [2]

* not all applications of RMT are spectral:

- combinatorics (Catalan # $C_k = \frac{\binom{2k}{k}}{k+1}$ = moments of ρ_{SC} (e.g. [5]
- triangulation of random surfaces → strings
- protein folding (biology)
- counting semi-classical orbits (quantum chaos)
- random permutation
- growth problems

→ for more details see chapters in [3]



Nuclear data ensemble
(1726 spacings)
(a)
P(s)
s
Sinai
billiard
(b)
P(s)
s
Anderson
model
(c)
P(s)
s

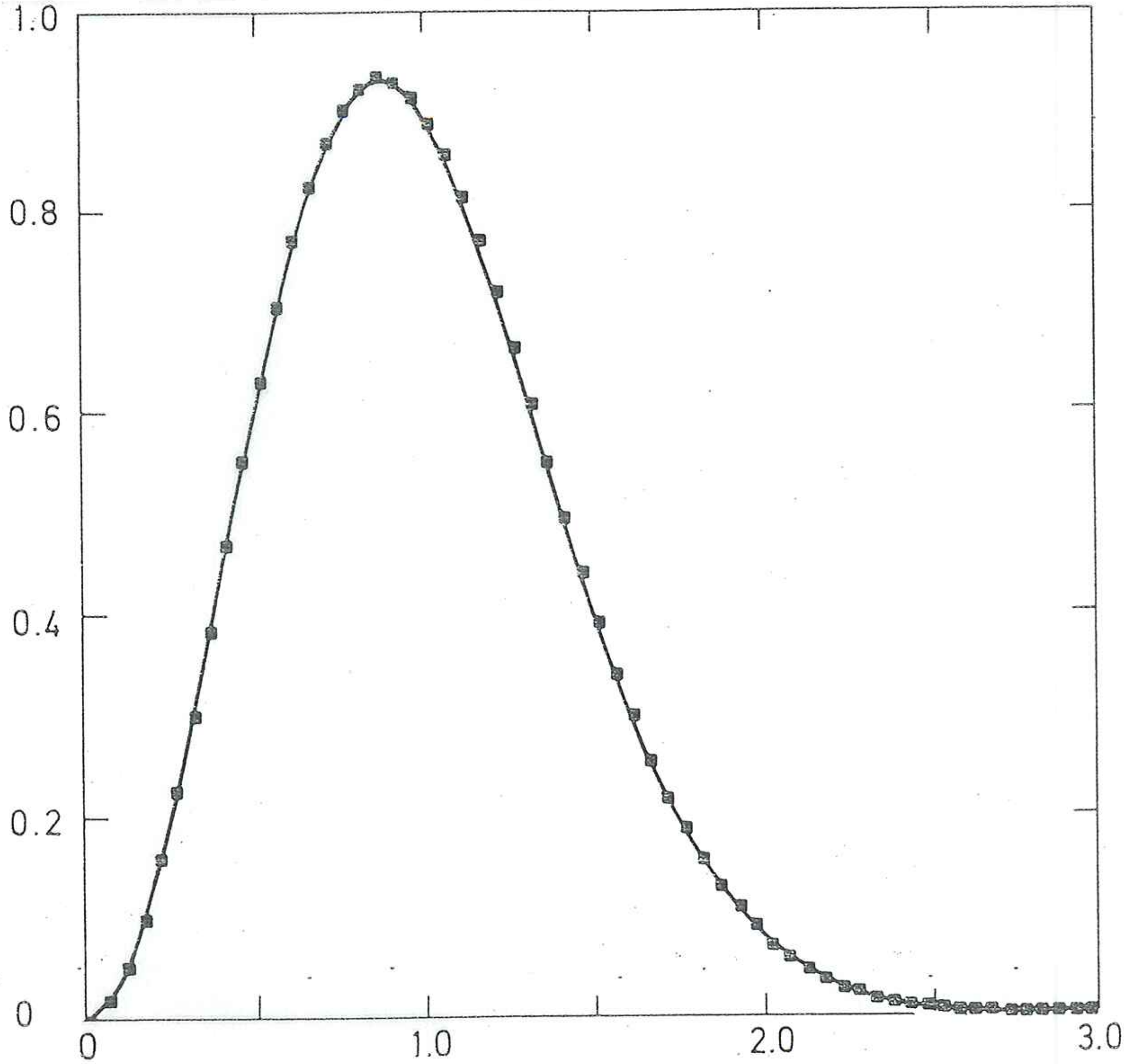


FIG. 1.14. The same as Figure 1.12 but for the 79 million zeros around the 10^{20}th zero. From Odlyzko (1989). Copyright ©, 1989, American Telephone and Telegraph Company reprinted with permission.

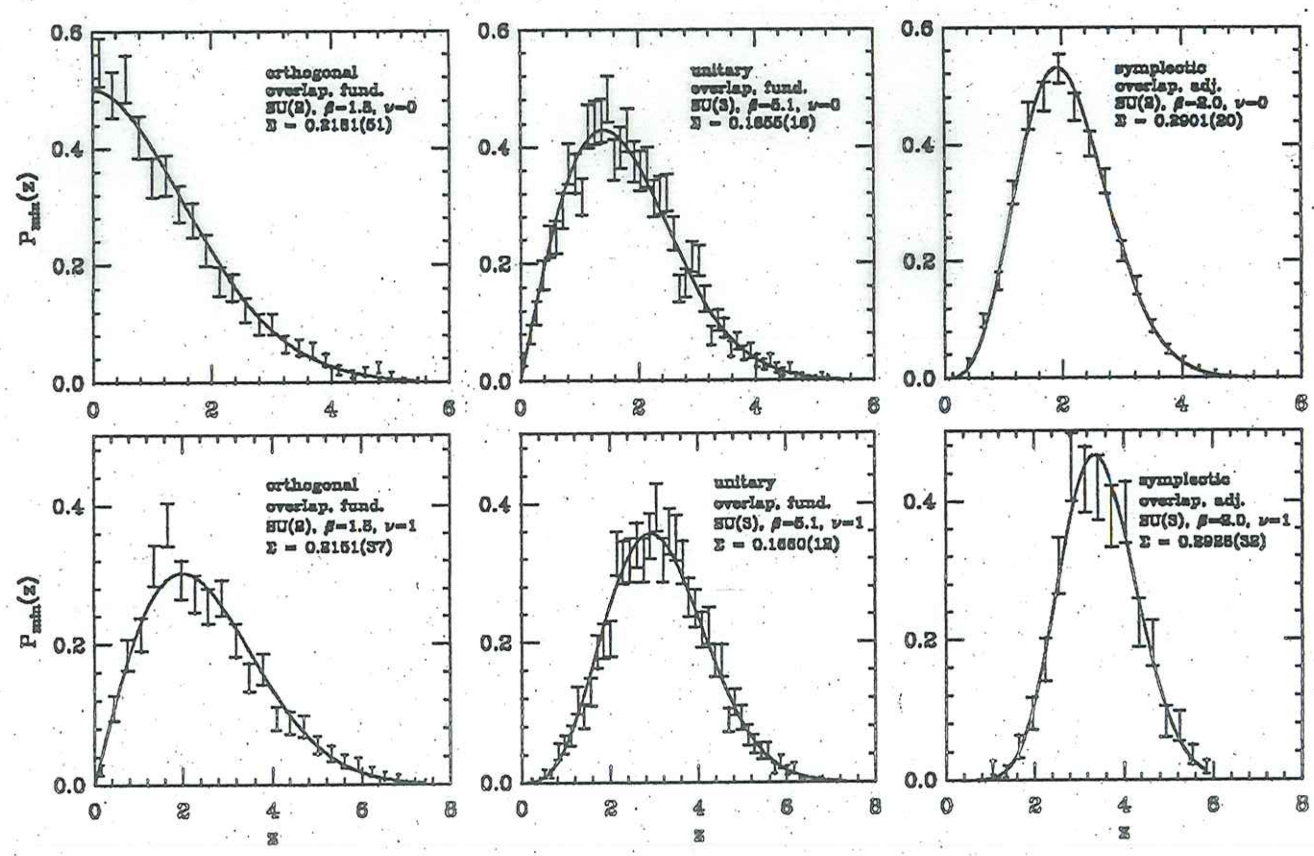


Figure 4. Plots of the distribution of the lowest eigenvalue for all three ensemble in the lowest two topological sectors. The curves are fits to the predictions from random matrix theory.

⇒ Universality ! – not only in the sense of many different applications,
– also in a mathematical sense = independence of the choice of distribution for the matrix elements (at $N\to\infty$, – ∃ different limits!)

- Wigner ensembles, H_{ij} different random variables:
  e.g. semi-circle remains
  not covered here, see chap 21, [3], [Tao+Vu, Erdös+Yau et al., ..]
- invariant ensembles $\exp[-\mathrm{Tr}\,V(H)]$ covered here

* not all different applications follow from 1 single MM → Symmetries

Classification of Matrix Models (see chap. 3, [3]: 1001.0722)

- we'll only consider $H=H^\dagger$ with ev $\in \mathbb{R}$ [$\in \mathbb{C}$ → D. Bernard + A. LeClair cond-mat/0110649, U. Magnea 0707.0418]

Dyson's 3 fold way – "Hamilton operators"

Wigner: sym of a QM system can be unitary U or anti-u $A=UK$, K complex conjugation, e.g. T time reversal is anti-unitary

↔ we may choose $H_{ij} \in$ field $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Dyson index (real d.o.f. per ...)

- $H_{ij} \in \mathbb{C}$ no symmetry $[H,T]\neq 0$ $\beta=2$
- $H_{ij} \in \mathbb{R}$ ⟷ time reversal $[H,T]=0$, $T^2=+1$ $\beta=1$
  real H (even spin)
- $H_{ij} \in \mathbb{H}$ ⟷ –"– $[H,T]=0$, $T^2=-1$ $\beta=4$
  pseudoreal H (odd spin)
  Kramer's degeneracy: each ev $\lambda_i = \lambda_i \mathbb{1}_2$ twice, matrix rep

of $\mathbb{H}$ quaternion: basis $(\mathbb{1},\vec{\sigma})$, H quaternion real: 4 real d.o.f. per H_{ij}
details [1] chap 2.4 Pauli

Dirac - Operators [Verbaarschot, hep-th/9401059]

- we consider random matrices $WW^\dagger$, $W_{ij} \in \mathbb{R}, \mathbb{C}, \mathbb{H}$

ev of Dirac-operator in chiral basis (chiral ensembles) $\not{D} = i\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$, Euclidean

- $\underbrace{W_{ij} \in \mathbb{C}}_{\beta=2}$ no antiunitary symmetric $\not{D} = i\not{\partial} + \not{A}$, $A_\mu \in U(1)$, $SU(N_c \geq 3)$ fund rep a

- $\underbrace{W_{ij} \in \mathbb{R}}_{\beta=1}$ $\exists$ real basis: $[\not{D}, \underbrace{C}_{\text{charge conj}} \sigma_2 K] = 0$ $A_\mu \in SU(2)$ fund

Pauli-Gürsey sym. charge conj (rep K by Pauli matrices)

- $\underbrace{W_{ij} \in \mathbb{H}}_{\beta=4}$ $\exists$ pseudo-real basis $[\not{D}, CK] = 0$, $(CK)^2 = -1$

$A_\mu \in SU(N_c)$ adj

* here N_c # of colours fixed, $\neq N$ size of $WW^\dagger$!

(later: relation to chiral symmetry breaking patterns)

* $\exists$ 4 more classes [Altland, Zirnbauer cond-mat/9602137 /9508026
cf. M. Caselle cond-mat/9610017

due to particle-hole symmetric,

$\Rightarrow$ 10 classes $\Leftrightarrow$ Riemannian symmetric spaces, chap 3 [3]
review M. Zirnbauer

(+ topological zero modes see review 1407.2131)

(+ many more: unitary UM etc, 2 or more matrix models etc.)

- We will now change basis from H (or $WW^\dagger$) to ev and derive the joint probability distribution function (jpdf) of the $\lambda_{i=1,\dots,N}$

  - explicitly for the 3 Dyson classes
  - result $\forall$ 10 classes

$\rightarrow$ we'll see the correlations between λ_i come up

From matrix elements to eigenvalues: [chap. 3, [1]]

- We consider the 3 Wigner-Dyson Ensembles with Gauß' distribution of matrix elements, where $H \in \mathbb{R}/\mathbb{C}/\mathbb{H}$ is diagonalised by an orthogonal / unitary / symplectic trafo: GOE/GUE/GSE

(notice that $\operatorname{Tr} H^2$ is invariant, so this applies to $\operatorname{Tr} V(H)$ too)

The Gaußian Orthogonal Ensemble (GOE)

$$H = H^T, \quad H_{ij} \in \mathbb{R} \quad \Rightarrow \quad N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2} \text{ indep. (real) matrix elements}$$

$$P_N^{(\beta)}(H) = \exp\left[-\frac{\beta}{4} \operatorname{Tr} H^2\right] \quad \text{distribution of} \quad \text{---}\text{''}\text{---}$$

$$\operatorname{Tr} H^2 = \operatorname{Tr} HH^T = \sum_{i,j=1}^N H_{ij} H^T_{ji} = \sum_{i=1} H_{ii}^2 + 2\sum_{i<j}^N H_{ij}$$

$$\Rightarrow \quad Z_N^{(\beta=1)} = \int dH\, P(H)^{(1)} = \int_{\mathbb{R}} \prod_{i=1}^N dH_{ii} \prod_{i<j} dH_{ij} \exp\left[-\frac{1}{4} \operatorname{Tr} H^2\right] \quad \text{partition function}$$

goal: compute Jacobian for change of variables

$$\left.\begin{aligned} &H = O \Lambda O^T, \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N) \quad N \text{ (real) variables} \\ &OO^T = \mathbb{1}_N = O^T O \quad N^2 - N - \frac{N(N-1)}{2} = \frac{N(N-1)}{2} \text{ real var} \end{aligned}\right\} \checkmark$$

trick: consider the invariant volume element

$$\operatorname{Tr}(dH\,dH) = (d\Lambda, dO)\, G \begin{pmatrix} d\Lambda \\ dO \end{pmatrix}, \quad G = \left(\frac{\partial H_{ij}}{\partial \lambda_i\, \partial O_{ij}}\right)$$

with the Jacobian being $J = \sqrt{\det G}$

- let's study dH for $H = S \Lambda S^\dagger$, S orth, unitary, sympl. for all 3 $\beta = 1, 2, 4$ with $SS^\dagger = \mathbb{1}_N = S^\dagger S$

$\Rightarrow \quad 0 = d(SS^\dagger) = dS\,S^\dagger + S\,dS^\dagger$, ditto $S^\dagger dS + dS^\dagger S = 0$

$\Rightarrow \quad dt \equiv S^\dagger dS$ ist antihermitian, $\underline{dt^\dagger = -dt}$ $= -S^\dagger dS$

- $\underline{dH} = dS\,\Lambda S^\dagger + S\,d\Lambda\,S^\dagger + S\Lambda\,dS^\dagger = S(S^\dagger dS\,\Lambda + d\Lambda + \Lambda\,dS^\dagger S)S^\dagger$

$\underline{= S(d\Lambda + [\underbrace{S^\dagger dS}_{dt}, \Lambda])\,S^\dagger}$ $\qquad [A,B] = AB - BA$

$\Rightarrow \operatorname{Tr}[dH\,dH] = \operatorname{Tr}[(d\Lambda + [dt,\Lambda])^2]$

$= \operatorname{Tr}[d\Lambda^2 + 2\,d\Lambda[dt,\Lambda] + [dt,\Lambda]^2]$, Tr is cyclic

