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AdS/CFT correspondence **Final exam**

Provide detailed solutions to the following set of questions, giving intermediate formulas, and explaining both the logic of the answers as well as the technical details of the calculations involved. Some intermediate computations may be carried out with the aid of computer programs, but the main steps should be reported in the answers; for example, statements such as "after some straightforward algrebra the result is" will not be regarded as satisfactory. The notation and conventions used in the problems adhere to those used in the lectures, but small variations may be present. Many, but not all, the necessary ingredients for working out the problems are provided in the text of the questions. You have access to the literature for looking up formulas that are not provided, for example the action and supersymmetry variations of type IIB supergravity. When using formulas that are not included in the text, write them clearly and provide the source (e.g. the arXive number of the paper or the title of the book), with equation numbers.

Problem 1: deformations of the D3-branes solution

Recall that the D3-brane solution of type IIB supergravity that was discussed in the lectures takes the form

$$\begin{aligned} \mathrm{d}s^2 &= H(r)^{-1/2}(-\mathrm{d}t^2+\mathrm{d}\vec{x}^2)+H(r)^{1/2}\left(\mathrm{d}r^2+r^2\mathrm{d}s^2(S^5)\right) && (0.1)\\ \mathrm{e}^{\phi} &= g_s && (0.2)\\ F_5 &= (1+*)\mathrm{d}\left(H(r)^{-1}\mathrm{vol}(\mathbb{R}^{1,3})\right) && (0.3)\end{aligned}$$

with

$$H(r) = 1+\frac{L^4}{r^4} \qquad (0.4)$$

and all other fields vanishing. This solves the equations of motion, supplemented with the self-duality condition $F_5 = *F_5$, as well as the Killing spinor equations (preserving $\frac{16}{32}$ supersymmetries).

(a) Verify that the black 3-brane configuration, given by

$$\begin{aligned} \mathrm{d}s^2 &= H(r)^{-1/2}(-f(r)\mathrm{d}t^2+\mathrm{d}\vec{x}^2)+H(r)^{1/2}\left(\frac{\mathrm{d}r^2}{f(r)}+r^2\mathrm{d}s^2(S^5)\right) && (0.5)\\ \mathrm{e}^{\phi} &= g_s && (0.6)\\ F_5 &= (1+*)\mathrm{d}\left(H(r)^{-1}\mathrm{vol}(\mathbb{R}^{1,3})\right) && (0.7)\end{aligned}$$

with

$$H(r) = 1+\frac{L^4}{r^4} \qquad f(r)=1-\frac{r_h^4}{r^4} \qquad (0.8)$$

is a solution of type IIB supergravity.

Consider the equation for a *parallel* spinor on a six-dimensional Riemannian space with metric g_{ij}, namely the equation

$$\nabla_i \xi \;=\; 0 \tag{0.9}$$

where recall that the spinorial covariant derivative is defined as

$$\nabla_i \;\equiv\; \partial_i + \frac{1}{4}\omega_i^{ab}\gamma_{ab} \tag{0.10}$$

Here we define such spaces as Calabi-Yau spaces.

(b) Show that the integrability condition of (0.9) implies the Ricci-flatness condition. Namely, show that if there exist a non-zero solution ξ to the equation (0.9) then the Ricci tensor of the metric g_{ij} obeys

$$R_{ij} \;=\; 0 \tag{0.11}$$

(c) Verify that replacing $\mathbb{R}^6$ with an arbitrary Calabi-Yau metric g_{ij} in the D3-brane solution is still a supersymmetric solution of type IIB supergravity. Namely, show that

$$\begin{aligned} \mathrm{d}s^2 &= H(y^i)^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + H(y^i)^{1/2}\left(g_{ij}\mathrm{d}y^i\mathrm{d}y^j\right) && (0.12)\\ \mathrm{e}^{\phi} &= g_s && (0.13)\\ F_5 &= (1+*)\mathrm{d}\left(H(y^i)^{-1}\mathrm{vol}(\mathbb{R}^{1,3})\right) && (0.14) \end{aligned}$$

with

$$\Box_{CY} H(y^i) \;=\; 0 \tag{0.15}$$

where $\Box_{CY}$ is the scalar Laplace operator computed with the Calabi-Yau metric g_{ij}, solves the equations of motion as well as the supersymmetry conditions of type IIB supergravity.

