DRAFT of the Lectures

Last one still missing

December 1, 2008

Contents

1 A crash course on fermions

A systematic analysis of the representations of the Lorentz group is beyond the scope of these lectures. Here we shall simply recall some basic facts about the spinor representations and the two-component notation.

The Lorentz group is the set of matrices Λ such that

$$\Lambda^m{}_\alpha \eta_{mn} \Lambda^n{}_\beta = \eta_{\alpha\beta}, \tag{1.1}$$

where

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1). \tag{1.2}$$

Its Lie algebra contains six hermitian generators: there are three J_i, corresponding to rotations, and three K_i, describing Lorentz boosts. These generators close the following Lie algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad [J_i, K_j] = i\epsilon_{ijk} K_k \qquad [K_i, K_k] = -i\epsilon_{ijk} J_k \tag{1.3}$$

This algebra possesses a quite simple mathematical structure that becomes manifest introducing the following linear combinations

$$J_i^\pm = \frac{1}{2}(J_i \pm iK_i). \tag{1.4}$$

In terms of the generators (1.4), the Lorentz algebra *roughly* separates into the direct sum of two conjugate $su(2)$ subalgebras

$$[J_i^{(\pm)}, J_j^{(\pm)}] = i\epsilon_{ijk} J_k^{(\pm)} \qquad [J_i^{(\pm)}, J_j^{(\mp)}] = 0, \tag{1.5}$$

where $J_i^{(\pm)}$ are not hermitian, being $J_i^{(+)\dagger} = J_i^{(-)}$. Technically, the algebra of the Lorentz group is a particular *real form* of the complexification of $su(2) \oplus su(2)$. At the level of finite dimensional representations, however, these subtleties can be neglected and we can classify the representations in terms of the two angular momenta associated to the two $su(2)$

$$(j_1, j_2) \qquad \text{where } j_1,\ j_2 = 0, 1/2, 1, \ldots, n/2, \ldots \quad \text{with } n \in \mathbb{N}. \tag{1.6}$$

Since $J_i = J^{(+)} + J^{(-)}$, the *rotational* spin content of the representation is given by the sum rule of the angular momenta

$$J = |j_1 - j_2|, |j_1 - j_2| + 1, \cdots, |j_1 + j_2|. \tag{1.7}$$

In this language, the spinor representation can be introduced by considering the universal covering of $SO(3,1)$, namely $SL(2,\mathbb{C})$. To see that $SL(2,\mathbb{C})$ is locally isomorphic to $SO(3,1)$ is quite easy. Introduce

$$\sigma^m = (-\mathbb{1}, \sigma^i), \tag{1.8}$$

where σ^i are the Pauli matrices. Then for every vector x^m, the 2×2 matrix $x^m\sigma_m$ is hermitian and its determinant is given by the Lorentz invariant $-x^m x_m$. Hence a Lorentz transformation must preserve the determinant and the hermiticity of this matrix. The action

$$\sigma^m x_m \mapsto A\sigma^m x_m A^\dagger \tag{1.9}$$

possesses both these properties if $|\det(A)| = 1$. Therefore, up to an irrelevant phase factor, we can choose the matrix A to belong to $SL(2,\mathbb{C})$ and write

$$\sigma_m x'^m = A\sigma_m x^m A^\dagger = \sigma_m \Lambda^m_{\ n}(A) x^n. \tag{1.10}$$

This means that we can associate a Lorentz transformation to each element of $SL(2,\mathbb{C})$. However, $\pm A$ generate the same Lorentz transformation, and thus the correspondence is not one to one, we have a *double-covering.*

Exercise: *Show that $SL(2,\mathbb{C})$ is a double covering of $SO(3,1)$ and that it is simply-connected.* SOLUTION:

[1. Double-covering]: If A and B in $SL(2,\mathbb{C})$ generate the same Lorentz transformation, for any x^m the following equality must hold

$$Ax^m\sigma_m A^\dagger = Bx^m\sigma_m B^\dagger,$$

Since A and B are invertible, the above condition can be equivalently written as

$$B^{-1}Ax^m\sigma_m = x^m\sigma_m B^\dagger A^{\dagger -1}$$

Choosing $x^m = (1,0,0,0)$, we find $B^{-1}A = B^\dagger A^{\dagger -1}$ and thus $B^{-1}A$ commutes with all the hermitian matrices. By the Schur lemma $B^{-1}A = \alpha\mathbb{1}$. Since $A, B \in SL(2,\mathbb{C})$, $\alpha^2 = 1$ which in turn implies $\alpha = \pm 1$.

[2. Simply-connected]: Given any element A of $SL(2,\mathbb{C})$, we can always find a matrix S such that $A = \exp(S)$ with $\mathrm{Tr}(S) = 2\pi i k$ and k an integer. Then the curve $A(t) = e^{tS}e^{-tQ}$, where $Q = \begin{pmatrix} 1 & 1 \\ 0 & 2\pi i k\end{pmatrix}$, possesses all the necessary properties: $A(0) = 1$, $A(1) = A$ and $\det(A(t)) = e^{t\mathrm{Tr}(S)}e^{-2\pi i k t} = 1$.

In this setting we can define spinors as the objects carrying the basic representation of $SL(2,\mathbb{C})$ (*fundamental* and *anti-fundamental*). Since the elements of $SL(2,\mathbb{C})$ are 2×2 matrices, a spinor is a two-complex object $\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$, which transforms under an element $\mathcal{M}_\alpha^{\ \beta}$ of $SL(2,\mathbb{C})$ as

$$\psi'_\alpha = \mathcal{M}_\alpha^{\ \beta}\psi_\beta. \tag{1.11}$$

Unlike for $SU(2)$ and similarly to $SU(3)$, the conjugate representation $\mathcal{M}^*$ is not equivalent to $\mathcal{M}$ and it provides a second possible spinor representation. An object in this representation is usually denoted by $\bar\psi_{\dot\alpha}$ and it is called *dotted spinor*. It transforms as

$$\bar\psi'_{\dot\alpha} = \mathcal{M}^{*\ \dot\beta}_{\dot\alpha}\psi_{\dot\beta}. \tag{1.12}$$

The *dot* on the indices simply recall that we are in a different representation. In the language of (1.6), we are dealing with $(1/2, 0)$ and $(0, 1/2)$ representations. [Technically, these are not representations of the full Lorentz, but only of its proper orthochronus part. The parity, in fact, exchanges these representations.] Since $\mathcal{M}$ is an unimodular matrix, we can construct a two invariant antisymmetric tensors

$$\epsilon_{\alpha\beta} \quad \text{and} \quad \epsilon_{\dot\alpha\dot\beta} \qquad \text{with} \qquad \epsilon_{12} = \epsilon_{\dot1\dot2} = -1. \tag{1.13}$$

Exercise: *Show that* $\epsilon_{\alpha\beta}$ *and* $\epsilon_{\dot\alpha\dot\beta}$ *are invariant tensors.*

Solution: Consider, for example, $\epsilon_{\alpha\beta}$. One has

$$\epsilon_{\alpha\beta} \mapsto \epsilon'_{\alpha\beta} = \mathcal{M}_\alpha{}^\rho \mathcal{M}_\beta{}^\rho \epsilon_{\rho\sigma} = \det(\mathcal{M})\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}.$$

Their inverse $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$, defined by

$$\epsilon_{\alpha\rho}\epsilon^{\rho\beta} = \delta_\alpha^\beta \qquad \epsilon_{\dot\alpha\dot\rho}\epsilon^{\dot\rho\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}, \tag{1.14}$$

are also invariant tensors. Moreover, (1.13) and (1.14) can be used to raise and lower the indices of the spinors

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta \qquad \bar\chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta} \qquad \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta \qquad \bar\chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}. \tag{1.15}$$

Exercise: *Show that* $\psi'^\alpha = \psi^\beta \mathcal{M}_\beta^{-1\alpha}$ *and* $\bar\psi'^{\dot\alpha} = \bar\psi^{\dot\beta}\mathcal{M}_{\dot\beta}^{*-1\dot\alpha}$

Solution: Consider, for example, ψ'^α. One has

$$\psi'^\alpha = \epsilon^{\alpha\beta}\psi'_\beta = \epsilon^{\alpha\beta}\mathcal{M}_\beta{}^\sigma\psi_\sigma = \epsilon^{\rho\sigma}\mathcal{M}_\rho^{-1\alpha}\psi_\sigma = \mathcal{M}_\rho^{-1\alpha}\psi^\rho,$$

where we have used that the invariance of the tensor $\epsilon^{\alpha\beta}$, *i.e.* $\mathcal{M}_\alpha{}^\rho\mathcal{M}_\beta{}^\sigma\epsilon^{\alpha\beta} = \epsilon^{\rho\sigma}$, in fact implies $\mathcal{M}_\beta{}^\sigma\epsilon^{\alpha\beta} = \epsilon^{\rho\sigma}\mathcal{M}_\rho^{-1\alpha}$.

Therefore (see exercise above) we can easily define spinor bilinear which are invariant under Lorentz transformations

$$\begin{aligned} \psi\chi &\equiv \psi^\alpha\chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\chi_\alpha\psi_\beta = \epsilon^{\beta\alpha}\chi_\alpha\psi_\beta = \chi^\beta\psi_\beta = \chi\psi \\ \bar\psi\bar\chi &\equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} = -\epsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta}\bar\psi_{\dot\alpha} = \epsilon^{\dot\beta\dot\alpha}\bar\chi_{\dot\beta}\bar\psi_{\dot\alpha} = \bar\chi_{\dot\beta}\bar\psi^{\dot\beta} = \bar\chi\bar\psi. \end{aligned} \tag{1.16}$$

Here we have used that spinors are Grassmannian variables and thus anticommute.
We shall also define an operation of Hermitian conjugation such that and given by

$$(\chi_\alpha)^\dagger = \bar\chi_{\dot\alpha}, \qquad (\bar\chi_{\dot\alpha})^\dagger = (\chi_\alpha) \qquad \text{and} \qquad (\psi_\alpha\chi_\beta)^\dagger = (\chi_\beta^\dagger\psi_\alpha^\dagger). \tag{1.17}$$

This implies, in turn, $(\psi\chi)^\dagger = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi$.

Notice that choices (1.11)-(1.12) and the eq. (1.10) fix the nature of the indices of σ^m. In this notation eq. (1.10) reads

$$\mathcal{M}_\alpha{}^\beta \mathcal{M}^*_{\dot\alpha}{}^{\dot\beta}\sigma^m_{\alpha\dot\alpha} = \Lambda(\mathcal{M})^m{}_n\sigma^n_{\alpha\dot\alpha}. \tag{1.18}$$

From the group theoretical point of view, (1.18) states that the matrices σ^m realize a one-to-one correspondence between the vector representation (v^m) and an object with two spinorial indices ($v_{\alpha\dot\alpha} = v_m\sigma^m_{\alpha\dot\alpha}$). Such object is called *bispinor*. [In terms of Lorentz representations, eq. (1.18) simply says "$(1/2,0)\otimes(0,1/2) = \underset{vect.\ rep.}{(1/2,1/2)}$".]