$[dt,\Lambda] = dt\Lambda - \Lambda dt$, cancels as $[d\Lambda,\Lambda] = 0$ for diag. matrices

$= \operatorname{Tr}[d\Lambda^2 + dt\Lambda dt\Lambda - dt\Lambda\Lambda dt - \Lambda dt dt\Lambda + \Lambda dt\Lambda dt]$

$= \operatorname{Tr}[d\Lambda^2 + 2\,dt\Lambda dt\Lambda - 2\Lambda^2 dt^2]$

$= \sum_{i=1}^N d\lambda_i^2 + 2\sum_{i,j=1}^N (dt_{ij}\lambda_j\,dt_{ji}\lambda_i - \lambda_i^2\,dt_{ij}\,dt_{ji})$ $\qquad dt_{ji} = -dt_{ij}^*$

$i=j$ gives 0 as $dt_{ii} = 0$

$= \sum_{i=1}^N d\lambda_i^2 + 2\sum_{i<j}^N \underbrace{(\lambda_i^2 + \lambda_j^2 - \lambda_j\lambda_i - \lambda_i\lambda_j)}_{(\lambda_i-\lambda_j)^2}\,dt_{ij}\,dt_{ij}^*$

$\Rightarrow$ we can read off the metric g which is diagonal

$\underline{\beta = 1:}$

$$\underset{\dim \frac{N(N+1)}{2}\times\frac{N(N+1)}{2}}{g^{(\beta=1)}} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 2(\lambda_1-\lambda_2)^2 & & \\ & & & & 2(\lambda_1-\lambda_3)^2 & \\ & & & & & \ddots \\ & & & & & & 2(\lambda_{N-1}-\lambda_N)^2 \end{pmatrix} \begin{array}{l} d\lambda_1 \\ \vdots \\ d\lambda_N \\ dt_{12} \\ dt_{13} \\ \\ dt_{N-1\,N} \end{array}$$

$$\Rightarrow Z_N^{(\beta=1)} = 2^{\frac{N(N-1)}{2}} \int_{\mathbb{R}} \prod_{i=1}^{N} d\lambda_i \prod_{k<l}^{N} |\lambda_k - \lambda_l| \underbrace{\int dO}_{\text{Vol of } O(N)} \, e^{-\frac{1}{4}\sum_{i=1}^{N} \lambda_i^2} \qquad \text{GOE}$$

also more general $e^{-\mathrm{Tr}\,V(H)} = e^{-\sum_{i=1}^{N} V(\lambda_i)}$

- for observables $F(H) = F(\Lambda)$ that are inv.

we have for expectation values

$$\langle F \rangle_\Lambda = \frac{2^{\frac{N(N-1)}{2}}}{Z_N^1} \int dO \int_{\mathbb{R}} \prod_{i=1}^{N} d\lambda_i \, e^{-\frac{\lambda_i^2}{4}} |\Delta_N(\Lambda)|^1 F(\Lambda)$$

cancels out

with $\boxed{\Delta_N(\Lambda) \equiv \prod_{N \geq k > l \geq 1} (\lambda_k - \lambda_l) = \det_{1 \leq k, l \leq N} [\lambda_k^{l-1}]}$ Vandermonde determinant

The Gaussian Unitary Ensemble

$\beta = 2$: $dt_{ij} = d\,\mathrm{Re}\, t_{ij} + i\, d\,\mathrm{Im}\, t_{ij}$ (indep) $\Rightarrow dt_{ij}$ has $\frac{N(N-1)}{2} \cdot 2$ real dof

$$\Rightarrow g_{N^2}^{(\beta=2)} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 2(\lambda_1-\lambda_2)^2 & \\ & & & & 2(\lambda_1-\lambda_2)^2 \\ & & & & & \ddots \end{pmatrix} \begin{array}{l} d\lambda_1 \\ \\ d\lambda_N \\ d\,\mathrm{Re}\, t_{12} \\ d\,\mathrm{Im}\, t_{12} \end{array}$$

$$\Rightarrow Z_N^{(\beta=2)} = 2^{\frac{N(N-1)}{2}} \int_{\text{coset}} dU \int_{\mathbb{R}} \prod_{i=1}^{N} d\lambda_i \, e^{-\frac{1}{2}\lambda_i^2} \Delta_N(\Lambda)^2 \qquad \text{GUE}$$

note: $H = H^\dagger$ has N^2 real dof, $U \in U(N)$ also N^2

$\Rightarrow H = U \Lambda U^\dagger$ with $U \in U(N)/U(1)^N$ = coset, its dim = $2N^2 - N - \frac{N(N-1)}{2} \cdot 2$

The Gaussian Symplectic Ensemble

$\beta = 4$: $\lambda_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix} = \lambda_i \mathbb{1}_2$, dt_{ij} is 2×2 matrix with 4 dof

$$\Rightarrow Z_N^{(\beta=4)} = 2^{2N(N-1)} \int dS \int_{\mathbb{R}} \prod_{i=1}^{N} d\lambda_i \, e^{-\lambda_i^2} \Delta_N(\Lambda)^4 \qquad \text{GSE}$$

The remaining 6 symmetry classes

We have seen $Z_N^\beta = 2^{\frac{\beta}{4}N(N-1)} \int dS \int_{\mathbb{R}} \prod_{i=1}^N d\lambda_i \, e^{-\frac{\beta}{4}\lambda_i^2} |\Delta_N(\lambda)|^\beta$ for $\beta = 1, 2, 4$
GOE, GUE, GSE

- in the chiral classes (chGOE etc.)
  we have that $WW^\dagger$ is positive definite $\Rightarrow \lambda_i \geq 0$ ev
  (note: the ev of $WW^\dagger$ are the singular values (sv) squared in the sv decomposition $W = U \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N}) V$ possible for any $W \neq W^\dagger$, $U, V \in U(N)$ ($O(N)$) for complex W_{ij})

result $Z^\beta_{N,ch} \equiv \int dW \, e^{-\frac{\beta}{4}\operatorname{Tr} WW^\dagger} = \text{const} \int_0^\infty \prod_{i=1}^N d\lambda_i \, \lambda_i^{\frac{\beta}{2}(\nu+1)-1} e^{-\lambda_i} |\Delta_N(\lambda)|^\beta$

integrate over all indep W_{ij}

where now W $N \times (N+\nu)$ rectangular, with ν zero eigenvalues

- the jpdf of the remaining 4 classes read [see Beenakker 1407.2131 for a review]

$$Z^\beta_{N,\text{other}} = \text{Const} \int_0^\infty \prod_{i=1}^N d\lambda_i \, \lambda_i^{\frac{\alpha}{2}-\frac{1}{2}} e^{-\lambda_i} |\Delta(\lambda)|^\beta$$

				name
with	$\beta = 1$	$\alpha = 1$		C I
	$\beta = 2$	$\alpha = 2$		C
	$\beta = 2$	$\alpha = 2m$	$m = 0(1)$ for N even (odd)	D ([1] chap 13 AS)
	$\beta = 4$,	$\alpha = 4m+1$,	——"——	class D III

dictionary:

	GUE	GOE	GSE	chGOE	chGUE	chGSE				
10 classes	A	AI	AII	BDI	AIII	CII	D, B	DIII	C	CI

Coulomb-gas interpretation & β-Ensembles

* all of the above 10 classes can be solved exactly for finite N (for β = 1, 2, 4 ?) using Hermite and Laguerre polynomials – as we will see later

∃ more general classes: setting $\beta > 0$ real in the GO/U/SE:

$$Z_N^{(\beta)} = \text{const} \int_{\mathbb{R}} \prod_{i=1}^{N} d\lambda_i \exp\left[-\frac{\beta}{4}\sum_{i=1}^{N}\lambda_i^2 + \frac{\beta}{2}\sum_{i\neq j}^{N} \ln|\lambda_i - \lambda_j|\right]$$

β-ensemble

- in 2 dimensions the Coulomb interaction is logarithmic!

  ⇒ the ev λ_i describe a set of Coulomb charges in 2d, restricted to the real line and subject to a harmonic potential, with inverse temperature β

(∃ truely 2d Coulomb picture when $\lambda_i \in \mathbb{C}$, c.f. chap. 15 [4])

- in the limit $\beta \to 0$ the ev λ_i become independent Gauß' random variables
  ⇒ spacing has Poisson statistics, → [Berry-Tabor 77]
  ($\beta = O(\frac{1}{N})$: transition)

- ∃ tridiagonal matrix rep for $\beta > 0$ [Dumitriu-Edelman math-ph/0206043]

- very little is known for ev correlation functions for $\beta \neq 1, 2, 4$: Z_N^β due to the Selberg integral e.g. 0710.3981,
  the tails of the largest ev far outside the support using large deviations (≠ TW tails),
  and most recently the generalisation of TW for general β
  Ramirez, Rider, Viray math/0607331

Saddle point approximation

- from the (so far conjectural) semi-circle law we see that rescaling $\lambda^2 \to N\lambda^2$, to get a limiting density on a compact support

$$\rho(\lambda) \simeq \frac{1}{\pi N}\sqrt{2N-\lambda^2} \longrightarrow \boxed{\rho_{sc}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2} \text{ on } \lambda \in [-\sqrt{2}, +\sqrt{2}]}$$

- this will change the potential term, but not the ln-term as N can be scaled out of the integral

$$\Rightarrow Z_N^{(\beta)} = \text{Const} \int_{\mathbb{R}} \prod_{i=1}^{N} d\lambda_i \, \exp\Big[-\frac{\beta}{2} \underbrace{S_N(\lambda)}_{\text{factors out}}\Big] = e^{\overset{\text{free energy}}{N^2 F}}$$

with action $\underline{S_N(\lambda) = N \sum_{j=1}^{N} V(\lambda_j) - \sum_{i \neq j} \ln|\lambda_i - \lambda_j|}$

where we now consider a general potential $\frac{1}{2}\lambda^2 \to V(\lambda)$

* obviously this integral is amenable to a saddle point approx. when $N \to \infty$

$$\frac{\partial S_N(\lambda)}{\partial \lambda_i} = N V'(\lambda_i) - \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \overset{!}{=} 0$$

which determines the continuum limit of the

finite-N density $\boxed{\rho_N(\lambda) = \frac{1}{N}\Big\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \Big\rangle}$, $\lim_{N\to\infty} \rho_N(\lambda) = \rho(\lambda)$

or directly in the continuum limit

$$\lim_{N\to\infty} \frac{1}{N^2} S_N = \int_\sigma d\lambda\, \rho(\lambda) V(\lambda) - \int_\sigma d\lambda\, d\lambda'\, \rho(\lambda)\rho(\lambda') \ln|\lambda - \lambda'|$$

with SP eq $\frac{\delta S}{\delta \rho(\lambda)} \overset{!}{=} 0 \Leftrightarrow V(\lambda) = 2\int_\sigma d\lambda' \rho(\lambda') \ln|\lambda - \lambda'| \quad \lambda \in \sigma$

$\partial_\lambda \Rightarrow$ (*) $\boxed{V'(\lambda) = 2 \, \fint_\sigma d\lambda' \frac{\rho(\lambda')}{\lambda-\lambda'}}$ saddle point equation in the continuum

(in principle we could have added a Lagrange multiplier α to the action, $\lim \frac{1}{N^2} S_N + \alpha \left(1 - \int_\sigma d\lambda \rho(\lambda)\right)$ to ensure normalisation

- Eq. (*) is a singular integral eq. that can be solved using a Theorem due to Tricomi. It applies for general $V(\lambda)$ as long as σ is a single interval.

* for $V(\lambda) = \frac{1}{2}\lambda^2$ the solution is the semi-circular density ρ_{sc}, holds for arbitrary β - as it dropped out of the SP eq. "universal"

* to leading order (LO) we have for $\boxed{Z_N^{(\beta)} \equiv e^{N^2 F^{(\beta)}}}$

with $F_0^{(\beta)} = -\lim_{N\to\infty} \min \left(\frac{1}{N^2} S_N\right)$, the planar limit of the free energy

* we could also analyse the SP eq. using the

$\boxed{\text{resolvent } W(p) \equiv \int d\lambda \frac{\rho_N(\lambda)}{p-\lambda}}$ (also called Stieltjes or Cauchy transform) $p \notin \mathbb{R}$

see e.g. hep-th/9306153 p 17, 18 to derive an eq. for $W_0(p)$ LO

invert when $N\to\infty$: $\frac{1}{x \pm i\varepsilon} = P.V.\frac{1}{x} \mp i\pi\delta(x) \Rightarrow \boxed{\rho(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0}\left(W_0(\lambda - i\varepsilon) - W_0(\lambda + i\varepsilon)\right)}$ $\lambda \in \sigma$

* in the following we'll derive a more powerful method, the loop eq. (or Schwinger-Dyson eq.) to get W, F & ρ for arbitrary potentials $V(\lambda)$ including higher order corrections

* the SP only allows to get $\rho_c(\lambda, \lambda')$, connected 2pt density, an $O\left(\frac{1}{N^2}\right)$ "

e.g. cond-mat/9310010

...ian: Random Matrices and 2d Quantum Gravity

...w Di Francesco, Ginsparg, Zinn-Justin hep-th/9306153

Euclidean Einstein Hilbert action in 2 dim

$$S = \int dx^2 \sqrt{\det g_{\mu\nu}} \left[\frac{-1}{4\pi G_N} R + \Lambda \right]$$

metric, Newton, Ricci scalar, cosmol. const

in 2 dim compact manifolds $\mathcal{M}$ can classified according to their

genus g = # of holes

with $\chi = 2-2g$

the Euler characteristic

→ $\mathcal{M}_{g,A}$ is characterised by g and its area A

in 2d the action above is topological (Gauss-Bonnet Thm)

$$\Rightarrow \quad S = -\frac{1}{G_N}(2-2g) + \Lambda A$$

- it is invariant under diffeomorphisms

pathintegral quantisation: quantum field $g_{\mu\nu}$, couplings G_N, Λ

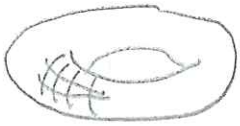
$$Z_{QGra}[G_N,\Lambda] = \int \frac{Dg_{\mu\nu}}{Vol(Diff)}\, e^{-S} = \sum_{g=0}^{\infty} \int_0^{\infty} dA\, e^{\frac{1}{G_N}\chi - \Lambda A} \int_{\mathcal{M}_{g,A}} \frac{Dg_{\mu\nu}}{Vol(Diff)}$$

* the last integral is difficult

→ continuum approach: Liouville theory

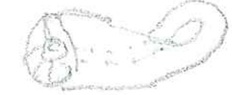
→ we'll choose a discrete approach using RMT:
discretise the surface using triangulations
generated by Feynman diagrams of 2dim MM "field theory"

" 4 "



we could generate such connected Feynman graphs from scalar 0 dim ϕ^4 Theory $\frac{\phi^2}{2} + \lambda \frac{\phi^4}{4!}$

- not enough structure to distinguish orientable / non orientable surfaces

e.g. Möbius, Klein bottle 

- # of diagrams grows too fast to count: non Borel summable series — E. Witten, 2d Gravity & intersection theory on moduli surfaces (see §4) 1991

try matrix valued 0-dim "FT": $H = H^\dagger$ $N \times N$ $H_{ij} \in \mathbb{C}$

$$Z = \int dH \, e^{-N \mathrm{tr}\left(\frac{H^2}{2} - \alpha H^4\right)}$$

def in a perturbative sense

$H \to U H U^\dagger$ $U \in U(N)$

$U(N)$ gauge theory

Feynman-rules (with 't Hoofts double line notation)