To study the supersymmetry conditions consider the following ansatz for the ten-dimensional Killing spinor

$$\epsilon \;=\; \psi \otimes \xi \tag{0.16}$$

where ψ is a spinor in Minkowski space obeying $\nabla_\mu \psi = 0$ and ξ is a spinor on the Calabi-Yau space.

(d) State the further necessary condition on the Calabi-Yau metric g_{ij} in order for the solution above to be interpreted as a stack of D3-branes placed at a Calabi-Yau singularity. Illustrate this condition by writing down an explicit example of Ricci-flat metric g_{ij}, that does *not* satisfy it.

(e) After picking an appropriate explicit solution to (0.15), write down and discuss the "near-horizon" limit of such solution, including the full isometry group of the near-horizon solution.

Problem 2: the Schwarzschild-AdS black hole

Consider the four-dimensional Schwarzschild-AdS black hole with line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) \tag{0.17}$$

where the coordinates $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ parameterise a round two-sphere and

$$f(r) = 1 - \frac{2GM}{r} + \frac{r^2}{L^2} \tag{0.18}$$

with M the mass of the black hole.

(a) Calculate the Hawking temperature T_H of the black hole by requiring regularity of the Euclidean metric obtained with the Wick rotation $t = -i\tau$. In particular, defining the position of the horizon of the Lorentzian metric as r_h, that is the largest root of the equation

$$f(r_h) = 0 \tag{0.19}$$

show that

$$T_H = \frac{L^2 + 3r_h^2}{4\pi r_h L^2} \tag{0.20}$$

(b) Verify the consistency of the Bekenstein-Hawking relation between the entropy of the black hole S_{BH} and the area of the horizon A,

$$S_{\mathrm{BH}} = \frac{A}{4G} \tag{0.21}$$

and the first law of thermodynamics

$$\mathrm{d}M = T_H \mathrm{d}S_{\mathrm{BH}} \tag{0.22}$$

(c) Compute the holographically renormalized on-shell action obtained as

$$I = \lim_{\epsilon\to\infty} S_{\mathrm{bulk}} + S_{\mathrm{GH}} + S_{\mathrm{ct}} \tag{0.23}$$

where

$$S_{\mathrm{bulk}} = -\frac{1}{16\pi G}\int_{r\le\epsilon} \mathrm{d}^4x\sqrt{|g|}\left(R + \frac{6}{L^2}\right) \tag{0.24}$$

is the bulk gravity action,

$$S_{\mathrm{GH}} = -\frac{1}{8\pi G}\int_{r=\epsilon} \mathrm{d}^3x\sqrt{|\gamma|}K[\gamma] \tag{0.25}$$

is the Gibbons-Hawking boundary term and

$$S_{\rm ct} = \frac{1}{8\pi G}\int_{r=\epsilon} {\rm d}^3x\sqrt{|\gamma|}\left(\frac{2}{L}+\frac{L}{2}R[\gamma]\right) \tag{0.26}$$

are the counterterms. Here γ_{ij} is the metric induced by (0.17) on a constant $r=\epsilon$ hypersurface; $K[\gamma]$ is the associated extrinsic curvature and $R[\gamma]$ its Ricci scalar. In particular, show that the result derived takes the form

$$I_E = -iI|_{t=-i\tau} = \frac{4\pi}{T_H}\frac{1}{8\pi GL^2}\left(GML^2-r_h^3\right) \tag{0.27}$$

$$= \frac{M}{T_H}-S_{\rm BH} \tag{0.28}$$

The computation can be carried out in Lorentzian signature and then analytically continued to Euclidean signature, with periodically identified $\tau\sim\tau+\frac{1}{T_H}$, at the end.