It is useful to introduce a second set of matrices, which are obtained by raising the indices of σ^m

$$\bar\sigma^{m\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta}. \tag{1.19}$$

Explicitly they are given by

$$\bar\sigma^m = (\mathbb{1}, -\sigma) \tag{1.20}$$

and satisfy the relation

$$\mathcal{M}^{*-1\dot\beta}_{\dot\alpha}\mathcal{M}^{-1\beta}_{\alpha}\bar\sigma^{m\dot\alpha\alpha} = \mathcal{M}^{*-1\dot\beta}_{\dot\alpha}\mathcal{M}^{-1\beta}_{\alpha}\epsilon^{\dot\alpha\dot\rho}\epsilon^{\alpha\rho}\sigma^m_{\rho\dot\rho} = \epsilon^{\beta\alpha}\epsilon^{\dot\beta\dot\alpha}\mathcal{M}_\alpha{}^\rho\mathcal{M}^*_{\dot\alpha}{}^{\dot\rho}\sigma^m_{\rho\dot\rho} = \Lambda^m{}_n(\mathcal{M})\bar\sigma^{n\dot\beta\beta}. \tag{1.21}$$

We have also the following relations

$$(\bar\sigma^m\sigma^n + \bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta} = -2\eta^{mn}\delta^{\dot\alpha}_{\dot\beta} \qquad (\sigma^m\bar\sigma^n + \sigma^n\bar\sigma^m)_\alpha{}^\beta = -2\eta^{mn}\delta^\beta_\alpha, \tag{1.22}$$

which can be easily proved.

Relations with Dirac spinors: The connection between the above representations and the usual four component notation for Dirac and Majorana fermions is easily seen if we use the so-called Weyl form of $\gamma-$matrices:

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar\sigma^m & 0 \end{pmatrix} \qquad \text{with} \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}. \end{pmatrix} \tag{1.23}$$

With this choice, the standard Lorentz generators separate into the direct sum of the two two-dimensional representations. In fact

$$\Sigma^{mn} = \frac{i}{4}[\gamma^m,\gamma^n] = i\begin{pmatrix} \frac{1}{4}(\sigma^m\bar\sigma^n - \sigma^n\bar\sigma^m)_\alpha{}^\beta & 0 \\ 0 & \frac{1}{4}(\bar\sigma^m\sigma^n - \bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} \equiv i\begin{pmatrix} (\sigma^{mn})_\alpha{}^\beta & 0 \\ 0 & (\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{1.24}$$

The upper representation corresponds to the undotted spinors, while the lower one to the dotted spinor. Therefore the four component Dirac spinor in two-component notation reads

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{1.25}$$

while the Majorana spinor is given by

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{1.26}$$

where $(\chi_\alpha)^\dagger = \bar{\chi}_{\dot{\alpha}}$. Consequently, the Dirac Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_D =& \bar{\Psi}_D(i\gamma^m\partial_m + M)\Psi_D = (\psi^\alpha \bar{\chi}_{\dot{\alpha}}) \begin{pmatrix} M\delta_\alpha^\beta & i\sigma^m_{\alpha\dot{\beta}}\partial_m \\ i\bar{\sigma}^{m\dot{\alpha}\beta}\partial_m & M\delta^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix} \begin{pmatrix} \chi_\beta \\ \bar{\psi}^{\dot{\beta}} \end{pmatrix} = \\ =& i\psi\sigma^m\partial_m\bar{\psi} + i\bar{\chi}\bar{\sigma}^m\partial_m\chi + M\psi\chi + M\bar{\psi}\bar{\chi}, \end{aligned} \tag{1.27}$$

while for Majorana fermions we have

$$\begin{aligned} \mathcal{L}_M =& \frac{i}{2}\chi\sigma^m\partial_m\bar{\chi} + \frac{i}{2}\bar{\chi}\bar{\sigma}^m\partial_m\chi + \frac{M}{2}\chi\chi + \frac{M}{2}\bar{\chi}\bar{\chi} = \\ =& \bar{\chi}\bar{\sigma}^m\partial_m\chi + \frac{M}{2}\chi\chi + \frac{M}{2}\bar{\chi}\bar{\chi} + \text{total divergence}. \end{aligned} \tag{1.28}$$

The Weyl action is simply obtained by setting $M = 0$ in (1.28).

Exercise: *Show that the following spinorial identities hold*

$$\chi^\alpha\chi^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\chi^2 \qquad \chi_\alpha\chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\chi^2 \qquad \bar{\chi}^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}^2 \qquad \bar{\chi}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^2 \tag{1.29a}$$

$$\chi\sigma^m\bar{\chi}\chi\sigma^n\bar{\chi} = -\frac{1}{2}\eta^{mn}\chi^2\bar{\chi}^2 \qquad \chi\sigma^{mn}\chi = \bar{\chi}\bar{\sigma}^{mn}\bar{\chi} = 0 \tag{1.29b}$$

$$\chi\sigma^n\bar{\psi} = -\bar{\psi}\bar{\sigma}^n\chi \tag{1.29c}$$

$$\bar{\psi}^{\dot{\alpha}}\chi^\alpha = \frac{1}{2}\chi\sigma_m\bar{\psi}\bar{\sigma}^{m\dot{\alpha}\alpha} = -\frac{1}{2}\bar{\psi}\bar{\sigma}_m\chi\bar{\sigma}^{m\dot{\alpha}\alpha} \qquad \chi_\alpha\bar{\psi}_{\dot{\alpha}} = \frac{1}{2}\bar{\psi}\bar{\sigma}_m\chi\sigma^m_{\alpha\dot{\alpha}} = -\frac{1}{2}\chi\sigma_m\bar{\psi}\sigma^m_{\alpha\dot{\alpha}} \tag{1.29d}$$

(Important technical exercise!!)

SOLUTION: The identities (1.29a) can be proven in the same way. Consider e.g.

$$\chi^\alpha\chi^\beta = \frac{1}{2}(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\beta_\rho\delta^\alpha_\sigma)\chi^\rho\chi^\sigma = \frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{\sigma\rho}\chi^\rho\chi^\sigma = -\frac{1}{2}\epsilon^{\alpha\beta}\chi^\rho\chi_\rho = -\frac{1}{2}\epsilon^{\alpha\beta}\chi^2$$

Instead, for the identities (1.29b) we can write

$$\chi\sigma^m\bar{\chi}\chi\sigma^n\bar{\chi} = -\chi^\alpha\chi^\beta\bar{\chi}^{\dot{\alpha}}\bar{\chi}^{\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}} = \frac{1}{4}\chi^2\bar{\chi}^2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}} = \frac{1}{4}\chi^2\bar{\chi}^2\sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{n\dot{\alpha}\alpha} = \frac{1}{4}\chi^2\bar{\chi}^2\text{Tr}(\sigma^m\bar{\sigma}^n) = -\frac{1}{2}\eta^{mn}\chi^2\bar{\chi}^2$$

(1.29c):

$$\chi\sigma^n\bar{\psi} = \chi^\alpha\bar{\psi}^{\dot{\alpha}}\sigma^n_{\alpha\dot{\alpha}} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\chi_\beta\bar{\psi}_{\dot{\beta}}\sigma^n_{\alpha\dot{\alpha}} = -\epsilon^{\beta\alpha}\epsilon^{\dot{\beta}\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\sigma^n_{\alpha\dot{\alpha}}\chi_\beta = -\bar{\psi}_{\dot{\beta}}\bar{\sigma}^{n\dot{\beta}\beta}\chi_\beta = -\bar{\psi}\bar{\sigma}^n\chi$$

(1.29d): The spinorial bilinear $\bar{\psi}^{\dot{\alpha}}\chi^\alpha$ is 2×2 matrix and thus it can be expanded in the basis $\bar{\sigma}^{m\dot{\alpha}\alpha}$, namely $\bar{\psi}^{\dot{\alpha}}\chi^\alpha = A_m\bar{\sigma}^{m\dot{\alpha}\alpha}$. Multiplying both sides of this identity by $\sigma^{n\alpha\dot{\alpha}}$ and taking the trace, we find

$$-\chi\sigma^n\bar{\psi} = A_m\text{Tr}(\sigma^n\bar{\sigma}^m) \qquad \Rightarrow \qquad A_n = \frac{1}{2}\chi\sigma^n\bar{\psi}.$$

The second identity can be proved in the same way.

Constructing any Lorentz representation: Consider the generic representation $(\frac{m}{2},\frac{n}{2})$.It can be constructed by means of the two fundamental representations $(1/2,0)$ and $(0,1/2)$. Since the two labels denoting the representation behave as angular momenta, we can write that

$$(m/2,n/2)=\left[\mathrm{sym}\bigotimes_{i=1}^{m}(1/2,0)\right]\otimes\left[\mathrm{sym}\bigotimes_{i=1}^{n}(0,1/2)\right]$$

This means that a field in the representation $(\frac{m}{2},\frac{n}{2})$ has m undotted indices $\alpha_1,\ldots,\alpha_m$ and n dotted indices $\dot{\alpha}_1,\ldots,\dot{\alpha}_n$

$$\chi_{\alpha_1\ldots\alpha_m;\dot{\alpha}_1\ldots\dot{\alpha}_n}. \tag{1.30}$$

Moreover there is a total symmetries in the indices $\alpha_1\ldots\alpha_m$ and in the indices $\dot{\alpha}_1\ldots\dot{\alpha}_n$.
For example the representation $(1/2,1/2)$ is described by the field $\chi_{\alpha\dot{\alpha}}$. This is a 2×2 matrix and it can be expanded in terms of the matrices $\sigma^m_{\dot{\alpha}\alpha}$:

$$\chi_{\alpha\dot{\alpha}}=V_m\sigma^m_{\dot{\alpha}\alpha}. \tag{1.31}$$

This shows that the representation $(1/2,1/2)$ corresponds to the four vectors.

2 From the Coleman-Mandula theorem to the supersymmetry algebra in D=4

The quest for a Lie-group unifying Poincarè invariance and internal symmetries in a non trivial way came to an end with the advent of the Coleman-Mandula theorem.

Theorem: *Let G be a connected symmetry group of the $S-$matrix and let us assume that the following 5 conditions hold*

- ASSUMPTION 1:*(Poincarè Invariance) The group G contains a subgroup isomorphic to the Poincarè group.*
- ASSUMPTION 2: *(Particle-finiteness) All the particles are representations with positive energy of the Poincarè group. Moreover for any M there is a finite number of particles with mass less than M.*
- ASSUMPTION 3: *(Weak elastic analyticity) The scattering amplitudes are analytic functions of the energy of the center of mass, s, and of the transferred momentum, t, in a neighborhood of the physical region with the exception of the particle-production thresholds.*

- Assumption 4: *(Occurrence of scattering) Given two one-particle states $|p\rangle$ and $|p'\rangle$, construct the two-particle state $|p,p'\rangle$. Then*
$$T|p,p'\rangle \neq 0$$
for almost any value of s.