Propagator (P)  Vertex (V)  Loop (L) 

$\langle H_{ik} H_{jl} \rangle \sim \frac{1}{N} \delta_{il} \delta_{jk}$ $\quad -\alpha N \quad N$

contribution of a graph: $(-\alpha)^V N^{V-P+L} \underset{\uparrow}{=} (-\alpha)^V N^{2-2g}$

Euler relation

recall $Z = e^{N^2 F}$, F generates connected diagrams $\Leftrightarrow Z_{\text{conn}}$

$$N^2 F[N, \alpha] = \sum_{g=0}^{\infty} N^{2-2g} \underbrace{\sum_{k=0}^{\infty} (-\alpha)^k F_{g,k}}_{\equiv F_g \text{ we'll compute the non-pert. in } \alpha} = \sum_{g=0}^{\infty} N^{2-2g} F_g$$

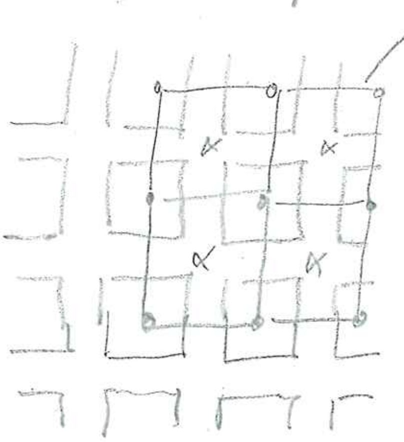
Identify this with the discretised path integral Z_{QGra}
for a discretised manifold $M_{g,A}$

area $A = k\cdot\varepsilon$, k plaquettes of equal size ε $\Rightarrow \int dA \to \sum_{k=0}^{\infty}$

$$\Rightarrow Z^{discrete}_{QGr} = \sum_{g=0}^{\infty}\sum_{k=0}^{\infty} \underbrace{\left(e^{G_N^{-1}}\right)^{2-2g}}_{\updownarrow\; N} \underbrace{\left(e^{-\Lambda\varepsilon}\right)^{k}}_{\updownarrow\; -\alpha} \underbrace{\int \frac{Dg_{\mu\nu}}{Vol(Diff)}}_{\updownarrow\; F_{g,k}}$$

to have $Z^{discrete}_{QGr} = N^2 F[N,\alpha]$

due to the above identification the discretised Riemann surface is given by the dual graph:

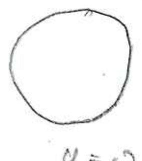


$V \to L$
$P \to P$
$L \to V$

as the MM vertices count the surfaces' plaquettes

solve the matrix model in the large-N = continuum limit (?)

$\Rightarrow$ leading order: genus $g=0$ contributes only, "planar limit"



$g=0$

$$\lim_{N\to\infty} F = F_0 + \frac{1}{N^2}F_1 + O\left(\frac{1}{N^4}\right)$$

$\Rightarrow$ we only get a contribution from higher genus $g\geq 1$ if F_g blows up to compensate N^{-2g}

we'll adjust our coupling $\alpha \to \alpha_c$ to achieve such a critical behavior

universality: we should get the same answer when triangulating with $\triangle$, $\hexagon$ etc.

...al behaviour $F_g[\alpha] \sim (\alpha_c - \alpha)^{(2-\gamma_{str})\frac{2-2g}{2}}$

...ith "string susceptibility" $\gamma_{str} = \frac{1}{12}\left(c-1-\sqrt{(c-1)(c-25)}\right)$, from continuum approach Liouville theory

<u>connection to conformal field theory</u>: central charge

<u>Virasoro-Algebra</u> $[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}$

L_m's are the generators of conformal trafos in 2d $\quad m,n \in \mathbb{Z}$

($L_m = -z^{m+1}\partial_z$ generators of $z \to z + \sum_{n \in \mathbb{Z}} a_n z^{n+1}$)

we'll see later that the matrix model partition function (see p. 28)

satisfies <u>Virasoro constraints</u> $L_m Z = 0 \quad m \geq -1$

where the L_m are realised as diff. operators wrt the (∞ many) coupling constants of V. $\quad V(x) = \sum_{n=1}^{\infty} \frac{g_n}{n} x^n$

<u>...mples</u>: $c = 0 \Rightarrow \gamma_{str} = -\frac{1}{2}$ ("pure gravity")

$c = 1 - \frac{6(p-q)^2}{pq}$, $(p,q) = (2m-1, 2) \Rightarrow \gamma_{str} = -\frac{1}{m}$

"gravity coupled to matter" (spins sitting on plaquettes)

the $\frac{1}{m}$ behaviour can be realised within the 1-matrix model by <u>tuning</u> the potential to a critical value

$\rho(\lambda) \sim (b-\lambda)^{m-\frac{1}{2}} (\lambda - a)^{\frac{1}{2}}$

to reach other critical exponent γ_{str} we need to go
a 2 matrix model with H_1, H_2 and tune to criticality there

each to the critical behaviour of F_g

mostly can enhance the contribution of higher genus $g \geq 1$ of

$$\lim_{\substack{\alpha \to \alpha_c \\ N \to \infty}} \underline{N(\alpha_c - \alpha)^{\frac{2-\gamma_{str}}{2}} = \text{const}} \qquad \underline{\text{double scaling limit}}$$

his indeed corresponds to a continuum limit as $\underline{A = k \cdot \varepsilon}$

and $\langle A \rangle = \varepsilon\, \alpha \frac{\partial}{\partial \alpha} \ln F_0 \sim \frac{\varepsilon}{\alpha_c - \alpha} \neq 0$, so $\varepsilon \to 0$, $\alpha \to \alpha_c$ together

Higher genus contributions in Matrix models: Loop Eq

we will focus on $\beta = 2$ here H complex Hermitian

→ comprehensive treatment hep-th/9302014 Ambjørn, Chekov, Kristjansen, Makeenko

RMT with $\beta = 1, 4$ have an expansion in $\frac{1}{N}$ (not $\frac{1}{N^2}$)
and describe non-oriented surfaces.
see e.g. cond-mat/9611214
results on the general-β case 1009.6007 using loop eqs.
here we postulate a $\frac{1}{N^{(2)}}$ expansion of F
this can be proven rigorously for $\beta > 0$ e.g. 1107.1167

recall some definitions:

part. funct. & free energy $Z[N,\{g_j\}] = e^{N^2 F} = \int dH\, e^{-N \operatorname{Tr} V(H)}$

potential $V(H) = \sum_{j=1}^{\infty} \frac{g_j}{j} H^j$, couplings provide sources for powers, may set all but few $\equiv 0$ at the end of calcul.

moments $\frac{1}{N}\langle \operatorname{Tr} H^j\rangle = \frac{1}{NZ}\int dH \operatorname{Tr} H^j e^{-N\operatorname{Tr} V} = -j\frac{\partial}{\partial g_j} F = -j\frac{\partial}{\partial g_j}\frac{\ln Z}{N^2}$

(N) object) $\rho_N(\lambda) = \frac{1}{N}\left\langle\sum_{\ell=1}^{N}\delta(\lambda-\lambda_\ell)\right\rangle \Rightarrow\ = \int d\lambda\, \lambda^j \rho_N(\lambda)$

resolvent $W(p) = \frac{1}{N}\sum_{k=0}^{\infty}\frac{\langle \operatorname{Tr} H^k\rangle}{p^{k+1}} = \frac{1}{N}\left\langle \operatorname{Tr}\frac{1}{p-H}\right\rangle$

has a genus expansion $= \sum_{g=0}^{\infty}\frac{1}{N^{2g}} W_g(p) \qquad = \int d\lambda \frac{\rho_N(\lambda)}{p-\lambda}$

just as the free energy $F = \sum_{g=0}^{\infty}\frac{1}{N^{2g}} F_g$

n-point resolvent $W(p_1,\dots,p_n) = N^{n-2}\sum_{k_1,\dots,k_n=1}^{\infty}\frac{\langle \operatorname{Tr} H^{k_1}\dots \operatorname{Tr} H^{k_n}\rangle_{conn}}{p_1^{k_1+1}\dots p_n^{k_n+1}}$

$= N^{n-2}\left\langle \operatorname{Tr}\frac{1}{p_1-H}\dots \operatorname{Tr}\frac{1}{p_n-H}\right\rangle_{conn}$

where connected means $W(p_1,p_2) = \left\langle \operatorname{Tr}\frac{1}{p_1-H}\operatorname{Tr}\frac{1}{p_2-H}\right\rangle - \left\langle\operatorname{Tr}\frac{1}{p_1-H}\right\rangle\left\langle\operatorname{Tr}\frac{1}{p_2-H}\right\rangle$

$= \langle \qquad \rangle_{conn}$

in the large-N limit

we expect factorisation $\lim_{N\to\infty}\left\langle \operatorname{Tr}\frac{1}{p_1-H}\operatorname{Tr}\frac{1}{p_2-H}\right\rangle = \left\langle\operatorname{Tr}\frac{1}{p_1-H}\right\rangle\left\langle\operatorname{Tr}\frac{1}{p_2-H}\right\rangle$
('t Hooft)

$\Rightarrow$ hence the (conn) n-point resolvent will give the non-trivial per order contributions.

the $W(p_1,\ldots,p_n)$ also all have a genus expansion

<u>loop-insertion operator</u> $\frac{d}{dV}(p) = -\sum_{j=1}^{\infty} \frac{j}{p^{j+1}} \frac{\partial}{\partial g_j}$ (generates extra trace)

can be used to generate all W from F

$$W(p) = \frac{1}{p} + \frac{d}{dV(p)} F \quad , \quad W(p_1,\ldots,p_n) = \frac{d}{dV}(p_1)\cdots\frac{d}{dV}(p_n) F \quad n \geq 2$$

and likewise for the higher genus contributions — exercise: check for $n=2$

<u>Goal:</u> derive an equation that allows to determine (iteratively) all $W_g(p)$ (or F_g) $\Rightarrow$ all $W_g(p_1,\ldots,p_n)$

<u>Assumptions</u>: we assume that we can make an expansion around $N=\infty$ such that the eigenvalues of H are supported on a <u>finite union of intervals σ</u> (= support of $\rho(\lambda)$)

<u>Derivation of the loop equation</u>:

- the partition function is invariant under the field redefinition $H \to H + \varepsilon H^k$ $\forall k = 0,1,2,\ldots$: $\left.\frac{dZ}{d\varepsilon}\right|_{\varepsilon=0} = 0$
- in order to generate an eq. containing $W(p)$ we redef

$$H \to H + \varepsilon \frac{1}{p-H} = H + \varepsilon \sum_{k=0}^{\infty} \frac{H^k}{p^{k+1}} \qquad p \notin \sigma \text{ (or } \mathbb{R} \text{ for } N \text{ finite)}$$

$$\Rightarrow \quad dH \to dH\left(1 + \varepsilon \left(\mathrm{Tr}\frac{1}{p-H}\right)^2 + O(\varepsilon^2)\right)$$

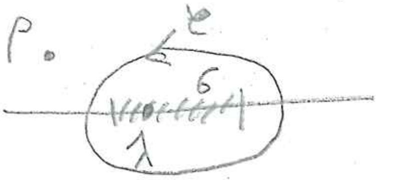
$$\mathrm{Tr}\, V(H) \to \mathrm{Tr}\, V(H) + \varepsilon\, \mathrm{Tr}\left(\frac{1}{p-H} V'(H)\right) + O(\varepsilon^2)$$

$$= \frac{1}{Z}\frac{dZ}{d\varepsilon}\Big|_{\varepsilon=0} = \left\langle \left(\mathrm{Tr}\,\frac{1}{p-H}\right)^2\right\rangle - N\left\langle \mathrm{Tr}\left(\frac{1}{p-H}\,V'(H)\right)\right\rangle \qquad \Big| \cdot \frac{1}{N^2}$$

$\Leftrightarrow$ Virasoro, p 28

we have $\underline{\frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{p-H}V'(H)\right)\right\rangle} = \int_\sigma d\lambda\, \rho_N(\lambda)\, \frac{V'(\lambda)}{p-\lambda}$

expansion around $N=\infty$:
assume all ev $\lambda \in \sigma$



$$= \int_\sigma d\lambda\, \rho_N(\lambda) \cdot \oint_C \frac{dw}{2\pi i}\,\frac{1}{w-\lambda}\,\frac{V'(w)}{p-w}$$

$$= \underline{\oint_C \frac{dw}{2\pi i}\, W(w)\, \frac{V'(w)}{p-w}} \quad \text{as } W(w) = \int_\sigma d\lambda\, \frac{\rho_N(\lambda)}{w-\lambda}$$

contour C encloses σ (one interval) and not p

$$\frac{1}{N^2}\left\langle \left(\mathrm{Tr}\,\frac{1}{p-H}\right)^2\right\rangle = \left(\frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{p-H}\right\rangle\right)^2 + \frac{1}{N^2}\left\langle \mathrm{Tr}\,\frac{1}{p-H}\,\mathrm{Tr}\,\frac{1}{p-H}\right\rangle_{conn}$$

$$= W(p)^2 + \frac{1}{N^2}W(p,p) = W(p)^2 + \frac{1}{N^2}\frac{d}{dV}(p)\,W(p)$$

$$\boxed{\oint_C \frac{dw}{2\pi i}\, W(w)\,\frac{V'(w)}{p-w} = W(p)^2 + \frac{1}{N^2}\frac{d}{dV}(p)\,W(p)}$$

Y. Makeenko 1991

<u>loop equation</u> for $W(p)$ around $N=\infty$

- to be solved iteratively in the $\frac{1}{N^2}$ genus expansion for $W(p)$

<u>starting point</u>: planar limit $N\to\infty$, determine $W_0(p)$

2nd term on RHS is subleading, $W(p,p) = O(1)$

$$\Rightarrow \quad \underline{\oint_C \frac{dw}{2\pi i}\, W_0(w)\,\frac{V'(w)}{p-w} = W_0(p)^2} \qquad \text{planar limit of loop eq.}$$

ar solution : pull C to ∞

$$\oint_{C}\frac{d\omega}{2\pi i}\frac{V'(\omega)}{p-\omega}W_0(p) = \oint_{C_p\cup C_\infty}\frac{d\omega}{2\pi i}\frac{V'(\omega)}{p-\omega}W_0 = V'(p)W_0(p) + \oint_{C_\infty}\frac{d\omega}{2\pi i}\frac{V'(\omega)}{p-\omega}W_0(\omega)$$

$\mathbb{C}$

$= \frac{1}{\omega} + O\left(\frac{1}{\omega^2}\right)$ also

$$\Rightarrow \quad 0 = W_0^2(p) - V'(p)W_0(p) - \oint_{C_\infty}\frac{d\omega}{2\pi i}\frac{V'(\omega)}{p-\omega}W_0(\omega)$$

quadratic in $W_0(p)$

$$\Rightarrow \quad W_0(p) = \frac{1}{2}V'(p) \pm \frac{1}{2}\sqrt{\underbrace{V'(p)^2 + 4\oint_{C_\infty}\frac{d\omega}{2\pi i}\frac{V'(\omega)}{p-\omega}W_0(\omega)}_{}}$$ polynomial

$W_0(p)$ is a complex valued function on $\mathbb{C}\setminus\sigma$ $\rightarrow$ need to choose branch of $\sqrt{\ }$ (subtle for σ multiple intervals)

$\lim_{p\to\infty} p\,W_0(p) = 1 \Rightarrow$ choose sign $\ominus$

the polynomial under the square root is at maximal of degree $2d-2$ when $V'(p)$ is of degree $d-1$ eg. for Gaussian $V(\lambda)=\lambda^2$ of degree 2

simplest Ansatz:

branch: $p(1-\frac{a}{p})^{\frac{1}{2}}(1-\frac{b}{p})^{\frac{1}{2}}$ for $|p|\to\infty$

$$W_0(p) = \frac{1}{2}\left(V'(p) - M(p)\sqrt{(p-a)(p-b)}\right)$$ $\Rightarrow$ has simple branch cut at $[a,b]=\sigma$

polynomial degree $d-2$, so const for Gaussian

$V =$  we may also get

cut Ansatz $\quad W_0(p) = \frac{1}{2}\left(V'(p) - M_2(p)\sqrt{(p-a)(p-b)(p-c)(p-d)}\right)$

degree $d-3 \qquad \sigma = [a,b]\cup[c,d]$

termination of $M(p)$ (1 cut)

$$(p) = \frac{V'(p) - 2W_0(p)}{\sqrt{(p-a)(p-b)}}$$, polynomial, so analytic on $\mathbb{C}$

still implicit as depends on $W_0(p)$

$$\Rightarrow M(p) = \int_{-\mathcal{C}_p} \frac{d\omega}{2\pi i} \frac{M(\omega)}{\omega - p} = \oint_{\mathcal{C}_\infty} \frac{d\omega}{2\pi i} \left(\frac{V'(\omega)}{(\omega-p)\sqrt{(\omega-a)(\omega-b)}} - \cancel{\frac{2W_0(\omega)}{(\omega-p)\sqrt{(\omega-a)(\omega-b)}}} \right)$$

↳ deform to $\mathcal{C}_\infty$

$$\text{as } \sim \frac{2}{\omega\cdot\omega\cdot\omega}\left(1 + O\left(\frac{1}{\omega^2}\right)\right), \oint_{\mathcal{C}_\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega^k} = \delta_{k,1}$$

$\Rightarrow$ dependence on $W_0(\omega)$ drops out, no longer implicit

* all that remains to be determined are the endpoints of σ a, b as functions of the coupling in $V(\lambda)$!