The minimum temperature for which the Schwarzschild-AdS black hole exists is

$$T_{\rm min} = \frac{\sqrt{3}}{2\pi L} \tag{0.29}$$

For $T\geq T_{\rm min}$ there exists another solution with finite temperature and the same asymptotic behaviour, which is "thermal AdS_4", given by Euclideanized global AdS_4, with metric

$${\rm d}s^2 = \left(1+\tfrac{r^2}{L^2}\right){\rm d}\tau^2+\frac{{\rm d}r^2}{1+\frac{r^2}{L^2}}+r^2\left({\rm d}\theta^2+\sin^2\theta{\rm d}\phi^2\right) \tag{0.30}$$

and τ identified periodically exactly as in the Schwarzschild-AdS black hole, namely as $\tau\sim\tau+\frac{4\pi r_hL^2}{L^2+3r_h^2}$.

(d) Compute the holographically renormalized Euclidean on-shell action of thermal AdS, following the same steps as for the Schwarzschild-AdS black hole, and show that it takes again the form (0.28). Discuss the Hawking-Page phase transition by comparing the values of the holographically renormalized on-shell actions for the two solutions and give the values of the critcal mass M_* and corresponding critical temperature T_* at which the transition occurs. In particular verify that $T_*>T_{\rm min}$.

Problem 3: $\mathbf{AdS_5\times S^5/\mathbb{Z}_2}$ and its dual SCFT

Recall that the metric on S^5 can be written in the following form

$$\begin{aligned}{\rm d}s^2(S^5) &= {\rm d}\sigma^2+\frac{1}{4}\sin^2\sigma\left({\rm d}\theta^2+\sin^2\theta{\rm d}\phi^2\right)+\frac{1}{4}\cos^2\sigma\sin^2\sigma({\rm d}\alpha+\cos\theta{\rm d}\phi)^2\\ &+ \frac{1}{9}\left({\rm d}\psi-\frac{3}{2}\sin^2\sigma({\rm d}\alpha+\cos\theta{\rm d}\phi)\right)^2\end{aligned} \tag{0.31}$$

where the ranges of the coordinates are $\sigma \in [0, \frac{\pi}{2}]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, $\alpha \in [0, 4\pi]$, $\psi \in [0, 6\pi]$. This is the "unit radius" Einstein metric on S^5, normalised with Ricci tensor

$$R_{ij} = 4g_{ij} \tag{0.32}$$

with Killing vectors generating the isometry group $SO(6)$. These are canonical coordinates that display the Sasaki-Einstein nature of S^5, which can be viewed as a $U(1)$ bundle fibered over the base space space $\mathbb{C}P^2$, with (Einstein) metric on the latter being the first line of (0.31). The $U(1) \subset SO(6)$ is generated by the Killing vector $3\frac{\partial}{\partial\psi}$.

Consider the space obtained by acting with a $\mathbb{Z}_2$ quotient on S^5, where in these coordinates the quotient acts by dividing by 2 the periodicity of the coordinate α, namely consider the above metric, with $\alpha \in [0, 2\pi]$.

(a) What is the isometry group of the quotient space $S^5/\mathbb{Z}_2$ so obtained?

It follows that $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_2$, namely

$$\mathrm{d}s^2 = \frac{r^2}{L^2}(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + \frac{L^2}{r^2}\left(\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^5/\mathbb{Z}_2)\right) \tag{0.33}$$

$$\mathrm{e}^{\phi} = g_s \tag{0.34}$$

$$F_5 = \frac{4}{L}\left(\mathrm{vol}(\mathrm{AdS}_5) + L^5\mathrm{vol}(S^5/\mathbb{Z}_2)\right) \tag{0.35}$$

with all the other fields of the $\mathrm{AdS}_5 \times S^5$ solution unchanged, is a solution to the equations of motion of type IIB supergravity. A more accurate analysis (that is not required in the problem) reveals that this solution preserves $\frac{16}{32}$ supersymmetry and therefore it must be dual to an $\mathcal{N} = 2$ SCFT in the IR.