- Assumption 5: *(Ugly technical hypothesis) There exists a neighborhood of the identity in G, such that every element in this neighborhood belongs to a one-parameter subgroup. Moreover, if x and y are two one-particle states whose wave-functions are test functions (for our distributions), the derivative*
$$\frac{1}{i}\frac{d}{dt}(x, g(t)y) = (x, Ay)$$
exists at $t=0$, and it defines a continuous function of x and y which is linear in y and anti-linear in x.

Then, G is locally isomorphic to the direct product of the Poincarè group with an internal symmetry group. The algebra of the internal symmetry group is the direct sum of a semisimple Lie algebra and of an abelian algebra.

The proof of this theorem is quite technical and it is far beyond the scope of these lectures. Here, to understand the origin of the theorem, we shall present a simple argument which illustrates why tensorial conserved charged are forbidden in interacting theories. Consider a spin 2 charge Q_{mn}, which we shall assume traceless ($Q^m_m = 0$) for simplicity. By Lorentz invariance, its matrix element on a one-particle state of momentum p and spin zero is

$$\langle p|Q_{mn}|p\rangle = \mathcal{A}(p_m p_n - \frac{1}{d}\eta_{mn}p^2). \tag{2.32}$$

Next, consider the scattering of two of these particles described by the asymptotic state $|p_1, p_2\rangle$. If the conserved charge is *local* (the integral of a local density), we can safely assume that for widely separated particles it holds

$$\langle p_1, p_2|Q_{mn}|p_1, p_2\rangle = \langle p_1|Q_{mn}|p_1\rangle + \langle p_2|Q_{mn}|p_2\rangle. \tag{2.33}$$

Then the scattering is constrained by the following conservation laws

$$\mathcal{A}(p_{1m}p_{1n} + p_{2m}p_{2n} - \frac{1}{d}\eta_{mn}(p_1^2 + p_2^2)) = \mathcal{A}(p'_{1m}p'_{1n} + p'_{2m}p'_{2n} - \frac{1}{d}\eta_{mn}(p'^2_1 + p'^2_2)) \tag{2.34a}$$
$$p_{1m} + p_{2n} = p'_{1m} + p'_{2n}. \tag{2.34b}$$

The only possible solutions of these equations are forward or backward scattering. There is no scattering in the other directions. This contradicts assumptions 3 and 4.

Exercise: *Show that the only solutions of* (2.34a) *and* (2.34b) *are forward or backward scattering.*

SOLUTION: Since we are assuming $\mathcal{A} \neq 0$ and $p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m^2$, the above equations are simply

$$
\begin{aligned}
&p_{1m} p_{1n} + p_{2m} p_{2n} = p'_{1m} p'_{1n} + p'_{2m} p'_{2n} \\
&p_{1m} + p_{2m} = p'_{1m} + p'_{2m}.
\end{aligned}
$$

In the center of mass reference frame, we can write $\vec{p}_1 = -\vec{p}_2$, $E_1 = E_2 = E$, $\vec{p'}_1 = -\vec{p'}_2$ and $E'_1 = E'_2 = E$. Thus the first equation implies

$$\vec{p}_{1i} \vec{p}_{1j} = \vec{p'}_{1i} \vec{p'}_{1j} \quad \text{with} \quad i, j = 1, \dots, d-1 \qquad \Rightarrow \qquad \vec{p}_{1i} = \pm \vec{p'}_{1i}.$$

Summarizing, the Coleman Mandula theorem states that all the conserved (bosonic) charges, except the Poincarè ones, commute with translations and possess spin zero. Therefore they cannot constrain the kinematics of the scattering, but only the internal conserved charges. All the multiplets for this symmetries will contains particle with the same mass and spin.

A natural question is if we can avoid the conclusion of this *no-go theorem* in some way. To explore this, let us look very carefully at possible situations where the hypotheses of the Coleman Mandula theorem break down. Naively, one might think that the *ugly technical hypothesis* is the natural candidate to be the *loop-hole* to elude the theorem. However this is not the case. In fact there are more interesting possibilities which are covered by the theorem and which are not due to the failure of the "*ugly technical hypothesis*"

- The theorem assumes that the symmetries are described by Lie algebra: *i.e.* the commutator of two symmetries of the $S-$matrix is again a symmetry. From the point of view of QFT, we are implicitly assuming that the conserved charges are bosonic objects, namely they carry an integer spin. Then the above theorem states that the only possible spin for the charge of an internal symmetry is zero.
  The structure of a QFT is richer. Next to bosonic objects, we have also fermionic quantities which obey anti-commutation relations; this fact naturally endows the algebra of fields with a graded structure:

  $$[\text{bosonic}, \text{bosonic}] = \text{bosonic} \qquad [\text{bosonic}, \text{fermionic}] = \text{fermionic}$$
  $$\{\text{fermionic,fermionic}\} = \text{fermionic}.$$

  In mathematics these structure are known as *graded Lie algebras* or more commonly as *superalgebras*. Is it possible to construct an interacting quantum field theory where the symmetries of the $S-$matrix close a graded-Lie algebra? A positive answer will imply the existence of conserved fermionic charges, evading in this way the Coleman Mandula theorem. In the following, we shall show that this is actually possible and this will lead us to construct the supersymmetry algebra and the supersymmetric theories.

- There is a second possible loop-hole in the Coleman Mandula theorem: it assumes to deal with point particles. Actually we can consider more general relativistic theories containing objects extended in p spatial dimensions ($p-$branes). These extended objects can carry conserved charges which are $p-$forms, $Q_{[m_1 m_2 \cdots m_p]}$. We shall not have the time to discuss this second possibility in the present lectures.

2.1 Graded Algebras

Before proceeding, we need to recall some basic facts on graded algebras. To begin with, we shall define a Z_2 graded vector space V. It is a vector space which decomposes into the direct sum of two vector subspaces

$$V = S_0 \oplus S_1 \tag{2.35}$$

All the elements in S_0 have grading zero and they are called *bosons* or *even elements*, while all the elements in S_1 have grading 1 and are called *fermions* or *odd elements*. Between two elements of S is defined a bilinear graded Lie-bracket operation or a bilinear graded commutator $[\cdot,\cdot\}$, which satisfies the following properties

- For all x and y in S the grading of the bracket $[x,y\}$ is $\eta_x + \eta_y|$ mod 2, where η_x and η_y are the grading of x and y respectively
- $[x,y\} = -(-1)^{n_x n_y}[y,x\}$
- $(-1)^{n_z n_x}[x,[y,z\}\} + (-1)^{n_x n_y}[y,[z,x\}\} + (-1)^{n_z n_y}[z,[x,y\}\} = 0 \qquad \text{(SuperJacobi)}.$

The graded structure entails that

$$[S_0, S_0] \subseteq \mathfrak{S}_0 \qquad \{S_1, S_1\} \subseteq S_0 \qquad [S_0, S_1] \subseteq S_1. \tag{2.36}$$

In other words S_0 is a standard Lie algebras. Moreover, since S_1 is invariant under the (adjoint) action of S_0, the fermionic sector carries a representation of S_0.

2.2 Supersymmetry algebra in D=4: LSH theorem

In this section we shall address the question of constructing the most general superalgebra containing the Poincarè algebra as a subalgebra of the bosonic sector S_0. We will begin by focussing on the fermionic sector S_1. It carries a representation $\mathfrak{R}$, in general reducible, of the Lorentz group. Let us decompose $\mathfrak{R}$ as the direct sum of irreducible representations

$$\mathfrak{R} = \bigoplus_{i=1}^{k} r_i \left(\frac{m_i}{2}, \frac{n_i}{2}\right) \tag{2.37}$$

where r_i denotes the multiplicity of the representation $(m_i/2, n_i/2)$. We shall assume that the representation $\mathfrak{R}$ is closed under hermitian conjugation: this implies that the representations $\left(\frac{m_i}{2}, \frac{n_i}{2}\right)$ and $\left(\frac{n_i}{2}, \frac{m_i}{2}\right)$ appear in (2.37) with the same multiplicity[1]. The generators transforming in the representation $\left(\frac{m_i}{2}, \frac{n_i}{2}\right)$ $(m_i > n_i)$ will be denoted with the symbol

$$Q^I_{\alpha_1 \ldots \alpha_{m_i}; \dot{\alpha}_1 \ldots \dot{\alpha}_{n_i}} \qquad I = 1, \ldots, r_i, \tag{2.38}$$

where the indices α_i and $\dot{\alpha}_i$ run over the values 1 and 2 and $Q^I_{\alpha_1 \ldots \alpha_{m_i}\ ;\ \dot{\alpha}_1 \ldots \dot{\alpha}_{n_i}}$ is totally symmetric both in the indices $\alpha_1 \ldots \alpha_{m_i}$ and $\dot{\alpha}_1 \ldots \dot{\alpha}_{n_i}$. The hermitian conjugate generator of $Q^I_{\alpha_1 \ldots \alpha_{m_i}; \dot{\alpha}_1 \ldots \dot{\alpha}_{n_i}}$ is indicated with

$$\overline{Q}^I_{\dot{\alpha}_1 \ldots \dot{\alpha}_{m_i}; \alpha_1 \ldots \alpha_{n_i}} \qquad I = 1, \ldots, r_i. \tag{2.39}$$

It obviously transforms in the representation $\left(\frac{n_i}{2}, \frac{mi}{2}\right)$ $(m_i > n_i)$. We shall now consider the anticommutator

$$\{Q^I_{\alpha_1 \cdots \alpha_{m_i}, \dot{\alpha}_1 \cdots \dot{\alpha}_{n_i}}, \bar{Q}^I_{\dot{\alpha}_1 \cdots \dot{\alpha}_{m_i}, \alpha_1 \cdots \alpha_{n_i}}\}. \tag{2.40}$$

The result of this graded Lie bracket must generically transform in the following direct sum of Lorentz representations

$$\left(\frac{m_i}{2}, \frac{n_i}{2}\right) \otimes \left(\frac{n_i}{2}, \frac{m_i}{2}\right) = \bigoplus_{i,j=|m_i-n_i|/2}^{(m_i+n_i)/2} (i, j). \tag{2.41}$$

However, if we choose all the indices equal to 1, the anticommutator

$$\{Q^I_{1 \cdots \alpha_1, \dot{1} \cdots \dot{1}}, \bar{Q}^I_{\dot{1} \cdots \dot{1}, 1 \cdots 1}\} \tag{2.42}$$

can only belong to the representation of maximal spin

$$\left(\frac{m_i + n_i}{2}, \frac{mi + n_i}{2}\right). \tag{2.43}$$

This can be checked by computing the eigenvalue of the $z-$component of $J^{(+)}$ and $J^{(-)}$: we find $\frac{m_i+n_i}{2}$ in both cases.