- before going to examples let's write the full solution to $W_0(p)$:

$$W_0(p) = \frac{1}{2}V'(p) - \frac{1}{2} \oint_{\mathcal{C}_\infty} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega-p)} \sqrt{\frac{(p-a)(p-b)}{(\omega-a)(\omega-b)}}$$

$$\overset{\shortparallel}{\frac{1}{2}} \oint_{\mathcal{C}_p} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \cdot \sqrt{\frac{(p-a)(p-b)}{(\omega-a)(\omega-b)}}$$

, combine contours to $\mathcal{C}$

$$\Rightarrow \boxed{W_0(p) = \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \sqrt{\frac{(p-a)(p-b)}{(\omega-a)(\omega-b)}}}$$

solution for $W_0(p)$ for arbitrary potential $V(p)$! Makeenko

- one can show that the same form is taken for multiple cuts Gl. hep-th/96...
- $\rho(\lambda)$ follows from $\text{Im}\, W_0(p)$ (4.5) or (4.4) ...

<u>examples:</u>

<u>Gaup</u> $V(p) = \frac{1}{2}p^2 \Rightarrow M(p) = \oint_{\mathcal{C}_\infty} \frac{d\omega}{2\pi i} \frac{\omega}{(\omega-p)\sqrt{(\omega-a)(\omega-b)}} = \oint_{\mathcal{C}_\infty} \frac{d\omega}{2\pi i} \frac{\omega}{\omega^2}\left(1-\frac{p}{\omega}\right)^{-1}\left(1-\frac{a}{\omega}\right)^{-\frac{1}{2}}\left(1-\frac{b}{\omega}\right)^{-\frac{1}{2}}$

$$= \oint_{\mathcal{C}_\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega} = 1$$

$\Rightarrow W_0(p) = \frac{1}{2}\left(p - 1\cdot\sqrt{(p^2-a^2)}\right)$ from symmetry $\sigma = [-a, a]$

$$W_0(p) = \frac{1}{p} + \frac{1}{N}\frac{\langle \mathrm{Tr} H\rangle^{\nearrow 0}}{p^2} + \frac{1}{N}\frac{\langle \mathrm{Tr} H^2\rangle}{p^3} + O\left(\frac{1}{p^4}\right)$$

$$= \frac{1}{2}\left(p - p\left(1-\frac{a^2}{p^2}\right)^{\frac{1}{2}}\right) = \frac{1}{2}\left(p - p + \frac{1}{2}\frac{a^2}{p} + \frac{1}{8}\frac{a^4}{p^3} + O\frac{1}{p^5}\right)$$

$\Rightarrow a = 2$, $\frac{1}{N}\langle \mathrm{Tr} H^2\rangle = 1$

$\boxed{W_0(p) = \frac{1}{2}\left(p - \sqrt{p^2-4}\right), \; \rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}}$ from discont. of $W_0(p)$ along σ

determination of Endpoints a, b : 2 Eqs for generic $V(p)$

$$W_0(p) = \frac{1}{p} + O\left(\frac{1}{p^2}\right) \quad \text{for } |p| \gg 1 ,$$

$$W_0(p) = \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\sqrt{(\omega-a)(\omega-b)}} \left(1 + \frac{1}{p}\left(\omega - \frac{1}{2}(a+b)\right) + O\left(\frac{1}{p^2}\right) \right)$$

$$\Leftrightarrow \boxed{\delta_{k,1} = \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{V'(\omega)\, \omega^k}{\sqrt{(\omega-a)(\omega-b)}}} \qquad k = 0, 1 \qquad \boxed{*}$$

example quartic potential : $\quad V(p) = \frac{g_2}{2} p^2 + \frac{g_4}{4} p^4$, $d = 4$, symmetric $\Rightarrow a = -b$

- 1-cut $\sigma = [-a, a]$:

determination of $M(p)$: we need to pick $O\left(\frac{1}{\omega}\right)$ in

$$\frac{V'(\omega)}{(\omega - p)\sqrt{\omega^2 - a^2}} = \frac{g_2 \omega + g_4 \omega^3}{\omega^2} \left(1 - \frac{p}{\omega}\right)^{-1} \left(1 - \frac{a^2}{\omega^2}\right)^{-\frac{1}{2}}$$

$$= \left(\frac{g_2}{\omega} + g_4 \omega\right)\left(1 + \frac{p}{\omega} + \frac{p^2}{\omega^2} + \frac{1}{2}\frac{a^2}{\omega^2} + O\left(\frac{1}{\omega^3}\right)\right)$$

$$\Rightarrow M(p) = \oint_{\infty} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - p)\sqrt{\omega^2 - a^2}} = g_2 + \left(p^2 + \frac{a^2}{2}\right) g_4 , \quad d - 2 = 2 \ \checkmark$$

$$\Rightarrow \boxed{W_0(p) = \frac{1}{2}\left(g_2 p + g_4 p^3 - \left(g_2 + g_4\left(p^2 + \frac{a^2}{2}\right)\right)\sqrt{p^2 - a^2} \right)} , \quad \rho(\lambda) = \ldots$$

in $\boxed{*}$ $k = 1$: $\quad 1 = \frac{1}{4} g_2 a^2 + \frac{3}{16} g_4 a^4$, ($k = 0$ identically 0)

* this is the one cut solution for the quartic potential

We have a second possibility here :

- 2 cuts $\sigma = [-b, -a] \cup [a, b]$ $\quad a < b \quad \Rightarrow M(p) = g_4 p$ of degree $4 - 3 = 1$

$$\Rightarrow W_0(p) = \frac{1}{2}\left(g_2 p + g_4 p^3 - g_4 p \sqrt{(p^2 - a^2)(p^2 - b^2)}\right)$$

Endpoints $a, b = ?$ modify $\boxed{*}$ to $\delta_{k,s} = \ldots$ for $k = 0, 1, \ldots, s$ for s cuts

* for a generic $V(\varphi)$ (non-symmetric) this provides only $s+1$ eqs. for $2s$ unknown endpoints

(for $s=2$ and V symmetric still ok from symmetry)

$\to$ the remaining $s-1$ eqs follow either by

- fixing filling fractions

$N = \sum_{i=1}^{s} N_s$

see Marino script

- requiring equal chemical potential between the cuts

$$0 = \int_{\bar{\sigma}_i} d\lambda\, M(\lambda) \sqrt{\prod_{j=0}^{2s} (\lambda - a_j)} \qquad i = 1, \dots, s-1$$

Jurkiewicz 90
G.A. hep-th/9606004

* the large-N limit for $s \geq 2$ cuts is more complicated as in general non-perturbative terms appear in addition to the genus expansion

(due to $N \in \mathbb{N}$ when $N \to \infty$, Jacobi theta fct : Bonnet, David, Eynard cond-mat/0003324

will come back to that

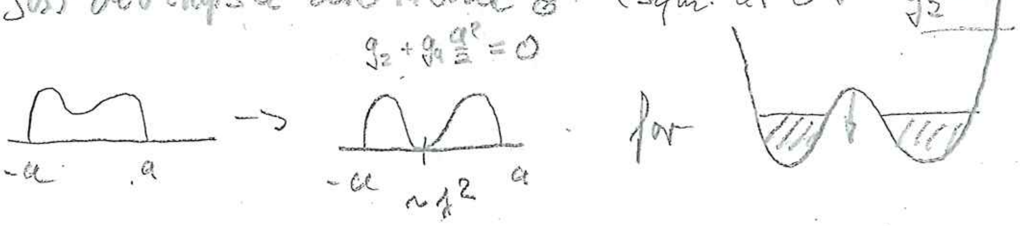
Phase transitions in the quartic potential :

1-cut : we need that $\rho(\lambda) = (g_2 + g_4(\lambda^2 + \frac{a^2}{2}))\sqrt{a^2 - \lambda^2}$)

remains positive on $\sigma \Rightarrow$ 2 possibilities for a transition

i) $\rho(\lambda)$ develops a zero inside σ (sym. at 0) $g_2 < 0$

$g_2 + g_4 \frac{a^2}{2} = 0$



$\to$ for generic $V(\varphi)$ $\rho(\lambda)$ can only vanish as $(\lambda - \lambda_c)^{2m}$, $m = 1, 2, \dots$ inside σ

ii) $\rho(\lambda)$ develops zero at endpoint $g_4 < 0$ (exercise: $g_4^c = -\frac{1}{12} g_2^2$)

$\overset{\text{crit}}{\rho}(\lambda) \sim (a^2 - \lambda^2)\sqrt{a^2 - \lambda^2}$

for $g_4 < 0$ the matrix integral is a formal one (non-convergent)

→ for generic $V(p)$ polynomial $g(1)$ can only vanish as $(a_i - 1)^{k+\frac{1}{2}}$ $k = 1, 2, \ldots$

* we have to see if this multicritical behaviour in g (and W_0) persists in higher genus contributions $W_{g\geq 1}$, $F_{g\geq 1}$

* if we want to generate different exponents we need a 2-matrix model

- Higher genus contributions to the resolvent

- once we have $W_g(p)$ we can compute $W_g(p_1, p_2, \ldots)$ using $\frac{d}{dV}$

insert $W(p) = \sum_{g=0}^{\infty} N^{-2g} W_g(p)$ into loop eq., compare orders

$$W(p)^2 = (\qquad)^2 = \sum_{g=0}^{\infty} \sum_{g'=0}^{g} W_{g'}(p) W_{g-g'}(p)$$

loop eq

$$\underline{(\hat{K} - 2 W_0(p)) W_g(p)}_{g'=0,g} = \underline{\sum_{g'=1}^{g-1} W_{g'}(p) W_{g-g'}(p) + \frac{d}{dV(p)} W_{g-1}(p)}$$

$g = 1, 2, \ldots$

where $\hat{K} W_g(p) = \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} W_g(\omega)$ is an integral operator

* observe that the RHS only depends on $W_{g'=0,1,2,\ldots g-1}$
thus determining $W_g(p)$ on the LHS IF we can invert $(\hat{K} - 2W_0)$
as an operator, e.g. $g=1$: $(\hat{K} - 2W_0(p)) W_1(p) = W_0(p,p)$

Strategy: find basis for $(\hat{K} - 2W_0)\, \chi_a^{(n)}(p) = \frac{1}{(p-a)^n}$ as RHS is a rational function,

so $W_0(p) \Rightarrow W_1(p), W_1(p,p) \Rightarrow W_2(p)$ etc.

* note that for $s>1$ $(\hat{K}-2W_0)$ has a non-trivial Kernel = zero modes

Universality – for general $V(\lambda)$, $\sigma=[b,a]$ $a>b$

change variables from $\{g_i\}$ couplings to $a, b, M_{a,b}^{(k)}$

moments $\boxed{M_a^{(k)} = \oint_{e} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega-a)^{k+\frac{1}{2}}(\omega-b)^{\frac{1}{2}}}} = g_{k+1} + g_{k+2}(\dots)\dots$ $k=1,2,\dots$

likewise for $a\leftrightarrow b$

* a quantity is called universal if for an arbitrary # of couplings g_i it only depends on a finite number of moments (= univ. parameters)

Example: $W_g(p) = \sum_{n=1}^{n_g} \left(A_g^{(n)} \chi_a^{(n)}(p) + B_g^{(n)} \chi_b^{(n)}(p) \right)$ $g\geq 1$

only depends on $a, b, M_{a,b}^{(n=1,\dots,3g-1)}$ for arbitrary V and are thus universal

[Ambjørn, Chekhov, Kristjansen, Makeenko hep-th/9302014]

* $W_0(p)$ is not universal, depends on a, b and all g_i explicitly!
and so does $\rho(\lambda)$
and $F_0 = \int\rho V - \int\int\rho\rho \ln$

(although ρ_{sc} is universal in a weak sense: $\forall\beta$, and changing Gauss random variables to non iid random var.))