(b) Compute the integrated volume $\mathrm{Vol}(S^5/\mathbb{Z}_2)$.

The supersymmetric gauge theory conjectured to be holographic dual to this solution has the following characteristics, expressed in the language of $\mathcal{N} = 1$ supersymmetric gauge theories:

1. Gauge group $G = SU(N)_1 \times SU(N)_2$.
2. Two chiral multiplets U_a, $a = 1, 2$ transforming in the bi-fundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$ of G. Two chiral multiplets V_a, $a = 1, 2$ transforming in the bi-fundamental representation $(\bar{\mathbf{N}}, \mathbf{N})$ of G. One chiral multiplet Φ_1 transforming in the adjoint of $SU(N)_1$ (and singlet under $SU(N)_2$). One chiral multiplet Φ_2 transforming in the adjoint of $SU(N)_2$ (and singlet under $SU(N)_1$).
3. A superpotential $W = \lambda(\mathrm{Tr}[\Phi_1\epsilon^{ab}U_aV_b] + \mathrm{Tr}[\Phi_2\epsilon^{ab}V_aU_b])$.

(c) Discuss the (internal) global symmetry group of the Lagrangian of this theory in the language of $\mathcal{N}=1$ supersymmetry and its relation to the isometry group of $S^5/\mathbb{Z}_2$ in the gravity dual.

(d) Compute the VMS obtained setting to zero the F-terms and the D-terms of the associated Abelian theory with $G = U(1)_1 \times U(1)_2$. Then discuss its relationship to the dual gravity solution. [**Hint**: consider first the D-term equations and exploit their similarity to the conifold theory; then consider the F-term equations.]

Recall that the NSVZ exact beta function associated to each Yang-Mills gauge coupling g of a gauge group factor is

$$\beta(\tfrac{g^2}{4\pi}) \quad = \quad -\frac{g^2}{8\pi}\frac{3T(G) - 3\sum_i T(\mathbf{r}_i)(1-R[X_i])}{1-\frac{g^2}{8\pi}T(G)} \tag{0.36}$$

where $R[X_i]$ are the R-charges of the fields X_i, transforming in some representation of the gauge group factor. For $SU(N)$, we have $T(G) = T(\text{adj}) = N$ and $T(\mathbf{N}) = T(\bar{\mathbf{N}}) = \frac{1}{2}$. Furthermore, the beta function associated to a superpotential coupling λ is proportional to

$$\beta(\lambda) \propto \sum_i R[X_i] - 2 \tag{0.37}$$

where the sum above is over all the chiral fields that participate to a superpotential terms with the given coupling.

(e) Assuming that this theory has a fixed point (that may be part of a so-called "conformal manifold" of fixed points) at which it is a SCFT, derive the R-charges of the six chiral fields, denoting them $R[X_i]$ with $X_i \in \{U_a, V_a, \Phi_1, \Phi_2\}$, by imposing conformal invariance and maximizing the trial central charge/Weyl anomaly coefficient a, given by

$$a \quad = \quad \frac{3}{32}\left(3\text{Tr}R^3 - \text{Tr}R\right) \tag{0.38}$$

Moreover, compute the a of the SCFT and compare it with $\text{Vol}(S^5/\mathbb{Z}_2)$ using the AdS/CFT dictionary.

Now consider the *mass deformation* given by adding to this theory the superpotential term

$$\delta W \quad = \quad \frac{m}{2}(\text{Tr}[\Phi_1^2] - \text{Tr}[\Phi_2^2]) \tag{0.39}$$

(f) Assuming that this deformed theory flows to an (a priori different from the above) SCFT in the IR, determine the exact R-charges and a anomaly coefficient of this deformed theory.

(g) After adding the mass deformation, the effective low energy theory may be derived by the process of "integrating out" the massive fields, namely solving the (non-Abelian) F-term equations for the massive fields in terms of the massless ones, and substituting them back into the super-potential. Carry out this computation and uncover the low energy theory. Discuss and interpret your findings.