Thus the result of the commutator (2.42) can only be a bosonic generator with this spin. However, due to Coleman-Mandula theorem, the bosonic generators have either spin 0 or belong to the Poincarè group,

$$M_{mn} \in (1, 0) \oplus (0, 1), \quad P_m \in (1/2, 1/2). \tag{2.44}$$

[1]Because of the definition (1.4) $J^{(+)\dagger} = J^{(-)}$. At the level representation this implies that

$$(m, n)^\dagger = (n, m)$$

Consequently, the anticommutator (2.42) can either vanish or be proportional only to the momentum P_m. If (2.42) vanishes and the representation of the graded algebra is realized on a Hilbert space of positive norm, we must conclude that $Q^I_{1\cdots 1,\dot{1}\cdots\dot{1}} = \bar{Q}^I_{\dot{1}\cdots\dot{1},1\cdots 1} = 0$. This, in turn, implies $Q^I_{\alpha_1\cdots\alpha_r,\dot{\alpha}_1\cdots\dot{\alpha}_r} = \bar{Q}^I_{\dot{\alpha}_1\cdots\dot{\alpha}_r,\alpha_1\cdots\alpha_s} = 0$ since $Q^I_{\alpha_1\cdots\alpha_r,\dot{\alpha}_1\cdots\dot{\alpha}_r}$ and $\bar{Q}^I_{\dot{\alpha}_1\cdots\dot{\alpha}_r,\alpha_1\cdots\alpha_s}$ belong to irreducible representations. Therefore we are left only with the second possibility, *i.e.* P_m. Then

$$\frac{m_i+n_i}{2} = \frac{1}{2}, \tag{2.45}$$

which is solved or by $m_i = 1/2$ and $n_i = 0$ or by $m_i = 0$ and $n_i = 1/2$. Since the two representations are hermitian conjugate one to each other, we can just consider the first choice. Thus the only admissible fermionic generators are

$$(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}) \qquad \text{with} \qquad I = 1, \ldots, N. \tag{2.46}$$

The above analysis also fixes the form of the anticommutator $\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\}$. In fact, the Lorentz implies

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = V^{IJ}\sigma^m_{\alpha\dot{\alpha}}P_m, \tag{2.47}$$

where V^{IJ} is an hermitian matrix. In fact by taking the hermitian conjugate of both sides in (2.47), we find

$$V^{IJ*}\sigma^m_{\alpha\dot{\alpha}}P^m = \{\bar{Q}^I_{\dot{\alpha}}, Q^J_\alpha\} = V^{JI}\sigma^m_{\alpha\dot{\alpha}}P_m. \tag{2.48}$$

By means of an unitary redefinition of the generators $Q^I = U^I{}_J Q^J$, we can bring V^{IJ} into diagonal form and write

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = \lambda_I \delta^{IJ}\sigma^m_{\alpha\dot{\alpha}}P_m. \tag{2.49}$$

Since the anticommutators $\{Q^I_1, \bar{Q}^I_{\dot{1}}\}$ and $\{Q^I_2, \bar{Q}^I_{\dot{2}}\}$ are positive definite, the numerical factor λ_I are positive and we can rescale the generators[2] so that

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\sigma^m_{\alpha\dot{\alpha}}P_m. \tag{2.50}$$

Next we shall examine the constraints on the commutators between fermionic generators and translations. Since the generators $P_{\alpha\dot{\alpha}} \equiv P_m\sigma^m_{\alpha\dot{\alpha}}$ belong to the representation $(1/2, 1/2)$ and the supersymmetry charge Q^I_α transforms in the $(1/2, 0)$, the result of the commutator will transform in the representation $(1, 1/2) \oplus (0, 1/2)$. In the absence of bosonic generators in the representation $(1, 1/2)$, the only possibility is

$$[P_{\alpha\dot{\alpha}}, Q^I_\beta] = K^I{}_J \epsilon_{\alpha\beta}\bar{Q}^J_{\dot{\alpha}}. \tag{2.51}$$

[2]It is sufficient to perform the following rescaling $Q^I_\alpha \mapsto \sqrt{\frac{\lambda^I}{2}}Q^I_\alpha \quad \bar{Q}^I_{\dot{\alpha}} \mapsto \sqrt{\frac{\lambda^I}{2}}\bar{Q}^I_{\dot{\alpha}}$

The presence of the invariant tensor $\epsilon_{\alpha\beta}$ in (2.51) ensures that the r.h.s. of (2.51) is a singlet with respect to the first $SU(2)$ of the Lorentz group. Similar arguments lead us to write

$$[P_{\alpha\dot{\alpha}}, \bar{Q}^I_{\dot{\beta}}] = \bar{K}^I{}_J \epsilon_{\dot{\alpha}\dot{\beta}} Q^J_\alpha, \tag{2.52}$$

where $\bar{K}^I{}_J = -(K^I{}_J)^*$ since $\bar{Q} = Q^\dagger$. The possible choices for the matrix $K^I{}_J$ are completely fixed by the *abelian* nature of translations. In fact this implies that $K^I{}_J$ can only vanish (see exercise below).

Exercise: *Show that $K^I{}_J = 0$.*

SOLUTION: Since translations close an abelian algebra, the graded Jacobi identity allows us to write

$$\begin{aligned}
0 =& 2\delta^{IJ}[P_{\beta\dot{\beta}}, [P_{\alpha\dot{\alpha}}, P_{\rho\dot{\rho}}]] = [P_{\beta\dot{\beta}}, [P_{\alpha\dot{\alpha}}, \{Q^I_\rho, \bar{Q}^J_{\dot{\rho}}\}]] = [P_{\beta\dot{\beta}}, \{Q^I_\rho, [P_{\alpha\dot{\alpha}}, \bar{Q}^J_{\dot{\rho}}]\}] + [P_{\beta\dot{\beta}}, \{\bar{Q}^J_{\dot{\rho}}, [P_{\alpha\dot{\alpha}}, Q^I_\rho]\}] = \\
=& \bar{K}^J{}_M \epsilon_{\dot{\alpha}\dot{\rho}} [P_{\beta\dot{\beta}}, \{Q^I_\rho, Q^M_\alpha\}] + K^I{}_M \epsilon_{\alpha\rho} [P_{\beta\dot{\beta}}, \{\bar{Q}^J_{\dot{\rho}}, \bar{Q}^M_{\dot{\alpha}}\}] = \\
=& \bar{K}^J{}_M \epsilon_{\dot{\alpha}\dot{\rho}} \{Q^M_\alpha, [P_{\beta\dot{\beta}}, Q^I_\rho]\} + \bar{K}^J{}_M \epsilon_{\dot{\alpha}\dot{\rho}} \{Q^I_\rho, [P_{\beta\dot{\beta}}, Q^M_\alpha]\} + K^I{}_M \epsilon_{\alpha\rho} \{\bar{Q}^M_{\dot{\alpha}}, [P_{\beta\dot{\beta}}, \bar{Q}^J_{\dot{\rho}}]\} + \\
&+ K^I{}_M \epsilon_{\alpha\rho} \{\bar{Q}^J_{\dot{\rho}}, [P_{\beta\dot{\beta}}, \bar{Q}^M_{\dot{\alpha}}]\} = \bar{K}^J{}_M K^I{}_S \epsilon_{\dot{\alpha}\dot{\rho}} \epsilon_{\beta\rho} \{Q^M_\alpha, \bar{Q}^S_{\dot{\beta}}\} + \bar{K}^J{}_M \epsilon_{\dot{\alpha}\dot{\rho}} K^M{}_S \epsilon_{\beta\alpha} \{Q^I_\rho, \bar{Q}^S_{\dot{\beta}}\} + \\
&+ K^I{}_M \epsilon_{\alpha\rho} \bar{K}^J{}_S \epsilon_{\dot{\beta}\dot{\rho}} \{\bar{Q}^M_{\dot{\alpha}}, Q^S_\beta\} + K^I{}_M \epsilon_{\alpha\rho} \bar{K}^M{}_S \epsilon_{\dot{\beta}\dot{\alpha}} \{\bar{Q}^J_{\dot{\rho}}, Q^S_\beta\} = \\
=& 2\bar{K}^J{}_M \delta^{MS} K^I{}_S \epsilon_{\dot{\alpha}\dot{\rho}} \epsilon_{\beta\rho} P_{\alpha\dot{\beta}} + 2\bar{K}^J{}_M \epsilon_{\dot{\alpha}\dot{\rho}} K^M{}_I \epsilon_{\beta\alpha} P_{\rho\dot{\beta}} + 2K^I{}_M \epsilon_{\alpha\rho} \bar{K}^J{}_S \epsilon_{\dot{\beta}\dot{\rho}} \delta^{MS} P_{\beta\dot{\alpha}} + 2K^I{}_M \epsilon_{\alpha\rho} \bar{K}^M{}_J \epsilon_{\dot{\beta}\dot{\alpha}} P_{\beta\dot{\rho}}.
\end{aligned}$$

Setting $\dot{\alpha} = \dot{\beta} = \dot{1}$, $\alpha = \beta = 1$ and $\rho = \dot{\rho} = 2$ in the above equation, we find $0 = 4(K\bar{K}^T)^{IJ} P_{1\dot{1}}$, which, in turn, implies $(K\bar{K}^T)^{IJ} = 0$. Since $\bar{K}^I{}_J = -(K^I{}_J)^*$, the previous condition is also equivalent to $(K\bar{K}^\dagger)^{IJ} = 0 \quad \Rightarrow \quad K = 0$.

Summarizing, we have shown that

$$[P_{\alpha\dot{\alpha}}, Q_\beta] = [P_{\alpha\dot{\alpha}}, \bar{Q}_{\dot{\beta}}] = 0. \tag{2.53}$$

Consider now the anticommutator of two fermionic charges

$$\{Q^I_\alpha, Q^J_\beta\}. \tag{2.54}$$

Lorentz invariance requires that the result of (2.54) only belongs to the representation $(0,0)$ and $(1,0)$, since $(1/2,0)\otimes(1/2,0) = (1,0)\oplus(0,0)$. Concretely, the r.h.s. of the anticommutator (2.54) must be a linear combination of the bosonic generators of spin zero, denoted by B^l (internal symmetries), and of the self-dual part $M_{\alpha\beta} = \sigma^{mn}_{\alpha\beta} M_{mn}$ of Lorentz generator M^{mn}. Therefore, we shall write

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} C^{IJ}_l B^l + Y^{IJ} M_{\alpha\beta}. \tag{2.55}$$

As Q^I_α commutes with P_m, Y^{IJ} must identically vanish. We are left with

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} C^{IJ}_l B^l \equiv \epsilon_{\alpha\beta} Z^{IJ}. \tag{2.56}$$

Taking the hermitian conjugate of (2.56), we find

$$\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} C^{IJ*}_l B^l \equiv \epsilon_{\alpha\beta} Z^{IJ}_+. \tag{2.57}$$

Next, we shall consider the commutator $[B_l, Q^I_\alpha]$. Its result must transform in the representation $(1/2, 0)$ of Lorentz group, therefore

$$[B_l, Q^I_\alpha] = (S_l)^I{}_L Q^L_\alpha. \tag{2.58}$$

The Jacobi identities for $[B_m, [B_l, Q^I_\alpha]]$ and $\{\bar{Q}^J_{\dot\alpha}, [B_l, Q^I_\alpha]\}$ imply that $-(S_l)^I{}_L$ must provide a unitary representation of the internal symmetries. Taking the complex conjugate of the commutator (2.58), we also find $[B_l, \bar{Q}^I_{\dot\alpha}] = -(S^*_l)^I{}_L \bar{Q}^L_{\dot\alpha}$.