• it can be checked explicitly that when going to a multicrit point, e.g. $\rho(\lambda)\sim(a-\lambda^2)^{\frac{3}{2}}$ by tuning $g_2, g_4 \to g_2^c, g_4^c$

the moments and thus $W_g(p) \sim (x-x_c)^{\frac{2-\gamma_{str}}{2}\chi}$ and that thus a double scaling limit can be defined

• multicrit points are also universal: higher order V with same criticality gives same W_g

Computation of $W_0(p,p) = \frac{d}{dV}(p)\, W_0(p)$: needed when computing $W_1(p)$

result: $\boxed{W_0(p,p) = \frac{(a-b)^2}{16(p-a)^2(p-b)^2}}$ universal

• we need $\frac{d}{dV}(p) V'(\omega) = -\sum_{j=1}^{\infty} \frac{j}{p^{j+1}} \frac{\partial}{\partial g_j} \sum_{\ell=1}^{\infty} g_\ell \omega^{\ell-1} = -\sum_{j=1}^{\infty} j \frac{\omega^{j-1}}{p^{j+1}}$

$= \partial_p \sum_{j=1}^{\infty} \frac{\omega^{j-1}}{p^j} = \partial_p \frac{1}{p-\omega}$

$\Rightarrow \frac{d}{dV}(p) W_0(p) = \frac{d}{dV}(p) \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \underbrace{\sqrt{\frac{(p-a)(p-b)}{(\omega-a)(\omega-b)}}}_{\equiv \sqrt{\;}}$

$= \underbrace{\frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{1}{(p-\omega)} \left(\partial_p \frac{1}{p-\omega}\right) \sqrt{\;}}_{\text{deform } \mathcal{C} \to \mathcal{C}_p \cup \mathcal{C}_\infty} + \frac{1}{2} \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \sqrt{\;} \cdot \frac{1}{2} \left(\frac{da}{dV}(p) \underbrace{\left(\frac{1}{p-a} - \frac{1}{\omega-a}\right)}_{= \frac{p-\omega}{(p-a)(\omega-a)}} + \frac{db}{dV}(p) \underbrace{\left(\frac{1}{p-b} - \frac{1}{\omega-b}\right)}_{= \frac{p-\omega}{(p-b)(\omega-b)}}\right)$

$= -\frac{1}{4} \partial_\omega^2 \sqrt{\;}\Big|_{p=\omega} + \frac{1}{4\sqrt{(p-a)(p-b)}} \left((p-b) \frac{da}{dV}(p) M_1^{(a)} + (p-a) \frac{db}{dV}(p) M_1^{(b)} \right)$

likewise

• $\frac{d}{dV}(p)$ of ⊠ : $0 = \frac{1}{2} \frac{d}{dV}(p) \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{V'(\omega)\, \omega^k}{\sqrt{(\omega-a)(\omega-b)}}$ $\quad$ $k = 0, 1$

$= \frac{1}{2} \left(a^k \frac{da}{dV}(p) M_1^{(a)} + b^k \frac{db}{dV}(p) M_1^{(b)} \right) + \partial_p \frac{p^k}{\sqrt{(p-a)(p-b)}}$

solving for $\frac{da}{dV}(p) M_1^{(a)}$, $\frac{db}{dV}(p) M_1^{(b)}$ and inserting above will give the result

$\Rightarrow \boxed{(\hat{K} - 2W_0(p))\, W_1(p) = W_0(p,p)}$ has rational function on RHS.

basis: $(\hat{K} - 2W_0(p))\, \chi_a^{(n)}(p) = \frac{1}{(p-a)^n}$ has sol. $\chi_a^{(1)}(p) = \frac{1}{M_1^{(a)}} \frac{1}{(p-a)\sqrt{(p-a)(p-b)}} = \frac{da}{dV}$ ...

for $n=1$, etc $\Rightarrow W_1(p)$, likewise one can integrate $W_1(p) = \frac{d}{dV}(p) F_1$ etc

$$\Rightarrow \quad F_1 = -\frac{1}{24}\ln\left(M_a^{(1)} M_b^{(1)}\right) - \frac{1}{6}\ln(a-b)$$

universal genus 1 free energy for arbitrary potential
. scales logarithmically at $\alpha = \alpha_c$

Loop eqs. and Virasoro constraints

back to top of page 20:

$$0 = \left\langle \left(\mathrm{Tr}\left(\frac{1}{p-H}\right)\right)^2 \right\rangle - N\left\langle \mathrm{Tr}\left(\frac{1}{p-H} V'(H)\right)\right\rangle$$

$$= \sum_{k=1}^{\infty} \frac{1}{p^{k+1}} \left\{ \sum_{\ell=0}^{k-1} \langle \mathrm{Tr} H^\ell \, \mathrm{Tr} H^{k-\ell-1} \rangle - N \langle \mathrm{Tr} H^k V'(H) \rangle \right\}$$

relabeling the couplings to $V(H) = \sum_{k=0}^{\infty} t_k H^k$ $\quad (t_k = \frac{g_k}{k}$, added $t_0)$

one obtains for $\vartheta(t) = e^{Z}$

$L_k \vartheta(t) = 0$ for $k \geq -1$ with

$$L_k = \frac{1}{N^2} \sum_{m+n=k} \frac{\partial^2}{\partial t_m \partial t_n} + \sum_{m=0} m\, t_m \frac{\partial}{\partial t_{m+k}}$$

$\Rightarrow$ L_k satisfy the Virasoro algebra (with $c=0$ = classical)

$$[L_m, L_n] = (m-n) L_{m+n}$$

Reminder: Density – Resolvent relation

- we had defined at finite-N the eigenvalue density
$$\rho_N(\lambda) = \frac{1}{N}\Big\langle \sum_{i=1}^{N} \delta(\lambda-\lambda_i)\Big\rangle = \frac{1}{N}\big\langle \mathrm{Tr}\,\delta(\lambda - H)\big\rangle$$

and resolvent
$$W(\lambda) = \frac{1}{N}\Big\langle \mathrm{Tr}\,\frac{1}{\lambda - H}\Big\rangle$$

at large-N
$$\rho(\lambda) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\big(W_0(\lambda - i\varepsilon) - W_0(\lambda + i\varepsilon)\big) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\mathrm{Im}\, W_0(\lambda+i\varepsilon)$$

higher order correlation functions

e.g. 2-point function
$$\rho_N(\lambda,\mu) = \frac{1}{N^2}\big\langle \mathrm{Tr}\,\delta(\lambda-H)\,\mathrm{Tr}\,\delta(\mu-H)\big\rangle .$$

$$\xrightarrow[N\to\infty]{} = \rho_{conn}(\lambda,\mu) + \rho(\lambda)\rho(\mu) + \underbrace{\delta(\mu-\lambda)\,\rho(\lambda)}_{\text{contact term}}$$

and we obtain the connected part from $W_0(p,q)$ [see eg. hep-th/9203009] via the discontinuity in both arguments

$$\rho_{conn}(\lambda,\mu) = \frac{\lim_{\varepsilon,\delta\to 0}}{(2\pi i)^2}\Big(W_0(\lambda-i\varepsilon,\mu-i\delta) - W_0(\lambda+i\varepsilon,\mu-i\delta) - W_0(\lambda-i\varepsilon,\mu+i\delta) + W_0(\lambda+i\varepsilon,\mu+i\delta)\Big)$$
$$= \frac{1}{\pi^2}\,\frac{-\mu\lambda + \frac{1}{2}(\mu+\lambda)(a+b) - ab}{(\mu-\lambda)^2\sqrt{(\lambda-a)(\mu-a)(b-\lambda)(b-\mu)}}$$

* How do we get individual ev distributions of their repulsion?
→ we have to study a different, microscopic limit (s);
what we've done so far: smooth fluctuation: macroscopic, global density
(also called wide correlators)

Exact Solution of Matrix Models for finite-N: Orthogonal Polynomials

goal: compute partition function, k-point density correl. funct. individual ev distributions → large-N limit(s)

application: Quantum Chromo Dynamics (QCD)
claim: low energy QCD = chiral Perturbation Theory (chPT) the effective field theory of the Goldstone Bosons is described by chiral RMT in the limit where chPT is dominated by the zero modes — this can be checked with Lattice QCD

* for this application we have to consider the 3 classes of RMT β-op.: ev of $WW^\dagger$ on $\mathbb{R}_+$

partition function
$$Z^{\beta}_{N,\nu} = \underset{\text{known, Vol}(O(S)U(N))}{\text{const}} \underbrace{\int_0^\infty \prod_{i=1}^N d\lambda_i\, \lambda_i^{\frac{\beta}{2}(\nu+1)-1} e^{-\lambda_i}}_{\int \prod_{i=1}^N d\lambda_i\, w(\lambda_i)} |\Delta_N(\lambda)|^\beta \qquad \beta=1,2,4$$

from now on general weight function $w(\lambda)$

$\nu = 0,1,2,\ldots$ zero ev from W $N\times(N+\nu)$ rectangular

k-point density correlation funct
$$R^{(\beta)}_{N,k}(\lambda_1,\ldots,\lambda_k) \equiv \frac{N!}{(N-k)!\, Z^{(\beta)}_N} \int d\lambda_{k+1}\ldots d\lambda_N \prod_{i=1}^N w(\lambda_i)\, |\Delta_N(\lambda)|^\beta$$

notice: * different normalisation $\int d\lambda\, R^{(\beta)}_{N,k=1}(\lambda) = N$

* no contact terms $R^{(\beta)}_{N,k=2}(\lambda,\mu) = w(\lambda)w(\mu) \int d\lambda_3 w(\lambda_3)\ldots d\lambda_N w(\lambda_N)\, |\Delta_N(\lambda)|^\beta \Big|_{\substack{\lambda_1=\lambda \\ \lambda_2=\mu}}$

does not contain $\delta(\lambda-\mu)\, R^{(\beta)}_{N,k=1}$

* these quantities can be computed using orthogonal polynomials

w.r.t. weight $w(\lambda)$: $\boxed{\int d\lambda\, w(\lambda)\, \tilde{P}_k(\lambda)\, \tilde{P}_\ell(\lambda) = \delta_{k\ell}\, h_\ell}$ for $\beta=2$ ($\beta=1,4$ skew op)

Properties of orthogonal polynomials (OP) of real variables

* can be constructed using Gram-Schmidt:

$$\tilde{P}_k(\lambda) = \frac{1}{c_k}\begin{vmatrix} m_0 & m_1 & & m_k \\ \vdots & & & \\ m_{k-1} & m_k & \dots & m_{2k-1} \\ 1 & \lambda & \dots & \lambda^{k-1} \end{vmatrix}, \quad c_k = \begin{vmatrix} m_0 & \dots & m_{k-1} \\ \vdots & & \\ m_{k-1} & \dots & m_{2k-2} \end{vmatrix} = \det_{0\le a,b\le k-1}[m_{a+b}]$$

Hankel det of m_a
(Toeplitz det: $\det m_{a-b}$)

or recursively (useful for $N\to\infty$):

$$\lambda \tilde{P}_k(\lambda) = \tilde{P}_{k+1}(\lambda) + a_k \tilde{P}_k(\lambda) + b_k \tilde{P}_{k-1}(\lambda)$$ 3-step recursion

proof: $\lambda \tilde{P}_k(\lambda) = \sum_{\ell=0}^{k+1} \alpha_\ell^{(k)} \tilde{P}_\ell(\lambda)$, $\alpha_\ell^{(k)} = h_\ell^{-1}\int d\lambda\, w(\lambda)\, \underbrace{\lambda \tilde{P}_k(\lambda)}\, \tilde{P}_\ell(\lambda)$
as $\tilde{P}_\ell(\lambda)$ complete set of functions, $\ell=0,1,\dots$; $O(\lambda^{\ell+1}) \perp$ to k if $\ell < k-1$

remark: • a_k, b_k can be expressed through h_k and d_k in $\tilde{P}_k(\lambda) = \lambda^k + d_k \lambda^{k-1} + O(\lambda^{k-2})$
(see e.g. Abramowitz-Stegun, chap 22, or exercise)

• $\frac{\partial}{\partial\lambda}\tilde{P}_k(\lambda)$ can also be expanded in $P_{\ell\le k-1}$, with recursion depth depending on $\deg V(\lambda) = d$

example Hermite: $w(\lambda) = e^{-\lambda^2}$ on $\mathbb{R}$, $h_k = \sqrt{\pi}\, 2^{-k} k!$, $a_k = 0$ from parity: $\forall\, V(\lambda) = V(-\lambda)$ sym, $w(\lambda) = e^{-V(\lambda)}$ on $\mathbb{R}$, $a_k = 0$

Orthonormal polynomials (for later purpose) $P_k(\lambda) \equiv \frac{\tilde{P}_k(\lambda)}{\sqrt{h_k}}$

$$\Rightarrow \delta_{k\ell} = \int d\lambda\, w(\lambda)\, P_k(\lambda) P_\ell(\lambda), \quad \lambda P_k(\lambda) = c_{k+1} P_{k+1}(\lambda) + a_k P_k(\lambda) + c_k P_{k-1}(\lambda)$$

with $c_k = \sqrt{\frac{h_k}{h_{k-1}}}$ (exercise show this), $a_k = \int d\lambda\, w(\lambda)\, \lambda\, P_k(\lambda)^2$

with $h_k = \int d\lambda\, w(\lambda)\, \tilde{P}_k(\lambda)^2 = \|\tilde{P}_k\|^2$ squared norms,

we choose monic normalisation $\tilde{P}_k(\lambda) = \lambda^k + O(\lambda^{k-1})$

(only condition on weight: all moments $m_k = \int d\lambda\, w(\lambda)\, \lambda^k < \infty$, $w(\lambda) > 0$ everywhere)

Examples: $w(\lambda) = e^{-\lambda^2}$ on $\mathbb{R}$ $\Rightarrow$ $\tilde{P}_n(\lambda) = 2^{-n} H_n(\lambda)$ Hermite poly.

$w(\lambda) = \lambda^{\nu} e^{-\lambda}$ on $\mathbb{R}_+$ $\Rightarrow$ $\tilde{P}_n(\lambda) = \frac{(-1)^n}{n!} L_n^{(\nu)}(\lambda)$ gen. Laguerre poly.

why? Vandermonde determinant:

$$\Delta_N(\lambda) = \prod_{1\le i<j\le N} (\lambda_j - \lambda_i) = \det_{1\le i,j\le N} \left[\lambda_i^{\,j-1}\right] = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & & \\ \vdots & & & & \\ 1 & \lambda_N & & & \lambda_N^{N-1} \end{vmatrix}$$

linear algebra: add columns from left to right

$$= \begin{vmatrix} 1 & q_1(\lambda_1) & \cdots & q_{N-1}(\lambda_1) \\ 1 & q_1(\lambda_2) & & \\ \vdots & & & \\ 1 & q_1(\lambda_N) & & q_{N-1}(\lambda_N) \end{vmatrix}$$

for any monic polynomials $q_k(\lambda) = \lambda^k + O(\lambda^{k-1})$

* for $\beta = 2$ we have $\Delta_N(\lambda)^2$, can use orthogonality when $q_k(\lambda) = \tilde{P}_k(\lambda)$

$$\Delta_N(\lambda) = \det_{1\le i,j\le N}\left[\tilde{P}_{j-1}(\lambda_i)\right] = \underbrace{\sum_{\sigma\in S_N}}_{\text{all perm}} \underbrace{(-)^{\sigma}}_{\text{sign}} \tilde{P}_{\sigma(1)-1}(\lambda_1)\, \tilde{P}_{\sigma(2)-1}(\lambda_2) \cdots \tilde{P}_{\sigma(N)-1}(\lambda_N)$$

$$\Rightarrow Z_N^{\beta=2} = \text{Const} \int \prod_{i=1}^{N} d\lambda_i\, w(\lambda_i) \sum_{\sigma,\sigma'} (-)^{\sigma+\sigma'} \tilde{P}_{\sigma(1)-1}(\lambda_1) \cdots \underbrace{\tilde{P}_{\sigma(N)-1}(\lambda_N)\, \tilde{P}_{\sigma'(1)-1}(\lambda_1)}_{\delta_{\sigma(N),\sigma'(N)}\, h_{\sigma(N)-1}} \cdots \tilde{P}_{\sigma'(N)-1}(\lambda_N)$$

$$= \text{Const} \sum_{\sigma} \underbrace{h_{\sigma(1)}\, h_{\sigma(2)} \cdots h_{\sigma(N)}}_{\prod_{j=1}^{N} h_{j-1}}$$

$$Z_N^{\beta=2} = \text{Const}\, N! \prod_{j=0}^{N-1} h_j$$

part. fund = prod of squared norms

(also true for $\beta = 1, 4$, diff. scalar product) later

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Hankel det of m_q (Toeplitz det: $\det m_{a-b}$)

or recursively (useful for $k\to\infty$):

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proof: $\lambda \tilde P_k(\lambda) = \sum_{\ell=0}^{k+1} \alpha_\ell^{(k)} \tilde P_\ell(\lambda)$ as $\tilde P_\ell(\lambda)$ complete set of functions, $\alpha_\ell^{(k)} = h_\ell^{-1}\int d\lambda\, w(\lambda)\, \lambda \tilde P_k(\lambda)\, \underbrace{\tilde P_\ell(\lambda)}$, $\ell = 0,1,\dots$

$O(\lambda^{\ell+1}) \perp$ to k if $\ell \le k-2$

remark: • a_k, b_k can be expressed through h_k and d_k in $\tilde P_k(\lambda) = \lambda^k + d_k \lambda^{k-1} + O(\lambda^{k-2})$.
(see e.g. Abramowitz–Stegun, chap. 22, or exercise)

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$\Rightarrow \delta_{k\ell} = \int d\lambda\, w(\lambda) P_k(\lambda) P_\ell(\lambda)$, $\lambda P_k(\lambda) = c_{k+1} P_{k+1}(\lambda) + a_k P_k(\lambda) + c_k P_{k-1}(\lambda)$

with $c_k = \sqrt{\frac{h_k}{h_{k-1}}}$ (exercise show this), $a_k = \int d\lambda\, w(\lambda)\, \lambda P_k(\lambda)^2$

- the kernel $\boxed{K_N(x,y) = \sum_{\ell=0}^{N-1} P_\ell(x)\, P_\ell(y)}$

* important object: building block $\forall$ $R_{k\text{-}point}$ [& important in approx of functions through OP]

<u>contraction properties</u>

- $\int d\lambda\, \omega(\lambda)\, K_N(\lambda,\lambda) = N$ (we'll see later $R^{k=1}_{N,\beta=2} = \omega(\lambda) K_N(\lambda,\lambda)$)
- $\int d\lambda\, \omega(\lambda)\, K_N(x,\lambda)\, K_N(\lambda,y) = K_N(x,y)$, $\cdot \int d\lambda\, \omega(x)\, K_N(\lambda,x) = 1$

<u>proofs:</u> orthonormality of OP (exercise)

<u>Christoffel - Darboux identity (i):</u> (only depends on degree N & $N-1$)

i) $K_N(x,y) = c_N \dfrac{P_N(x) P_{N-1}(y) - P_N(y) P_{N-1}(x)}{x-y}$, $x \neq y$, $c_N = \sqrt{\dfrac{h_N}{h_{N-1}}}$

ii) $K_N(x,x) = c_N \left(P_N'(x)\, P_{N-1}(x) - P_N(x)\, P_{N-1}'(x) \right)$ convenient for large-N analysis!

proof i): $(x-y)\, K_N(x,y) = \sum_{\ell=0}^{N-1} \left(x P_\ell(x) P_\ell(y) - P_\ell(x)\, y P_\ell(y) \right)$ use 3step rec. & telescopic sum

$= c_N \left(P_N(x)\, P_{N-1}(y) - P_{N-1}(x)\, P_N(y) \right)$

ii) Taylor expand in limit $x \to y$ $\square$ $\Rightarrow K_N(x,y) = c_N \dfrac{\det \begin{vmatrix} P_N(x) & P_N(y) \\ P_{N-1}(x) & P_{N-1}(y) \end{vmatrix}}{\Delta_{N=2}(x,y)}$

<u>Relating OP and kernel to average characteristic polynomials</u>

- $P_N(\lambda) = \langle \det(\lambda - H) \rangle_{\#\text{of ev}=N} = \dfrac{1}{N! \prod_{j=0}^{N-1} h_j} \int \prod_{i=1}^{N} d\lambda_i\, \omega(\lambda_i)\, (\lambda-\lambda_i)\, |\Delta_N(\lambda)|^2$

[Heine-formula, 19th century] = N-fold integral rep for OP for general weight function $\omega(x)$!

check $\lambda \to \infty$ $\langle \det(\lambda - H) \rangle_N = \langle \lambda^N \rangle_N + O(\lambda^{N-1}) = \lambda^N \cdot 1 + O(\lambda^{N-1})$

so ok

proof: absorb product in Vandermonde – frequent trick!

$$\text{def } \lambda_{N+1} = \lambda \Rightarrow \prod_{i=1}^{N}(\lambda-\lambda_i)\underbrace{\prod_{j>i}^{N}(\lambda_j-\lambda_i)}_{\Delta_N(\lambda)} = \Delta_{N+1}(\lambda) = \det\left[\tilde{P}_{j-1}(\lambda_i)\right]_{1\le i,j\le N+1}$$

$$\Rightarrow \langle \det(\lambda-H)\rangle = \frac{1}{N!\prod_{j=0}^{N-1}h_j}\int\prod_{i=1}^{N}d\lambda_i\,w(\lambda_i)\sum_{\substack{\sigma\in S_{N+1}\\ \sigma'\in S_N}}(-)^{\sigma+\sigma'}\underbrace{\tilde{P}_{\sigma(1)-1}(\lambda_1)\ldots\tilde{P}_{\sigma(N)-1}(\lambda_N)\,\tilde{P}_{\sigma(N+1)-1}(\lambda_{N+1})\,\tilde{P}_{\sigma'(1)-1}(\lambda_1)\ldots\tilde{P}_{\sigma'(N)-1}(\lambda_N)}_{\delta_{\sigma(1),\sigma'(1)}\,h_{\sigma(1)-1}}$$

* we can only saturate all pairs when the degree of $\tilde{P}(\lambda_{N+1})$ is N, else one δ gives zero

$$= \frac{1}{N!\prod_{j=0}^{N-1}h_j}\sum_{\sigma\in S_N}\tilde{P}_N(\lambda_{N+1})\underbrace{\prod_{j=1}^{N}h_{\sigma(j)-1}}_{\prod h_j} = \tilde{P}_N(\lambda) \qquad \square$$

* notice: for $w(\lambda)=e^{-\lambda^2}$ Hermite $\exists$ "dual" single-integral rep

$$H_N(\lambda) = \frac{2^N}{\sqrt{\pi}}\int_{-\infty}^{\infty}dt\,e^{-t^2}(\lambda+it)^N = \ldots = \langle \det(\lambda-\underset{\substack{\uparrow\\ N\times N\\ \text{matrix}}}{H})^N\rangle_\lambda$$

⇒ much more usefull for $N\to\infty$ saddle point evaluation

(one can show this using supersymmetry, $\langle\det\rangle\sim$ Fermions)

* $\underline{h_N\,K_{N+1}(\lambda,\mu) = \langle\det(\lambda-H)\det(\mu-H)\rangle_N}$ — P. Zinn-Justin cond-mat/9703033

check $\lambda,\mu\gg 1$: LHS $\approx h_N\,P_N(\lambda)P_N(\mu) = h_N\frac{\lambda^N}{\sqrt{h_N}}\frac{\mu^N}{\sqrt{h_N}} = (\lambda\mu)^N$ RHS

* absorb each product $\prod_{i=1}^{N}(\lambda-\lambda_i)$, $\prod_{i=1}^{N}(\mu-\lambda_i)$ in a different

$$\text{RHS} = \frac{1}{N!\prod_{j=0}^{N-1}h_j}\int\prod_{i=1}^{N}d\lambda_i\,w(\lambda_i)\sum_{\sigma,\rho\in S_{N+1}}(-)^{\sigma+\rho}\tilde{P}_{\sigma(1)-1}(\lambda_1)\ldots\underbrace{\tilde{P}_{\sigma(N)-1}(\lambda_N)\tilde{P}_{\sigma(N+1)-1}(\lambda)\,\tilde{P}_{\rho(1)-1}(\lambda_1)}_{h_{\sigma(1)}\delta_{\sigma(1),\rho(1)}}\ldots\tilde{P}_{\rho(N)-1}(\lambda_N)\tilde{P}_{\rho(N+1)-1}(\mu)$$

→ now all contractions with δ_{ij} can be satisfied whenever the degree of $\tilde{P}(\lambda)$ and $\tilde{P}(\mu)$ is equal, not necessary N

$$\Rightarrow \text{RHS} = \frac{1}{N!\prod_{j=0}^{N}h_j} \underbrace{\sum_{\sigma\in S_N}}_{\substack{\text{perm of } N \\ \text{variables } \in\{1,\dots,N\}}} \sum_{j=0}^{N} \tilde{P}_j(\lambda)\tilde{P}_j(\mu) \prod_{\substack{\ell=1 \\ \sigma(\ell)\neq j}}^{N} h_{\sigma(\ell)-1} \quad \leftarrow (N+1)\cdot N! \text{ terms}$$

$$= \frac{1}{N!\prod_{j=0}^{N-1}h_j} \sum_{\sigma\in S_N} \underbrace{\prod_{\ell=1}^{N-1} h_{\sigma(\ell)-1}}_{\prod_{\ell=0}^{N} h_\ell} \sum_{j=0}^{N} \frac{\tilde{P}_j(\lambda)\tilde{P}_j(\mu)}{h_j} = h_N K_N(\lambda,\mu)$$

$\square$

* note: $\exists$ closed formulas for $\left\langle \prod_{\ell=1}^{k} \det(\mu_\ell - H) \right\rangle_N$, we'll need them later (and for vectors)

<u>determination of the k-point density function $R_{N,k}$</u>

using $\det A = \det A^T$, $\det A \det B = \det A\cdot B$
and the fact that one can pull common factors in or out of rows or columns of det's we have

$$\Delta_N(\lambda)^2 = \det_{1\le i,j\le N}\left[\tilde{P}_{j-1}(\lambda_i)\right]^2 = \prod_{j=0}^{N-1} h_j \det_{i,j=1..N}\Big[\underbrace{P_{j-1}(\lambda_i)}_{A_{ij}}\Big]^2 \quad N\times N$$

$$= \prod_{j=0}^{N-1} h_j \det_{i..N} AA^T = \prod_{j=0}^{N-1} h_j \det_{1\le i,k\le N}\Big[\sum_{j=1}^{N} A_{ij}A_{kj}\Big], \quad \sum_{j=1}^{N} P_{j-1}(\lambda_i)P_{j-1}(\lambda_k) = K_N(\lambda_i,\lambda_k)$$

$\Rightarrow$ <u>the jpdf = integrand of Z can be written as:</u>

$$P_N^{\beta=2}(\lambda) = \prod_{j=1}^{N} w(\lambda_j)\,\Delta_N(\lambda)^2 = \prod_{j=0}^{N-1} h_j \det_{1\le i,k\le N}\left[w(\lambda_i)^{\frac12} w(\lambda_k)^{\frac12} K_N(\lambda_i,\lambda_k)\right] \quad \leftarrow N\times N \text{ det}$$

* such integrands that can be written as a det are called <u>determinantal point process</u>

* ev repell each other as for $\lambda_i = \lambda_k$ the integrand is zero
(= zero probab. to find equal ev)

Theorem Dyson–Mehta [see e.g. [illegible] Thm 5.1.4]

For a weighted kernel $K(x,y) \left(= w^{\frac{1}{2}}(x)\, w^{\frac{1}{2}}(y)\, K_N(x,y)\right)$ satisfying the the contraction property with normalisation $\int dx\, K(x,x) = c$ it holds

$$\boxed{\int dx_N \det_{1\le i,j\le N}[K(x_i,x_j)] = (c-N+1)\det_{1\le i,j\le N-1}[K(x_i,x_j)]}$$

* ⇒ the size of the det is reduced by one
* applying the theorem to the det of $R^{(\beta=2)}_{N,k}(\lambda_1,\dots,\lambda_N)$ successively $N-k$ times (with $c=N$) we obtain a reduction of $(N-k)$-integrals of an $N\times N$ det to

$$\Rightarrow \boxed{R^{(\beta=2)}_{N,k}(\lambda_1,\dots,\lambda_k)} = \det_{1\le i,j\le k}\left[w(\lambda_i)^{\frac{1}{2}} w(\lambda_j)^{\frac{1}{2}} K_N(\lambda_i,\lambda_j)\right]$$

$$= \prod_{l=1}^{k} w(\lambda_l) \det_{1\le i,j\le k}\left[K_N(\lambda_i,\lambda_j)\right] \leftarrow \text{det of size } k\times k$$

exact solution for arbitrary weights

⇒ the large-N limit can be taken, only contained in $P_N(\lambda)$, $P_{N-1}(\lambda)$ through Christoffel–Darboux!

Examples $R^{(1)}_{N,1}(\lambda_1) = w(\lambda_1) K_N(\lambda_1,\lambda_1)$ density

$R^{(2)}_{N,2}(\lambda_1,\lambda_2) = w(\lambda_1)w(\lambda_2)\left[K_N(\lambda_1,\lambda_1)K_N(\lambda_2,\lambda_2) - K_N(\lambda_1,\lambda_2)^2\right]$

← connected part, no contact term $\delta(\lambda_1-\lambda_2)$

$\boxed{\text{proof}}$: $\det_{N\times N}[K(x_i,x_j)] = \sum_{\sigma\in S_N} (-)^{\sigma} K(x_1,x_{\sigma(1)}) \cdots K(x_N,x_{\sigma(N)})$

• all $N!$ permutations can be reached exchanging 2nd arguments with

* first take all $(N-1)!$ permutations (including the identity) with $\sigma(N)=N$ ⇒ $\int dx_N\, K(x_N,x_N) = c$,

  the rest is a det of size $(N-1)\times(N-1)$

  ⇒ 1st term in the theorem $c \cdot \det_{1\le i,j\le N-1} K(x_i,x_j)$

* the remaining $(N-1)(N-1)!$ permutations are reached with 1 pair exchang

$$\left(K(x_1,x_{\sigma(1)}) \dots K(x_j,x_{\sigma(j)}) \dots K(x_{N-1},x_{\sigma(N-1)})\, K(x_N,x_N)\right)$$

∃ $(N-1)$ positions to do such an exchange, take in the j-th and the pair exchange gives an extra (-1), we thus get

$\Rightarrow \int dx_N\, K(x_j, x_N)\, K(x_N, x_{\sigma(j)}) = K(x_j, x_{\sigma(j)})$ for each such term

$\Rightarrow$ we get $(-1) \sum_{\sigma \in S_{N-1}} K(x_1, x_{\sigma(1)}) \ldots K(x_{N-1}, x_{\sigma(N-1)}) \cdot (N-1)$ from each position j

$= -(N-1) \det_{1 \le i,j \le N-1} \left[K(x_i, x_j) \right]$

<u>collecting all factors in $R_{N,k}^{(\beta=2)}$</u>

$$R_{N,k}^{(\beta=2)}(\lambda_1, \ldots, \lambda_k) = \left(\frac{1}{N! \prod_{j=0}^{N-1} h_j} \right) \frac{N!}{(N-k)!} \int d\lambda_{k+1} \ldots d\lambda_N \prod_{j=0}^{N-1} h_j \det_{N \times N} \left[\omega(\lambda_i)^{\frac{1}{2}} \omega(\lambda_j)^{\frac{1}{2}} K_N(\lambda_i, \lambda_j) \right]$$

each $\int$ gives
$1\,(N-(N-1)+1) \ldots$
$\cdot (N-(k+1)-1) = (N-k)!$

$$= \frac{(N-k)!}{(N-k)!} \det_{k \times k} \underbrace{\left[\omega(\lambda_i)^{\frac{1}{2}} \omega(\lambda_j)^{\frac{1}{2}} K_N(\lambda_i, \lambda_j) \right]}_{= \sum_{\ell=0}^{N-1} \psi_\ell(\lambda_i) \psi_\ell(\lambda_j)}$$

* <u>Notation</u>: sometimes $\omega^{\frac{1}{2}}(x) P_k(x) \equiv \psi_k(x)$ is called <u>wave function</u>,
with <u>$\int dx\, \psi_k(x) \psi_\ell(x) = \delta_{k\ell}$</u>, with <u>weighted kernel</u> $K(x,y) = \sum_{\ell=0}^{N-1} \psi_\ell(x) \psi_\ell(y)$

How can we determine all $R_{N,k}^{(\beta)}$ for $\beta = 1$ and 4?