Exercise: *Show that $-(S_l)^I{}_L$ provides a unitary representation of the internal symmetries.*

SOLUTION: The Jacobi identities for $[B_m, [B_l, Q^I_\alpha]]$ implies

$$\begin{aligned}0 =&[B_m, [B_l, Q^I_\alpha]] + [B_l, [Q^I_\alpha, B_m]] + [Q^I_\alpha, [B_m, B_l]] = (S_l)^I{}_L (S_m)^L{}_R Q^R_\alpha - (S_m)^I{}_L (S_l)^L{}_R Q^R_\alpha + i c_{lm}{}^k (S_k)^I{}_R Q^R_\alpha = \\ =&\left((S_l)^I{}_L (S_m)^L{}_R - (S_m)^I{}_L (S_l)^L{}_R + i c_{lm}{}^k (S_k)^I{}_R\right) Q^R_\alpha,\end{aligned}$$

where we have used that the bosonic generators B_l close a Lie algebra: $[B_l, B_m] = i c_{lm}{}^k B_k$. Since the generators Q^I_α are linearly independent, the matrices $-(S_l)^I{}_K$ must provide a representation of the internal symmetries. The Jacobi identity for $\{\bar{Q}^J_{\dot\alpha}, [B_l, Q^I_\alpha]\}$ implies that the matrices S_l are hermitian and thus the representation is unitary:

$$0 = \{\bar{Q}^J_{\dot\alpha}, [B_l, Q^I_\alpha]\} - [B_l, \{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\}] - \{Q^I_\alpha, [\bar{Q}^J_{\dot\alpha}, B_l]\} = 2((S_l)^I{}_J - (S^*_l)^J{}_I) P_{\alpha\dot\alpha}.$$

Now, we have all the ingredients to characterize the bosonic generators $Z^{LM} = C^{LM}_l B^l$ and $Z^{LM}_+ = C^{LM*}_l B^l$ appearing in the r.h.s of (2.56) and (2.57). To begin with, the Jacobi identities for $[B_l, \{Q^I_\alpha, Q^J_\beta\}]$ require that Z^{LM} generators form an invariant subalgebra (*i.e.* an *ideal*) of the spin zero bosonic sector

$$\begin{aligned}0 =&[B_l, \{Q^I_\alpha, Q^J_\beta\}] + \{Q^I_\alpha, [Q^J_\beta, B_l]\} - \{Q^J_\beta, [B_l, Q^I_\alpha]\} = \\ =&\epsilon_{\alpha\beta}\left([B_l, Z^{IJ}] - (S_l)^J{}_K Z^{IK} - (S_l)^I{}_K Z^{KJ}\right).\end{aligned} \tag{2.59}$$

Moreover, the generators Z^{IJ} commute with all the fermionic charges: $[\bar{Q}^K_{\dot\alpha}, Z^{IJ}] = [Z^{IJ}, Q^M_\alpha] = 0$. In fact from Jacobi identity

$$0 = [Q^I_\alpha, \{Q^J_\beta, \bar{Q}^K_{\dot\alpha}\}] + [Q^J_\beta, \{\bar{Q}^K_{\dot\alpha}, Q^I_\alpha\}] + [\bar{Q}^K_{\dot\alpha}, \{Q^I_\alpha, Q^J_\beta\}] = \epsilon_{\alpha\beta}[\bar{Q}^K_{\dot\alpha}, Z^{IJ}] \tag{2.60}$$

it immediately follows that $[\bar{Q}^K_{\dot\alpha}, Z^{IJ}] = 0$. Instead if we set $[Q^K_\alpha, Z^{IJ}] = H^{KIJ}_R Q^R_\alpha$, the Jacobi identities

$$0 = \{\bar{Q}^M_{\dot\alpha}, [Q^K_\alpha, Z^{IJ}]\} - \{Q^K_\alpha, [Z^{IJ}, \bar{Q}^M_{\dot\alpha}]\} + [Z^{IJ}, \{\bar{Q}^M_{\dot\alpha}, Q^K_\alpha\}] = 2H^{KIJ}_M P_{\alpha\dot\alpha}, \tag{2.61}$$

implies $H^{KIJ}_M = 0$ and thus $[Z^{IJ}, Q^M_\alpha] = 0$. A simple but very important consequence of this property is that

$$[Z^{IJ}, Z^{LM}] = \frac{1}{2}\epsilon^{\alpha\beta}[Z^{IJ}, \{Q^L_\alpha, Q^M_\beta\}] = 0. \tag{2.62}$$

In other words, the generators Z^{LM} do not only form an invariant subalgebra, but this invariant subalgebra is also abelian! Moreover, from the Coleman-Mandula theorem, we know that the bosonic generators of spin zero close a Lie algebra that is the direct sum of a semisimple Lie algebra and of an abelian algebra $\mathfrak{C}$. Therefore $Z^{LM} \in \mathfrak{C}$, since the other component is semisimple and

$$[Z^{LM}, B_l] = 0. \tag{2.63}$$

Thus, Z^{LM} and (in the same way) $Z_+^{LM})$ commute with all the generators of the graded Lie algebra. For this reason, they are called *central charges.* They will play a fundamental role in the second part of these lectures. Note that the *central charges* cannot be arbitrary; eq. (2.59) provides a strong constraint on their form, $(S_l)^J{}_K Z^{IK} + (S_l)^I{}_K Z^{KJ} = 0$. Namely, they must be an invariant tensor of the representation given by S_l.

Let us summarize the result of this section: we have argued that the most general graded algebra in $D = 4$, whose bosonic sector contains the Poincarè algebra and respects the Coleman-Mandula theorem, is given by

$$[P_m, P_n] = 0 \qquad [M_{ab}, P_m] = i(\eta_{am} P_b - \eta_{bm} P_a) \tag{2.64a}$$

$$[M_{ab}, M_{mn}] = i(\eta_{am} M_{bn} + \eta_{bn} M_{am} - \eta_{mb} M_{an} - \eta_{na} M_{bm}) \tag{2.64b}$$

$$[P_m, Q_\alpha^L] = [P_m, \bar{Q}_{\dot{\alpha}}^L] = 0 \qquad [P_m, B_l] = [P_m, Z^{IJ}] = 0 \tag{2.64c}$$

$$\{Q_\alpha^M, \bar{Q}_{\dot{\alpha}}^N\} = 2\delta^{MN} \sigma_{\alpha\dot{\alpha}}^m P_m \qquad \{Q_\alpha^M, Q_\beta^N\} = \epsilon_{\alpha\beta} Z^{MN} \qquad \{\bar{Q}_{\dot{\alpha}}^M, \bar{Q}_{\dot{\beta}}^N\} = \epsilon_{\dot{\alpha}\dot{\beta}} Z_+^{MN} \tag{2.64d}$$

$$[Z^{LM}, Q_\alpha^J] = [Z^{LM}, \bar{Q}_{\dot{\alpha}}^J] = 0 \qquad [Z^{LM}, Z^{IJ}] = [Z^{LM}, B_l] = 0 \tag{2.64e}$$

$$[B_l, B_m] = i c_{lm}{}^k B_k \qquad [B_l, Q_\alpha^I] = (S_l)^I{}_L Q_\alpha^L \qquad [B_l, \bar{Q}_{\dot{\alpha}}^I] = -(S_l^*)^I{}_L \bar{Q}_{\dot{\alpha}}^L. \tag{2.64f}$$

where $Z^{LM} = C_l^{LM} B^l$ e $Z_+^{LM} = C_l^{LM*} B^l$.

It is worth mentioning that this is *not* the most general superalgebra, if we admit the presence of extended objects as well. The anticommutator of the supercharges can contain

$$\{Q_\alpha^M, \bar{Q}_{\dot{\alpha}}^N\} = 2\delta^{MN} \sigma_{\alpha\dot{\alpha}}^m P_m + \mathcal{Z}_m^{MN} \sigma_{\alpha\dot{\alpha}}^m \qquad \{Q_\alpha^M, Q_\beta^N\} = \epsilon_{\alpha\beta} Z^{MN} + \sigma_{\alpha\beta}^{mm} \mathcal{Z}_{mn}^{MN} \tag{2.65}$$

where $\mathcal{Z}_m^{MN}$ and $\mathcal{Z}_{mn}^{MN}$ are respectively traceless and symmetric in the indices M an N. The first central extension can appear in a theory containing strings, the second one in a theory containing a domain wall.

3 Representations of the supersymmetry algebra

3.1 General properties

In this subsection, using the general structure of the susy algebra we shall establish some basic properties of supersymmetric theories:

1. Since the full susy algebra contains the Poincarè algebra as a subalgebra, any representation of the full susy algebra also gives a representation of the Poincarè algebra, although in general a reducible one. Since each irreducible representation[3] of the Poincarè algebra corresponds to a particle, an irreducible representation of the susy algebra in general corresponds to several particles.

2. All the particles belonging to the same irreducible representation possess the same mass. In fact the Poincarè quadratic Casimir $P_m P^m$ is also a Casimir of the supersymmetry algrebra, since $[P, Q] = [P, \bar{Q}] = 0$.

3. An irreducible representation of the susy algebra contains both bosonic and fermionic particles. In fact, if $|\Omega\rangle$ is a state, $\bar{Q}|\Omega\rangle$ and/or $Q|\Omega\rangle$ is also a state. The difference in spin between $|\Omega\rangle$ and $\bar{Q}|\Omega\rangle$ (or $Q|\Omega\rangle$) is 1/2.