* we need a different scalar product

$\beta = 2$ we had $\langle f, g \rangle_2 \equiv \int dx\, \omega(x) f(x) g(x)$

for $\beta = 1, 4$ we need an anti-symmetric, skew product (not pos. definite)

$$\beta = 4 \quad \langle f, g \rangle_4 = \int dx\, \omega(x) \left(f(x) g'(x) - f'(x) g(x) \right) = - \langle g, f \rangle_4$$

$$\beta = 1 \quad \langle f, g \rangle_1 = \int dx\, dy\, \omega(x) \omega(y)\, \epsilon(x-y)\, f(x) g(y) = - \langle g, f \rangle_1$$

<u>with step function $\epsilon(x)$</u> $= \begin{cases} +\frac{1}{2} & x > 0 \\ -\frac{1}{2} & x < 0 \end{cases}$

partition function for $\beta = 4$:

- It can be shown that

size $2N \times 2N$ $\qquad$ $q_j(\lambda) = \lambda^j + \dots$ arb. monic

$$\Delta_N(\lambda)^4 = \prod_{j>i}^N (\lambda_j - \lambda_i)^4 = \det_{\substack{1\le i\le N\\ 1\le j\le 2N}} \left[\lambda_i^{j} \quad j\lambda_i^{j-1} \right] = \det_{\substack{1\le i\le N\\ 1\le j\le 2N}} \left[q_j(\lambda_i) \quad q_j'(\lambda_i) \right]$$

- for $N = 2m$ even, $m \in \mathbb{N}$ one can define for $A = -A^T$ $N\times N$ antisymmetric

the Pfaffian determinant

$$\mathrm{Pf}[A] = \sqrt{\det[A]} = \frac{1}{(2^m m!)} \sum_{\substack{\sigma\in S_N\\ \text{all } N! \text{ perm}\\ \text{ordinary}}} (-)^\sigma A_{\sigma(1)\sigma(2)} \dots A_{\sigma(2m-1)\sigma(2m)}$$

no index twice, unlike in $\det A$: $A_{i\sigma(i)}$

$$= \sum_{\substack{\sigma\in S_N\\ \text{ordered s.th. } \sigma(1)<\sigma(2)\\ \sigma(3)<\sigma(4)\\ \vdots}} (-)^\sigma A_{\sigma(1)\sigma(2)} \dots A_{\sigma(2m-1)\sigma(2m)}$$

examples: $N=2$ $\quad \mathrm{Pf}[A] = A_{12} \quad \left(= \sqrt{\begin{vmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{vmatrix}} \text{ given } A_{12} > 0 \right)$

$N=4$: $\mathrm{Pf}[A] = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$

or [3] chap. 5

applying the 2nd de Bruijn integral formula (1955) we can solve $Z_N^{\beta=4}$ & $R_{k,N}^{\beta}$

$$\int dx_1 \dots dx_m \det_{\substack{1\le i\le m\\ 1\le j\le 2m}} \left[\varphi_j(x_i) \quad \psi_j(x_i) \right] = m!\, \mathrm{Pf}_{2m\times 2m} \left[\int dx\, \varphi_i(x)\psi_j(x) - \varphi_j(x)\psi_i(x) \right]$$

proof [3] chap. 5

$$\Rightarrow Z_N^{(\beta=4)} = \text{const} \int dx_1 \dots dx_N \prod_{i=1}^N w(x_i) \det_{\substack{1\le i\le N\\ 1\le j\le 2N}} \left[\tilde{Q}_j(x_i) \quad \tilde{Q}_j'(x_i) \right]$$

choose $\tilde{Q}_j(x)$ orthogonal wrt $\langle . \rangle_4$ to get δ_{kj} inside Pf.

$$\boxed{\begin{aligned} \langle \tilde{Q}_{2j}, \tilde{Q}_{2k+1} \rangle_4 &= h_j^{(\beta=4)} \delta_{kj} = -\langle \tilde{Q}_{2k+1}, \tilde{Q}_{2j} \rangle_4 \\ \langle \tilde{Q}_{2j}, \tilde{Q}_{2k} \rangle_4 &= 0 = \langle \tilde{Q}_{2j+1}, \tilde{Q}_{2k+1} \rangle_4 \end{aligned}}$$

- how to pull factors inside $Pf[A]$, $A = -A^T$

$c \cdot Pf A = c\sqrt{\det A} = \sqrt{c^2 \det A}$ pull c into j'th row and column $\rightarrow A \cdot j$ to keep

- we can divide and multiply by the norms of the $\tilde{Q}_j$, then apply $\det B \to$ resulting matrix antisym.

$$\Rightarrow Z_N^{(\beta=4)} = \text{const}\, N! \prod_{j=0}^{N-1} h_j^{(4)}\, Pf\left[\int d\mu(x) \frac{\tilde{Q}_i(x)}{\sqrt{h_i}} \frac{\tilde{Q}_j'(x)}{\sqrt{h_j}} - \frac{\tilde{Q}_j'(x)}{\sqrt{h_j}} \frac{\tilde{Q}_i(x)}{\sqrt{h_i}}\right]$$

$$Pf\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} = \pm 1$$

Example:

Laguerre weight $w(x) = x^{\alpha} e^{-x}$ on $\mathbb{R}_+$, $\alpha > -1$

$\tilde{Q}_{2j+1}(x) = -L_{2j+1}^{(\alpha)}(x) + L_{2j}^{(\alpha)}(x)$

$\tilde{Q}_{2j}(x) = L_{2j}^{(\alpha)}(x) - L_{2j-1}^{(\alpha)}(x) - \frac{2j+\alpha-1}{(2j-1)} \tilde{Q}_{2j-2}(x)$, $h_j^{(4)} = \frac{\Gamma(2j+1+\alpha)}{(2j)!}$

[1] chapter 1?

* one can also define a kernel from these skew OP and compute all $R_{N,k}^{(4)}$ in terms of a $Pf[\text{kernel}]$ [3] chapter 5

* the skew OP (and their kernel) enjoy a $\langle \det \rangle$ representation

$\tilde{Q}_{2j}(x) = \langle \det(x - H) \rangle_j$ not unique, drops out in antisym. $\langle \cdot \rangle_4$

$\tilde{Q}_{2j+1}(x) = \langle \det(x - H)(\mathrm{Tr} H + x + \text{const}) \rangle_j$

B. Eynard, cond-mat/0012046

* note that due to Kramers' degeneracy / quaternionic structure we have $\det(x - H) = \prod_{j=1}^{N} (x - \lambda_j)^2$

partition function for $\beta = 1$

- we have to deal with $|\Delta_N(\lambda)|$ in the jpdf

$\Rightarrow$ after ordering ev $\lambda_N \geq \lambda_{N-1} \geq \dots \geq \lambda_1$ we can drop the absolute value and replace $\det[\tilde{Q}_{i-1}(\lambda_j)] = \Delta_N(\lambda)$

$$Z_N^{(\beta=1)} = \text{const} \int d\lambda_1 \dots d\lambda_N\, \omega(\lambda_1) \dots \omega(\lambda_N)\, |\Delta_N(\lambda)|^1 \quad , \text{restrict } \int \text{to } [\ ]$$

$$= \text{const}\, N! \int d\lambda_1 \dots \int d\lambda_N \prod_{i=1}^{N} \omega(\lambda_i) \det[\tilde{Q}_{i-1}(\lambda_j)]$$
$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$$

use de Bruijn integral formula (proof [3] chap 6)

$$\underset{x_1 < x_2 < \dots < x_N}{\int dx_1 \dots \int dx_N} \det_{1 \leq i,j \leq N} [\varphi_i(x_j)] = \mathrm{Pf}_{1 \leq i,j \leq N} \left[\int dx\, dy\, \varepsilon(x-y)\, \varphi_i(x) \varphi_j(y) \right]$$

---

here we assume that N is even (N odd see [1])

- choosing the monic polynomials $\tilde{Q}_i(x)$ to be skew orthogonal wrt $\langle , \rangle_1$, $\langle \tilde{Q}_{2j}, \tilde{Q}_{2k+1} \rangle_1 = h_j^{(\beta=1)} \delta_{kj}$, other 0

we obtain again $Z_N^{(1)} = \text{const}\, N! \prod_{j=1}^{N} h_{j-1}^{(1)}$

---

and the same remarks as for $\beta = 4$ regarding correl. functions and the rep of the polynomials hold true, with the same reference and $\tilde{Q}_{2j}(x) = \langle \det(x - H) \rangle_{2j}$

Interlude: Random Matrix Limit of QCD

- QCD with # of colours $N_c = 2, 3$ (or more) fixed, choose rep. (fund/adj) and N_f flavours of quarks (often $N_f = 2, 3$ light ones)

$$Z_{QCD} = \int [d\Psi][dA]\, e^{-\int d^4x \left(\mathcal{L}_{\psi} + \mathcal{L}_{YM}\right)} \qquad \text{path integral}$$

with $\mathcal{L}_{YM} \sim \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$

$$\mathcal{L}_{\psi} \sim \sum_{f=1}^{N_f} \bar{\Psi}_f \left(\not{D}(A) + m_f \right) \Psi_f = \bar{\Psi} \left(\not{D} + \underbrace{M}_{\mathrm{diag}(m_1, \ldots, m_{N_f})} \right) \Psi$$

the Ψ's are Fermions and anticommute $\Rightarrow$ can integrate them out

using Grassmann variables: $\chi_{k=1,\ldots,n}$, $\chi_k \in \mathbb{C}$

anti-commutator $\{\chi_k, \chi_l\} = \chi_k \chi_l + \chi_l \chi_k = 0$

Integration $\int d\chi = 0$, $\int \chi\, d\chi = 1$, $\{\chi_k, d\chi_l\} = 0$

e.g. $n = 1$, $a \in \mathbb{R}$ bosonic

$$\int d\chi \int d\chi^* \, e^{\chi a \chi^*} = \int d\chi \int d\chi^* (1 + \chi a \chi^* + 0) = - \int \chi d\chi \int \chi^* d\chi^* a = -a$$

$n = 2$: $\int d\chi d\chi^* d\xi d\xi^* \, e^{\binom{\chi}{\xi}^T A \binom{\chi^*}{\xi^*}}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ bosonic

expand exp to 2. order: $(\chi a + \xi c, \chi b + \xi d) \begin{pmatrix} \chi^* \\ \xi^* \end{pmatrix} = \chi a \chi^* + \xi c \chi^* + \chi b \xi^* + \xi d \xi^*$

$$= \ldots = ad - cb = \underline{\underline{\det A}}$$

true in general (for Gauss-$\int$ over bosonic var. we get $(\det A)^{-1}$)

$\Rightarrow$ we can formally integrate out the $\Psi_f(x)$ above ($n = \infty$ or on lattice)

* can we also use this in RMT? yes, to compute e.g. $\langle \det(x - H) \rangle$, $e^{-\mathrm{Tr} H^2}$ are

$\Rightarrow \quad Z_{QCD} = \int [dA] \prod_{f=1}^{N_f} \det(\not{D}(A) + m_f)\, e^{-S_{YM}[A]}$

extension: add $i \int d^4x \, \mathrm{Tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} \cdot \theta = i\nu\theta$ topological term to S_{YM}

ν winding number

Atiyah-Singer index theorem: $\nu = \#$ of zero eigenvalues of $\not{D}$
$(= \nu_L - \nu_R$ see below$)$

$\Rightarrow \quad Z_{QCD} = \sum_{\nu \in \mathbb{Z}} e^{+i\theta\nu} \int [dA]_\nu \prod_{f=1}^{N_f} \det{}_\nu [\not{D}(A) + m_f]\, e^{-S_{YM}[A]} \equiv \sum_{\nu \in \mathbb{Z}} e^{i\theta\nu} Z_\nu$

Fourier mode, ν fixed

<u>chiral symmetry breaking</u>:

in Euclidean space $\gamma^{0,1,2,3}$ are Hermitian, $\gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ in chiral basis

def <u>projection operators</u> $P_{L/R} = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$, $P_{L/R}^2 = P_{L/R}$

$P_L P_R = 0 = P_R P_L$

$P_L + P_R = \mathbb{1}$

$\Rightarrow \quad \psi = P_L \psi + P_R \psi = \psi_L + \psi_R$

L left and R right handed ψ,
its $\#$ of zero modes $\nu_{L,R}$

chiral symmetry $\{\not{D}, \gamma_5\} = 0$ $\left(\Rightarrow \not{D} = i\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}\right.$ block off diag$\left.\right)$
and antiherm. $\not{D}^\dagger = -\not{D}$

$\Rightarrow$ <u>kinetic term</u>

$\bar{\psi} \not{D} \psi = \bar{\psi}_R \not{D} \psi_R + \bar{\psi}_L \not{D} \psi_L$ is invariant under

global rotation $U_L(N_f) \times U_R(N_f)$

$\psi_{L/R} \to U_{L/R} \psi_{L/R}$, $U_{L,R} \in U_{L/R}(N_f)$

$*$ this is true for $N_c = 3, 4, \ldots$ fund

for e.g. $N_c = 2$ fund the symm. group is larger $U(2N_f)$

- the mass term explicitly breaks chiral symmetry

$$\bar\psi M \psi = \bar\psi_R M \psi_L + \bar\psi_L M \psi_R$$

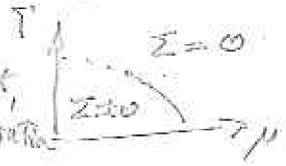
which is still invariant for $M = m\mathbb{1}$ if $U_L = U_R$

- at low energy the vacuum expectation value (vev) $\Sigma = |\langle \bar\psi\psi\rangle| \neq 0$ and spontaneously breaks chiral symmetry just as the mass term :

phase diagram cartoon

$$\boxed{U_L(N_f) \times U_R(N_f) \to U_V(N_f)}$$

analogy Ferromagnet, $\Sigma \to$ magnetisation



[Vafa-Witten: Vector symm. remain unbroken, there is a $U(1)$ sym broken by an anomaly :]

$$\underline{U_A(1) \times SU_L(N_f) \times SU_R(N_f) \times U_V \to U_V(1) \times SU_V(N_f)}$$

$\exists$ 3 chiral sym. breaking patterns for $U(N_c)$ fund/adj. gauge theories [M. Peskin 1980], these correspond to the 3 chiral RMT classes $\beta = 1, 2, 4$ of Verbaarschot

- "naive" matrix model replacement for QCD Shuryak, Verbaarschot hep-th/9212088

fix ν, replace: (precise limit later!)

$$Z_\nu^{QCD} \to Z_\nu = \int dW \prod_{f=1}^{N_f} \det[\not{D} + m_f]\, e^{-\mathrm{Tr}\, V(WW^\dagger)}$$

universality: choose Gaussian

where W $N \times N+\nu$ $W_{ij} \in \mathbb{C}$ for QCD with $N_c = 3$ ν zeromodes, fixed

$$\not{D} = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} \Rightarrow \det[\not{D} + m_f] = m_f^{\,\nu} \prod_{k=1}^{N} (\Lambda_k + m_f^2)$$

ev λ_k of $\not{D}$ come in pairs $\lambda_k = \pm\sqrt{\Lambda_k}$, $\Lambda_k > 0$ ev of $WW^\dagger$

(from now on let's rotate the ev $i\mathbb{R}$ of $\not{D}^\dagger = -\not{D}$ to $\mathbb{R}$)

$\Rightarrow$ diagonalisation of W $\Rightarrow$ Jacobian $|\Delta|^2$

$$Z_\nu = \text{const} \int_0^\infty \prod_{i=1}^{N} \left(d\lambda_i \, \lambda_i^{\nu} \prod_{f=1}^{N_f} (\lambda_i + m_f^2) e^{-\lambda_i} \right) \Delta_N(\lambda)^2$$

---

here $\not{D}$ has $2N(+\nu)$ ev and is like a lattice regularised Dirac op which would have $2N \sim V$ $(\sim L^4)$ ev

$\Rightarrow$ take the limit $N \to \infty$ at the end (N_c is fixed!)