4. An irreducible representation of the susy algebra with a finite number of particles contains the same number of fermions and bosons.
   Proof: Let us denote the fermionic number operator with N_F. It counts the number of particles with half-integer spin present in a given state. Starting from N_F, we shall define the operator $(-1)^{N_F}$, which is 1 on bosonic states (*state of integer spin*) and -1 on fermionic states (*state of half-integer spin*). The defining property of this operator implies that it commutes with all the supersymmetry charges, *i.e.*
   $$(-1)^{N_F} Q^M_\alpha + Q^M_\alpha (-1)^{N_F} = 0 \quad \text{e} \quad (-1)^{N_F} \bar{Q}^M_{\dot{\alpha}} + \bar{Q}^M_{\dot{\alpha}} (-1)^{N_F} = 0. \tag{3.66}$$
   Consider the subspace W generated by the states of fixed momentum p_m. Since we are dealing with representations containing a finite number of particles, this subspace is finite dimensional and on this subspace we can compute the following trace
   $$\begin{aligned} \mathrm{Tr}_W(P_m(-1)^{N_F}) &= \mathrm{Tr}_W((-1)^{N_F}\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\}) = \mathrm{Tr}_W((-1)^{N_F}(Q^I_\alpha \bar{Q}^J_{\dot{\alpha}} + \bar{Q}^J_{\dot{\alpha}} Q^I_\alpha)) = \\ &= \mathrm{Tr}_W(-Q^I_\alpha (-1)^{N_F} \bar{Q}^J_{\dot{\alpha}} + Q^I_\alpha (-1)^{N_F} \bar{Q}^J_{\dot{\alpha}}) = 0. \end{aligned} \tag{3.67}$$
   This, in turn, implies
   $$p_m \mathrm{Tr}_W((-1)^{N_F}) = p_m(n_B - n_F) = 0, \tag{3.68}$$

[3] Here, we obviously mean the representation with $p^2 \geq 0$ and $p^0 > 0$.

namely the number of fermionic (n_F) and bosonic (n_B) particles is the same.

5. The energy of a supersymmetric theories is greater or equal to zero. In fact the susy algebra allows us to write the hamiltonian of theory as follows

$$H = P^0 = \frac{1}{4}[\{Q_1^I, \bar{Q}_{\dot{1}}^I\} + \{Q_2^I, \bar{Q}_{\dot{2}}^I\}] = \frac{1}{4}[\{Q_1^I, (Q_1^I)^\dagger\} + \{Q_2^I, (Q_2^I)^\dagger\}], \tag{3.69}$$

where we used that $(Q_{1,2}^I)^\dagger = \bar{Q}_{\dot{1},\dot{2}}^I$. Consequently the Hamiltonian is positive

$$\langle\psi|H|\psi\rangle = \frac{1}{4}\left[\left|\left|Q_1^I|\psi\rangle\right|\right|^2 + \left|\left|\bar{Q}_{\dot{1}}^I|\psi\rangle\right|\right|^2 + \left|\left|Q_2^I|\psi\rangle\right|\right|^2 + \left|\left|\bar{Q}_{\dot{2}}^I|\psi\rangle\right|\right|^2\right] \geq 0. \tag{3.70}$$

Let $|\Omega\rangle$ be the vacuum of a supersymmetric theory. If supersymmetry is not spontaneously broken, the vacuum is annihilated by all the charges Q and $\bar{Q}$, and thus the vacuum energy vanishes:

$$\langle\Omega|H|\Omega\rangle = \frac{1}{4}\left[\left|\left|Q_1^I|\Omega\rangle\right|\right|^2 + \left|\left|\bar{Q}_{\dot{1}}^I|\Omega\rangle\right|\right|^2 + \left|\left|Q_2^I|\Omega\rangle\right|\right|^2 + \left|\left|\bar{Q}_{\dot{2}}^I|\Omega\rangle\right|\right|^2\right] = 0. \tag{3.71}$$

Vice versa, if the vacuum energy is different from zero eq. (3.71) implies that there is at least one supersymmetric charge, which does not annihilate the vacuum. Namely, the supersymmetry is spontaneously broken.
A remark about this result is in order. In (3.71) there is no sum over the index I; this entails that supersymmetries are either all broken or all preserved. In this reasoning there is however a potential loop-hole, which can be used (and it has been used) to evade such a result. In fact this type of argument assumes implicitly that Poincarè invariance is present in the system. If one now considers other systems where part of the Poincarè invariance is preserved and part is broken, a partial breaking of the supersymmetry can also occur.

3.2 Representations without central charges

In this section we wish to construct all the possible unitary representations of the supersymmetry algebra. To achieve this goal, we shall use the Wigner method of induced representations. This method consists of two steps: (a) Firstly, we choose a reference momentum p^m. We find the subalgebra G, which leaves p^m unaltered, and construct a representation of this subalgebra on the states with momentum p^m. (b) Secondly, we (literally) boost the representation of the subalgebra H up to a representation of the full susy algebra. In the following we shall enter into the details of this second part of the procedure, since it is very similar to the one for the Poincarè group.
Since the $M^2 = -P_m P^m$ is a Casimir, we can consider the case of massless and massive representations separately.

3.2.1 Massless representations

In a representation where all the particles are massless ($M^2 = -P^2 = 0$), a natural choice as reference momentum is provided by $p_m = (-E, 0, 0, E)$. The subalgebra G then contains the following elements:

Lorentz. The Lorentz transformations preverving the above reference momentum are generated by $J = M_{12}$, $S_1 = M_{01} - M_{13}$ and $S_2 = M_{02} - M_{23}$. These three operators close the algebra of the Euclidean group E_2 in two dimensions. However, in any unitary representation with a finite number of particle states, the generators S_1 and S_2 must be represented trivially, *i.e.* $S_1 = S_2 = 0$. Therefore the only surviving generator is J, which is identified with the well-known *helicity* of the massless particles.

Exercise: *Show that the the reference momentum $p_m = (-E, 0, 0, E)$ is preserved by three generators $J = M_{12}$, $S_1 = M_{01} - M_{13}$ and $S_2 = M_{02} - M_{23}$, which close the algebra of E_2.*

SOLUTION: Consider a generic combination of the Lorentz generators $\omega^{mn} M_{mn}$. It will preserve the reference momentum if and only if

$$0 = \omega^{mn}[P^r, M_{mn}]|\psi\rangle = -2i\omega_{rn} P_n|\psi\rangle \quad \Rightarrow \quad \omega^{rn} p_n = E(\omega^{r3} - \omega^{r0}) = 0.$$

This condition constrains the form of the coefficients ω^{mn}. On finds $\omega^{03} = 0$, $\omega^{13} = -\omega^{01}$, $\omega^{23} = -\omega^{02}$, namely the most general element of the Lorentz algebra which leave the momentum p_m intact is $\omega^{mn} M_{mn} = \omega^{01}(M_{01} - M_{13}) + \omega^{02}(M_{02} - M_{23}) + \omega^{12} M_{12}$. This element is a linear combination of the generators $J \equiv M_{12}$, $S_1 \equiv M_{01} - M_{13}$ and $S_2 \equiv M_{02} - M_{23}$, which close the following algebra

$$[J, S_1] = i(M_{02} - M_{23}) = iS_2 \quad [J, S_2] = -i(M_{01} - M_{13}) = -iS_1 \quad [S_1, S_2] = -iM_{12} + iM_{12} = 0.$$

This is the algebra of the euclidean group in two dimensions.

Exercise: *Show that S_1 and S_2 are represented trivially ($S_{1,2} = 0$) in any unitary finite dimensional representation.*

SOLUTION: We choose a basis for the vector spaces carrying the representation so that $J|\lambda\rangle = \lambda|\lambda\rangle$. If we define $S_\pm = S_1 \pm iS_2$, these operators satisfy the commutation rule $[J, S_\pm] = [J, S_1 \pm iS_2] = iS_2 \pm S_1 = \pm S_\pm$. This implies

$$JS^n_\pm|\lambda, p, \sigma\rangle = (\lambda \pm n) S^n_\pm|\lambda, p, \sigma\rangle.$$

The representation will contain an infinite number of states with different values of J unless there exists an integer n such that $S^n_\pm = 0$. Since $S^\dagger_+ = S_-$ and $[S_+, S_-] = 0$, S_+ is a normal operator and it can be diagonalized. We must conclude that $S_+ = 0 \quad \Rightarrow S_- = S_1 = S_2 = 0$.

Supersymmetry charges: All the supersymmetry charges preserve the reference momentum p_m, since they commute with the generator P_m. In this subspace, their anticommutation relations take the simplified form

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2\delta^{IJ}\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} + 2\delta^{IJ}\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} = 4E\delta^{IJ}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.72a}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = 0. \tag{3.72b}$$

Internal symmetries: All the generators B_l obviously leave the momentum p_m unchanged. Now, we shall construct a unitary representation of the subalgebra G. Let $\mathcal{H}$ be the Hilbert space carrying the representation. We shall choose a basis of $\mathcal{H}$ so that J is diagonal

$$J|\lambda, p, \sigma\rangle = \lambda|\lambda, p, \sigma\rangle, \tag{3.73}$$

i.e. a basis formed by states of given helicity. In (3.73), p denotes the reference momentum, while σ stands for the other possible labels of the state. The action of the supersymmetry charges on this basis can be easily determined by proceeding as follows. Consider first Q^I_2 and $\bar{Q}^I_{\dot 2}$. Since $\{Q^I_2, \bar{Q}^J_{\dot 2}\}$ vanishes and $(\bar{Q}^I_{\dot 2} = (Q^I_2)^\dagger)$ in any unitary representation, Q^I_2 must be realized by the null operator. Then we are left with the reduced algebra

$$\{Q^I_1, \bar{Q}^J_{\dot 1}\} = 4E\delta^{IJ} \qquad \{Q^I_1, Q^J_1\} = \{\bar{Q}^I_{\dot 1}, \bar{Q}^J_{\dot 1}\} = 0 \tag{3.74}$$

If we define the operators

$$a^I = \frac{1}{\sqrt{4E}} Q^I_1, \qquad (a^I)^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot 1}. \tag{3.75}$$

the above algebra becomes that of N fermionic creation and annihilation operators

$$\{a^I, (a^J)^\dagger\} = \delta^{IJ}, \qquad \{a^I, a^J\} = \{(a^I)^\dagger, (a^J)^\dagger\} = 0. \tag{3.76}$$

The representations of this algebra are constructed in terms of a Fock vacuum, namely a state such that

$$a^I|\lambda, \Omega\rangle = 0 \qquad \text{per } I = 1, \ldots, N. \tag{3.77}$$

Since $[J, (a^I)^\dagger] = \frac{1}{2}(a^I)^\dagger$, the Fock vacuum can be also chosen to be an eigenstate[4] of the helicity J:

$$J|\lambda, \Omega\rangle = \lambda|\lambda, \Omega\rangle. \tag{3.78}$$

All the other states of the representation can be generated by acting with creation operators on the vacuum. A generic state will be of the form

$$|I_1; \cdots; I_n\rangle = (a^{I_1})^\dagger \cdots (a^{I_n})^\dagger |\Omega, \lambda\rangle, \tag{3.79}$$