* we first need to compute the OP to solve Z_ν and $R_{nk}^{(1)}$ and then take $N \to \infty$

$N_f = 0$ quenched: $\tilde{P}_k(\lambda) = \frac{(-1)^k}{k!} L_k^{(\nu)}(\lambda)$

$N_f \neq 0$ unquenched: we know that for an arbitrary weight, e.g. $w^{(N_f)}(\lambda) = \prod_{f=1}^{N_f} (\lambda + m_f^2) \lambda^{\nu} e^{-\lambda}$

$$\tilde{P}_N(\lambda) = \langle \det(\lambda - WW^\dagger) \rangle_N = \frac{\langle \det(\lambda - WW^\dagger) \prod_{f=1}^{N_f} \det(m_f^2 + WW^\dagger) \rangle}{\langle \prod_{f=1}^{N_f} \det(\quad) \rangle}$$

quenched averages wrt $w^{(0)}(\lambda)$

so e.g. for $N_f = 1$

$$\tilde{P}_N^{(N_f=1)}(\lambda) = \frac{K_N(\lambda, -m_f^2)}{\tilde{P}_N^{(0)}(-m_f^2)} (\lambda + m_f^2)$$

quenched kernel & polynomial, Laguerre

* it is always possible to compute the RHS in terms of det's of the quenched polynomials

see e.g. Damgaard, Nishigaki hep-th/9711023

also OP for $(x+a)w(x)$: book of Szegö on OP)

Why should we care for the density of $\not{D}$ in QCD?

$\Rightarrow$ Banks–Casher relation $\rho_{\not{D}}(0) \sim \Sigma$ 1980.
in QCD

generate Σ by ∂_m ($N_f = 1$) before integrating out v

$$\underline{\Sigma} = \partial_m \log Z = \partial_m \log \int \ldots \prod_k m^\nu (x_k^2 + m^2)$$

$$= \left\langle \sum_k \frac{2m}{x_k^2 + m^2} \right\rangle_{QCD} + \ldots \quad \text{neglect } \nu \; ;$$

for this to make sense in QCD we need to UV & IR regularise V

1) $\lim\limits_{V \to \infty} = \frac{1}{V} \int dx\, \rho_{\not{D}}(x) \frac{2m}{x^2 + m^2}$

2) $\lim\limits_{m \to 0} = \underline{\frac{\pi \rho_D(0)}{V}}$, $\lim\limits_{m \to 0} \frac{2m}{x^2 + m^2} = \pi \delta(x)$
chiral limit

= explicit breaking remains

* order of limits does not commute! (like in spont. magnetisation of ferromagnet)

* small Dirac ev ≈ 0 are responsible for non perturbative effect: chiral symm. breaking (chSB)! $\to$ study on lattice

* small ev are spaced with $\frac{1}{V}$ $\neq$ spacing of free ev $\sim \frac{1}{L}$

take microscopic large-N limit $N x^2 = \hat{x}^2$ const: $\frac{\langle \nu \rangle}{N_u} \to J_1(\hat{x})$ Bessel
($N \to \infty$, $x \to 0$)
in RMT & compare with lattice

$\Rightarrow$ for smallest QCD Dirac ev agree (density, better compute distribution of individual ev, easier to normalise on lattice)

* the key to understand why this works is the low energy effective field theory of QCD: chiral Perturbation Theory

chPT and the epsilon regime = RMT limit

* breaking of a global symmetry $\Rightarrow$ $\exists$ Goldstone bosons

here: lightest particles are $\pi^\pm, \pi^0$, considering $N_f = 2$ light quarks u, d

general N_f: $U(x) \in SU_L(N_f) \times SU_R(N_f) / SU_V(N_f)$ for $N_c = 3, 4, ..$ fund. fixed

$\in SU(2N_f)/Sp(2N_f)$ for $N_c = 2$ fund

$\in SU(N_f)/SO(N_f)$ N_c adj

with $U(x) = \exp i \xi(x) \frac{\sqrt{2}}{F_\pi}$, e.g. $N_f = 2$ generators $\xi(x) = \sum_{a=1}^{3} \pi^a(x) \sigma_a$ Pauli matrices

chiral Lagrangian, low energy constants F_π, Σ (LEC)

$$\mathcal{L}_{eff} = \frac{F_\pi^2}{4} \partial_\mu U(x) \partial^\mu U(x)^\dagger - \Sigma M (U(x) + U^\dagger(x)) + \text{higher order in } U\text{'s and } \partial U\text{'s (+ nonlin. ...)}$$

(perturbative:)

expansion of $U(x)$ in powers of $\xi(x)$ $\Rightarrow$ standard kinetic term and mass term with $\underline{m_\pi^2 F_\pi^2 = 2 \Sigma m_q}$ Gell-Mann Oakes Renner relation

$\Rightarrow$ need to know LEC to convert m_q to m_π, and/or other phys. observables

* physical limit $\quad L \gg \frac{1}{m_\pi}$, $L^4 = V$ size of box

box $\gg$ Compton wavelength

Counting $m_\pi \sim p \sim \frac{1}{L}$ $\Rightarrow$ in path integral

$$Z_{chPT} = \int dU(x)\, e^{-S[\mathcal{L}_{eff}]}$$

integral in Fourier space ok $O\left(\frac{1}{V(p^2 + m_\pi^2)}\right) \sim O\left(\frac{L}{VL^0}\right) \sim O\left(\frac{1}{L^2}\right)$ small

small fluctuations

- unphysical limit $\boxed{L \ll \frac{1}{m_\pi}}$ Gasser, Leutwyler 1984, 85

with counting $m_\pi \sim \frac{1}{L^2}$, p, "Epsilon regime $\varepsilon = \frac{1}{L}$"

$\Rightarrow$ • $p=0$ modes in integral: $\mathcal{O}\left(\frac{1}{V(0^2+m_\pi^2)}\right) = \mathcal{O}(1)$ large fluct.

need to treat nonperturbative !

• $p \neq 0$ modes perturbative as before

$\Rightarrow$ reparametrise $U(x) = U_0 \exp(i\xi_p(x))$

propagating modes; constant modes

$$Z_\varepsilon = \int [d\xi_p(x)]\, e^{-\int d^4x \frac{1}{2}\partial_\mu \xi_p(x) \partial^\mu \xi_p(x)} \cdot \int_{SU(N_f)} dU_0 \exp\left[V \frac{\Sigma}{2} \mathrm{Tr}\, M (U_0 + U_0^\dagger)\right]$$

all other higher order non-linear terms are subleading

* factorisation in Gauß free field theory times non-perturbative group integral $\leftarrow$ equiv to RMT at $N \to \infty$

* 1-loop corrections within chPT preserve this factorised structure, simply renormalising F_π, Σ by $\frac{1}{V}$ terms

* in chPT (ε-regime or not) we can compute $\pi\pi$ correlation functions. How do Dirac op eigenvalues / densities relate to that?

first check agreement of Z_{RMT} & $Z_{\varepsilon chPT}$ (dropping Gauß free fields)

• Agreement of chiral RMT and the zero mode sector of ε-ChPT: partition function

in Z_{QCD} with θ term we had $Z = \underbrace{\sum_\nu e^{i\theta\nu} \int \ldots \prod_{f=1}^{N_f} m_f^\nu (x_k^2 + m_f^2)}_{= \sum_\nu e^{i\theta\nu} Z_\nu}$ combine $m_f e^{i\theta/N_f}$

$$\Rightarrow \quad Z_\nu = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-i\nu\theta} \int_{SU(N_f)} dU_0}_{\text{combine to } U(N_f)} \exp\left[\frac{V}{2}\Sigma\, \mathrm{Tr}\left(M \underbrace{e^{i\theta/N_f} U_0}_{} + h.c.\right)\right]$$

$$Z_\nu^{LS} = \int_{U(N_f)} dU_0 \, \det{}^\nu U_0 \exp\left[\frac{1}{2}\mathrm{Tr}(\hat M U_0 + \hat M U_0^\dagger)\right], \quad \hat m_f = V\Sigma m_f$$

this group integral was computed by Brower, Rossi, Tan 1981 / Leutwyler, Smilga 1992

• a derivation of this and other unitary group integrals using a character expansion is found in hep-th/0007161

$$\boxed{Z_\nu^{LS} = \frac{\det_{N_f \times N_f}\left[\hat m_f^{g-1} I_{\nu+g-1}(\hat m_f)\right]}{\Delta_{N_f}(\hat m_f^2)}} \xrightarrow[\text{equal mass}]{} \det\left[I_{\nu+i-j}(\hat m)\right]$$

$$\overset{!}{=} \lim_{\substack{N\to\infty \\ Nm = \hat m}} \left(Z_{RMT}^{(N_f)} = \int dW_{N\times(N+\nu)} \prod_{f=1}^{N_f} \det\begin{pmatrix} m_f & iW \\ iW^\dagger & m_f\end{pmatrix} e^{-\mathrm{Tr}\, WW^\dagger}\right)$$

example: $N_f = 2$ $\quad Z_{RMT}^{(N_f=2)} = \langle \det(m_1^2+WW^\dagger)\det(m_2^2+WW^\dagger)\rangle (m_1 m_2)^\nu \sim K_N(-m_1^2,-m_2^2)(m_1 m_2)^\nu$

$$= \frac{c_N (m_1 m_2)^\nu}{(m_2^2-m_1^2)}\left(L_N^{(\nu)}(-m_1^2)\,L_{N-1}^{(\nu)}(-m_2^2) - L_N^{(\nu)}(-m_2^2)\,L_{N-1}^{(\nu)}(-m_1^2)\right)$$

• large-N asymptotics:

using $\boxed{\lim_{n\to\infty} n^{-\nu} L_n^{(\nu)}\left(-\frac{x^2}{n}\right) = x^{-\nu} I_\nu(2x)}$, $\quad \hat m_{1,2}^2 = 4N m_{1,2}^2$ $\quad (\downarrow \infty \; \downarrow 0)$

and $m\partial_m I_\nu(m) - \nu I_\nu(m) = m I_{\nu+1}(m)$ for the Taylor expansion

we obtain the above Z_ν^{LS}

Result: $Z_\nu^{FB} = \frac{1}{\hat{m}_2^2 - \hat{m}_1^2} \begin{vmatrix} I_\nu(\hat{m}_1) & \hat{m}_1 I_{\nu+1}(\hat{m}_1) \\ I_\nu(\hat{m}_2) & \hat{m}_2 I_{\nu+1}(\hat{m}_2) \end{vmatrix}$

both from unimode sector of ε-chPT and chiral random matrix

- in passing we have computed the limiting density for the (quenched) chGUE in the microscopic scaling limit $N x^2 = \hat{x}^2$ fixed

$$\lim_{N\to\infty} \left(R^{(1)}_{N,\nu}(\hat{x}^2 = N x^2) = \omega(\hat{x}^2) K_N(+\hat{x}^2, +\hat{x}^2) \right) = \frac{\hat{x}}{2}\left(J_\nu(\hat{x})^2 - J_{\nu+1}(\hat{x}) J_{\nu-1}(\hat{x}) \right)$$

Bessel density $= \rho_\nu(\hat{x})$

finite-N chGUE

zooming into origin

* How would we compute such densities from ε chPT? (without knowing M)

- using $\det(A) = e^{\mathrm{Tr}\ln(A)} \Rightarrow \partial_z \det(z-H) = \mathrm{Tr}\frac{1}{z-H}\det(z-H)$

⇒ adding more source (±) quarks to $\int$ chPT — need bosonic for this

we can generate the resolvent $W(z) = \frac{1}{N}\langle \mathrm{Tr}\frac{1}{z-H}\rangle$, taking the discontinuity gives $\rho(\hat{x})$

via replicas: $W(z) = \frac{1}{N}\lim_{n\to 0} \partial_z \left\langle \frac{1}{n} \det{}^n(z-H) \right\rangle$

⇒ need to compute $U(N_f + n)$-$\int$, take limits

* Such a procedure is not unique in general if we only compute $n \in \mathbb{N}$; using the fact that Z satisfies the

Toda-lattice eq. $\quad Z_\nu^{N_f}(x)\,(x\partial_x)^2 \ln Z_\nu^{N_f} = c x^2 Z_\nu^{N_f+1}(x)\, Z_\nu^{N_f-1}(x)$,

that is its relation to integrable systems makes $n \in \mathbb{R}$ unique.

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Supersymmetry: $W(z) = \frac{1}{N} \partial_j \left\langle \frac{\det(z+j-H)}{\det(z-H)} \right\rangle \Big|_{j=0}$

$\Rightarrow$ we need to add bosonic quarks & compute $U(N_f+1\,|\,1)$ - $\int$

* has been done for $N_f = 0, 1, 2$ to generate $W(z)$ and $\rho(\lambda)$) and all for k-point distr.

Gap probabilities and individual eigenvalue distributions

* so far we have only computed the k-point density correl. funct. $R_{N,k}$. What is the probability A_I that e.g. an interval, say $I = [-s, s]$ is empty of eigenvalues? Let's look at $\beta = 2$:

$$A_I = \frac{1}{Z} \int_{\mathbb{R}\setminus I} \prod_{i=1}^N dx_i \; P_N^{(\beta=2)}(x_1, \ldots, x_N) \quad , \text{write} \int_{\mathbb{R}\setminus I} dx = \int_{\mathbb{R}} dx (1 - \chi_I(x))$$

characteristic function

$$= \frac{1}{N!} \int_{\mathbb{R}} \prod_{i=1}^N dx_i \det_{1\le i,j\le N} \left[\varphi_{j-1}(x_i) \right]^2 \quad , \varphi_{j-1}(x) = \left[(1-\chi_I(x))\,\omega(x)\right]^{\frac{1}{2}} \frac{P_{j-1}(x)}{\sqrt{h_{j-1}}}$$

$$\overset{(*)}{=} \det_{1\le i,j\le N} \left[\int_{\mathbb{R}} dx\, \varphi_{i-1}(x)\varphi_{j-1}(x) \right] = \det \Big[\int_{\mathbb{R}} dx \underbrace{(1-\chi_I(x))}_{1 \text{ gives } \delta_{ij}} \omega(x) P_{i-1}(x) P_{j-1}(x) \Big]$$

$$A_I = \det_{1\le i,j\le N} \left[\delta_{ij} - \int_I dx\, \omega(x) P_i(x) P_j(x) \right] \quad \text{Fredholm determinant}$$

where we have used the Andréief integral identity

$$(*) \quad \boxed{\int dx_1 \ldots dx_N \det_{1,\ldots,N} \left[\varphi_j(x_i) \right] \det_{1\ldots N} \left[\psi_j(x_i) \right] = N! \det_{1\le i,j\le N} \left[\int dx\, \varphi_i(x) \psi_j(x) \right]}$$

with $\varphi_j(x), \psi_j(x)$ integrable

* with the 2 de Bruijn formulas this completes our set of integral identities for $\beta = 1, 2, 4$