[4]Because of the commutation relation $[J, (a^I)^\dagger] = \frac{1}{2}(a^I)^\dagger$, the fermionic operators a^I map eigenstates of J into eigenstates of J.

where n runs from 1 to N, the numbers of supersimmetries. The states, by construction, are completely antisymmetric in the indices (I_r). Therefore their total number is finite and it can be determined as follows. Let us fix the number of creation operators acting on the vacuum to be n. Then a simple exercise in combinatorics shows that the number of states containing n creation operators is $\binom{N}{n}$. Now the total number d of states is obtained by summing over n the previous result

$$d = \sum_{n=0}^{N} \binom{N}{n} = (1+1)^N = 2^N. \tag{3.80}$$

The above states are associated to representations of fixed helicity. In fact, as

$$[J, (a^I)^\dagger] = \frac{1}{2}(a^I)^\dagger, \tag{3.81}$$

we find that

$$\begin{aligned} J|I_1; \cdots ; I_n\rangle &= J(a^{I_1})^\dagger \cdots (a^{I_n})^\dagger |\Omega, \lambda\rangle = [J, (a^{I_1})^\dagger \cdots (a^{I_n})^\dagger]|\Omega, \lambda\rangle + \lambda |I_1; \cdots ; I_n\rangle = \\ &= \left(\lambda + \frac{n}{2}\right) |I_1; \cdots ; I_n\rangle. \end{aligned} \tag{3.82}$$

Moreover, since adding a creation operator simply raises the helicity by 1/2, the representation will always contains 2^{N-1} bosonic states (states of integer helicity) and 2^{N-1} fermionic states (states of half-integer helicity), independently of the helicity possessed by the Fock vacuum.

A systematic classification of the states belonging to a given representation can be given by means of the so-called $\mathcal{R}-$symmetry, the group of transformations which leaves invariant the anticommutators of the spinorial charges. In mathematical terms this is called an *automorphism* of the supersymmetry algebra. In some cases this automorphism can be also promoted to be an internal symmetry of the supersymmetric theory.
The $U(N)$ transformations $a^I \mapsto U^I{}_J a^J$ and $a^{\dagger I}{}_J \mapsto U^{*I}{}_J a^{\dagger J}$ provide a natural automorphism for the fermionic algebra (3.76). At the infinitesimal level, this $\mathcal{R}-$symmetry is generated by $M^{IJ} = [a^{\dagger I}, a^J]$. Since these generators commute with the helicity J ($[J, M^{RS}] = 0$), the states of fixed helicity carry a representation of this $\mathcal{R}-$symmetry. For example, the states of helicity $\lambda + n/2$ realize a representation $\mathfrak{R}$, which is the totally antisymmetric tensor product of n anti-fundamental representations of $U(N)$: $\mathfrak{R} = \bigwedge_{i=1}^{n}(\bar{N})$.
Although the $\mathcal{R}-$symmetry group introduced above is a powerful tool for organizing the states of the representation, it is not the largest automorphism of the fermionic algebra. In fact, if we define $q_I = (a^I + (a^I)^\dagger)$ and $q_{N+I} = i((a^I)^\dagger - a^I)$ for $I = 1, \ldots, N$, the fermionic algebra can be rewritten as follows

$$\{q_a, q_b\} = 2\delta_{ab} \qquad \text{with} \quad a = b = 1, ..., 2N. \tag{3.83}$$

This is the Euclidean Clifford algebra of dimension $2N$, whose largest automorphism is the $SO(2N)$, generated by $R_{ab} = \frac{i}{4}[q_a, q_b]$. Any irreducible supermultiplet will also carry a representation of this automorphism. However, this automorphism cannot be an internal symmetry of an interacting supersymmetric field theory, since its multiplets contain particles of different helicity, $[J, R_{ab}] \neq 0$. This possibility is forbidden by the Coleman Mandula theorem, $SO(2N)$ being a bosonic symmetry.

The largest automorphism which preserves the helicity of the states is given by the $U(N)$ discussed previously, and in fact this $\mathcal{R}-$symmetry can be realized as an internal symmetry of an interacting supersymmetric field theory.

Exercise: *Show that the largest isomorphism commuting with the helicity is $U(N)$.*

SOLUTION: The most general operator M acting linearly on the fermionic algebra must be a bilinear in a^I and $a^{\dagger I}$, namely

$$M = s_{IJ}[a^I, a^J] + r_{IJ}[a^{I\dagger}, a^{J\dagger}] + \omega_{IJ}[a^{I\dagger}, a^J].$$

This operator commutes with the helicity generator $J = \frac{1}{2}\sum_I a^{I\dagger}a^I + \lambda\mathbb{1}$ if and of if $s_{IJ} = r_{IJ} = 0$. We are left with $M = \omega_{IJ}[a^{I\dagger}, a^J]$. It is easy to check that these operators generate $U(N)$.

Some relevant examples: N=1 SUPERSYMMETRY: The generic supermultiplet of $N = 1$ is formed by two states: a vacuum of helicity λ and one excited state of helicity $\lambda + 1/2$

$$|\lambda, \Omega\rangle \qquad a^\dagger|\lambda, \Omega\rangle. \tag{3.84}$$

For $N = 1$, the $\mathcal{R}-$symmetry is simply $U(1)$ and thus we can associate to each state an $\mathcal{R}-$charge. The form of the $U(1)$ generator is determined up to an additive constant, which can be identified with the charge of the vacuum. It is given by $\mathcal{R} = \sum_I a^{I\dagger}a^I + n\mathbb{1}$. The charges of the above two states are n and $n+1$ respectively.

The multiplet (3.84) is not CPT-invariant; in fact the state CPT$|\lambda, \Omega\rangle$ has helicity $-\lambda$ and $\mathfrak{R}$-charge $-n$ and thus it cannot belong to the above irreducible multiplet. CPT-invariance is a mandatory symmetry of relativistic local quantum field theories, therefore we cannot construct a theory containing just the multiplet (3.84). This difficulty can be easily circumvented by considering a reducible representation. We shall add to (3.84) the multiplet generated by the vacuum $|-\lambda + 1/2, \Omega\rangle$ e with $\mathcal{R}-$charge $-n-1$. Then the total (reducible) supermultiplet is CPT-invariant and is given by

$$\underset{(-n-1)}{|-\lambda - 1/2, \Omega\rangle} \qquad \underset{(-n)}{a^\dagger|-\lambda - 1/2, \Omega\rangle} \qquad \underset{(n)}{|\lambda, \Omega\rangle} \qquad \underset{(n+1)}{a^\dagger|\lambda, \Omega\rangle}. \tag{3.85}$$

The possibile $CPT-$invariant $N = 1$ multiplet are summarized in the table below

$(\lambda,\lambda')\diagdown(j)$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$(0,-1/2)$ mult. chirale	·	·	·	1	2	1	·	·	·
$(1/2,-1)$ mult. vettoriale	·	·	1	1	·	1	1	·	·
$(1,-3/2)$	·	1	1	·	·	·	1	1	·
$(3/2,-2)$ mult. supergravità	1	1	·	·	·	·	·	1	1

Next we consider the case of the $N=2$ supermultiplets. We choose a Fock vacuum $|\lambda,\Omega\rangle$ with helicity λ. Under the $\mathcal{R}-$symmetry group $U(2)$, it possesses an $\mathcal{R}-$charge n with respect to the $U(1)$ and it transforms in the trivial representation of the $SU(2)$. Then the generic supermultiplet is

$$\underset{(1,n)}{|\lambda,\Omega\rangle} \qquad \underset{(2,n+1)}{(a^I)^\dagger|\lambda,\Omega\rangle} \qquad \underset{(1,n+2)}{\frac{1}{2}\epsilon_{IJ}(a^I)^\dagger(a^J)^\dagger|\lambda,\Omega\rangle}. \tag{3.86}$$

In the notation $(\cdot,\cdot)$, the first entry denotes the representation of the $SU(2)$ of $\mathfrak{R}-$symmetry, while the second one is the $U(1)$ $\mathfrak{R}-$symmetry. We have two singlets of $SU(2)$, one of helicity λ and one of helicity $\lambda+1$, and a doublet of $SU(2)$ of helicity $\lambda+1/2$.

Let us illustrate some very important examples:

(A): For $\lambda=-1$ we have a singlet of helicity -1, a doublet of spinors of helicity $-1/2$ and finally a singlet of helicity zero. This multiplet is not CPT-conjugate. To have a multiplet which is closed under CPT, we shall add the multiplet generated by a vacuum of helicity 0. It contains two singlets with $\lambda=0$ and $\lambda=1$ respectively and a doublet with spin 1/2. Summarizing, we have the so-called N=2 VECTOR MULTIPLET:

- One massless vector
- Two massless spinors forming a doublet of SU(2)
- Two massless scalars, which are singlets of SU(2)

Sometimes it is convenient to break this multiplet into multiplets of the $N=1$ supersymmetry: 1 (N=2 vector multiplet) = 1 (N=1 vector multiplet) $\oplus$ 1 (N=1 chiral multiplet).

(B): For $\lambda=-1/2$, the supermultiplet contains a state of helicity $-1/2$, an SU(2) doublet of helicity 0 and a second singlet of helicity 1/2. Such irreducible representation might appear CTP-conjugate, but this is not the case. In fact, the two particles of spin 0 have to be represented in terms of two real fields if they are CTP self-conjugate. However two real fields cannot be an $SU(2)$ doublet. Again we can overcome this difficulty by adding a second multiplet of the same type. The total content of the supermultiplet is the

- Two spinors
- Two complex scalar forming an $SU(2)$ doublet

This supermultiplet is known as HYPERMULTIPLET. In the language of $N = 1$ supersymmetry the hypermultiplet corresponds to $2 \times$ (N=1 chiral multiplet) .
(C): Finally we shall consider the case of a supersymmetry $N = 4$. We shall choose a vacuum of helicity -1, the only vacuum yielding a consistent field theory in the absence of gravity. Moreover it will transform in the trivial representation of $\mathcal{R}-$symmetry $SU(4)$. Then the states of the multiplets are

$$\begin{array}{ccc} \underset{(-1,1)}{|-1,\Omega\rangle} & \underset{(-1/2,\bar{4})}{(a^I)^\dagger|-1,\Omega\rangle} & \underset{(0,6)}{(a^I)^\dagger(a^J)^\dagger|-1,\Omega\rangle} \\ \underset{(1/2,4)}{\frac{1}{3!}\epsilon_{IJKL}(a^I)^\dagger(a^J)^\dagger(a^K)^\dagger|-1,\Omega\rangle} & & \underset{(1,1)}{\frac{1}{4!}\epsilon_{IJKL}(a^I)^\dagger(a^J)^\dagger(a^K)^\dagger(a^L)^\dagger|-1,\Omega\rangle} \end{array} \tag{3.87}$$

In the notation $(\cdot,\cdot)$ the first entry is the helicity of the state, while the second one denotes the relevant representation of $SU(4)$. This multiplet is CTP-selfconjugate and is called N=4 VECTOR MULTIPLET :

- A massless vector which is singlet of $SU(4)$
- Four massless spinors transforming in the fundamental of $SU(4)$
- 6 real scalars transforming in the **6** of $SU(4)$: Recall in fact that $\mathbf{4}\wedge\mathbf{4} \sim \mathbf{6}$. The **6** corresponds to an antisymmetric two-tensor Φ_{IJ} of $SU(4)$. This representation is real since for an antisymmetric tensor Φ_{IJ} we can define the following $SU(4)$ invariant reality conditions

  $$\Phi_{IJ} = \frac{1}{2}\epsilon_{IJKL}(\Phi^\dagger)_{KL} \qquad \Phi_{IJ} = -\frac{1}{2}\epsilon_{IJKL}(\Phi^\dagger)_{KL}.$$

  For example we could use the first one to define our six scalars.

This multiplet in terms of the $N = 1$ or $N = 2$ multiplets decomposes as follows:

$$\begin{array}{lll} 1 \ \ \text{(N=1 vector multiplet)} & \oplus & 3 \ \ \text{(N=1 chiral multiplet)} \\ 1 \ \ \text{(N=2 vector multiplet)} & \oplus & 1 \ \ \text{(N=2 hypermultiplet)} \end{array}$$

The number of supersymmetries present in a consistent field theory cannot be arbitrary. In fact consistent field theories cannot describe particles with helicity strictly greater than 2. This requirement translates into the following constraint

$$\lambda + N/2 \leq 2 \quad \lambda \geq -2 \qquad \Rightarrow N \leq 8, \tag{3.88}$$

namely the maximal number of supersymmetries is 8. If we neglect the gravitational interaction, the maximal helicity allowed in a field theory is 1 and thus we have the more restrictive limit

$$\lambda + N/2 \leq 1 \quad \lambda \geq -1 \qquad \Rightarrow N \leq 4. \tag{3.89}$$

For completeness, before considering the massive multiplets, we give a table of the most common supergravity multiplets

N	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	1	1	·	·	·	·	·	1	1
2	1	2	1	·	·	·	1	2	1
4	1	4	6	4	2	4	6	4	1
8	1	8	28	56	70	56	28	8	1

3.2.2 Massive representations

For massive particles the natural choice for the reference momentum is $p_m = (-M, 0, 0, 0)$.

Lorentz. The Lorentz transformations preserving this reference momentum are those generated by $J_1 = M_{23},\ J_2 = M_{31}\ ,\ J_3 = M_{12}$ and they close the $SU(2)$ algebra of spatial rotations

$$[J_i, J_k] = i\epsilon_{ikl} J_l. \tag{3.90}$$

Supersimmetry charges. Since the super-charges commute with the momentum, they all leave the reference momentum unaltered. The fermionic algebra in the rest frame takes the simplified form

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2\delta^{IJ} \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} = 2M\delta^{IJ}\delta_{\alpha\dot\alpha}, \quad \{Q^I_\alpha, Q^J_\beta\} = \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = 0. \tag{3.91}$$

Internal symmetries Again all the generators B_l do leave the momentum unaffected.

The Hilbert space carrying the representation of the massive supermultiplet can be decomposed into the direct sum of representations of the group $SU(2)$ of the spatial rotation. Namely, we shall write

$$\mathcal{H} = \bigoplus_s n_s \mathcal{H}_s \tag{3.92}$$

Each subspace $\mathcal{H}_s$ carries a representation of spin s and n_s is the number of times that this representation appears in the above decomposition. For each $\mathcal{H}_s$, we choose a basis such that

$$|s, s_z, \{i\}\rangle \quad \text{con} \quad J^2|s, s_z, \{i\}\rangle = s(s+1)|s, s_z, \{i\}\rangle \quad J_3|s, s_z, \{i\}\rangle = s_z|s, s_z, \{i\}\rangle, \tag{3.93}$$

where the additional label $\{i\}$ denotes the other possible properties of the state. We are ready now to investigate the action of the supersymmetry generators on this basis. To begin with, the supersymmetry charges transform as follows under action of this $SU(2)$

$$[Q^I_\alpha, J_i] = \frac{1}{2}(\sigma_i Q^I)_\alpha \qquad [\bar{Q}^I_{\dot{\alpha}}, J_i] = -\frac{1}{2}(\bar{Q}^I \sigma_i)_{\dot{\alpha}}. \tag{3.94}$$

In other words the charge Q transforms in the fundamental of $SU(2)$ $\mathbf{2}$, while $\bar{Q}$ in the anti-fundamental $\bar{\mathbf{2}}$. For $SU(2)$ these two representations are equivalent and thus we shall drop the distinction between dotted and undotted indices. If we redefine the charges as follows

$$a^I_\alpha = \frac{1}{\sqrt{2M}} Q^I_\alpha, \quad \text{e} \quad (a^I_\beta)^\dagger = \frac{1}{\sqrt{2M}} \bar{Q}^I_{\dot{\beta}}, \tag{3.95}$$

they will satisfy the algebra

$$\{a^I_\alpha, (a^J_\beta)^\dagger\} = \delta^{IJ}\delta_{\alpha\beta} \quad \{a^I_\alpha, a^J_\beta\} = \{(a^I_\alpha)^\dagger, (a^J_\beta)^\dagger\} = 0. \tag{3.96}$$

Again, the supersymmetric charges will close the algebra of fermion creation and annihilation operators. Eq. (3.96) has an obvious automorphism: it is invariant under the $U(N)$ transformations $a^I \mapsto U^I{}_J a^I$ and $a^{I\dagger} \mapsto U^{*I}{}_J a^{I\dagger}$. Since this automorphism does not act on the Greek indices, it will commute with the spin and thus all the states of a given spin will realize a representation of the group $U(N)$. In other words $U(N)$ must be a part of the $\mathcal{R}-$symmetry group.

Any representation of this fermionic algebra can be constructed starting from a set of Fock vacua defined by

$$a^I_\alpha|\Omega, i\rangle = 0 \quad \forall \;\; I = 1, \dots, N \quad \alpha = 1, 2. \tag{3.97}$$

This set of vacua $|\Omega, i\rangle$ must carry a representation of the spatial rotation. In fact

$$a^I_\alpha J_k|\Omega, i\rangle = [a^I_\alpha, J_k]|\Omega, i\rangle = \frac{1}{2}(\sigma_k a^I)_\alpha|\Omega, i\rangle = 0. \tag{3.98}$$

We choose the subspace of vacua $|\Omega, i\rangle$ to support an irreducibile representation of spin s and consequently we shall use the notation

$$|s, s_z, \Omega\rangle \quad \text{con} \quad s_z = -s, \dots, s. \tag{3.99}$$

In general the space of Fock vacua can also carry a representation R of the $\mathcal{R}-$symmetry group. In this case we shall use the following notation for the vacua

$$|s, s_z, R, \Omega\rangle \tag{3.100}$$

Then the representation of the supersymmetry algebra is generated by all the states of the form

$$|(I_1, \alpha_1); \cdots; (I_n, \alpha_n)\rangle = (a^{I_1}_{\alpha_1})^\dagger \cdots (a^{I_n}_{\alpha_n})^\dagger |s, s_z, R, \Omega\rangle, \tag{3.101}$$

where n run from 1 to N, the number of supercharges. The state, by construction, is completely antisymmetric in the indices (I_r, α_r). In the following, we shall consider the case where the vacuum has spin 0 and is invariant under $\mathcal{R}-$symmetry.

The classification of massive multiplets is more involved than the massless case. A way to determine their properties and to classify them systematically is to use the $R-$symmetry group. The largest automorphism of the algebra (3.96) is not $U(N)$. Let us redefine the basis of the algebra as follows

$$\Gamma^\ell = (a_1^\ell + (a_1^\ell)^\dagger) \quad \Gamma^{\ell+N} = (a_2^\ell + (a_2^\ell)^\dagger) \quad \Gamma^{\ell+2N} = i(a_1^\ell - (a_1^\ell)^\dagger) \quad \Gamma^{\ell+3N} = i(a_2^\ell - (a_2^\ell)^\dagger). \tag{3.102}$$

These hermitian operator will close a Clifford algebra of dimension $4N$

$$\{\Gamma^M, \Gamma^N\} = 2\delta^{MN} \qquad \text{con} \qquad M, N = 1, \ldots, 4N. \tag{3.103}$$

Therefore the largest automorphism is the $SO(4N)$, generated by $A^{RS} = \frac{i}{4}[\Gamma^R, \Gamma^S]$. This automorphism is particularly useful for determining the dimension of the multiplet. In fact the Clifford algebra possess only one irreducible representation of dimension 2^{2N}, which corresponds to the spinor representation of $SO(4N)$. The states of this spinor representation can be decomposed into two sets of different chirality. These two sets are the eigenspaces of the projectors $1/2(\mathbb{1} \pm \Gamma^{4N+1})$, where $\Gamma^{4N+1} = \prod_{L=1}^{4N} \Gamma^L$. If we choose a vacuum with fixed chirality ($\Gamma^{4N+1}|\Omega\rangle = (-1)^s|\Omega\rangle$) the eigenvalue of Γ^{4N+1} will simply distinguish between states with an even and an odd number of fermionic creation operators, namely between bosonic and fermionic states[5]. Since both the chiral subspaces have dimension 2^{2N-1}, all the massive supermultiplet will have the same number of boson and fermions.

If the vacuum has spin s and it carries a representation R of the $\mathcal{R}-$symmetry group the dimension of the multiplet is

$$2^{2N} \times (2s+1) \times \dim(R). \tag{3.104}$$

The automorphism is not suitable for an actual classification of the states belonging to the multiplet. Since it does not commute with the spatial rotation generated by

$$J_1 = \frac{1}{2}\sum_{\ell=1}^{N} (a_1^\ell)^\dagger a_2^\ell + (a_2^\ell)^\dagger a_1^\ell, \quad J_2 = -\frac{i}{2}\sum_{\ell=1}^{N} (a_1^\ell)^\dagger a_2^\ell - (a_2^\ell)^\dagger a_1^\ell, \quad J_3 = \frac{1}{2}\sum_{\ell=1}^{N} (a_1^\ell)^\dagger a_1^\ell - (a_2^\ell)^\dagger a_2^\ell, \tag{3.105}$$

its multiplets contain particles of different spin and thus it cannot be promoted to be an internal symmmetry of a quantum field theory. For our goal, it will be more effective to consider the

[5] Observed that

$$\Gamma^{4N+1}|(I_1, \alpha_1); \cdots; (I_n, \alpha_n)\rangle = (-1)^{s+n}|(I_1, \alpha_1); \cdots; (I_n, \alpha_n)\rangle.$$

largest automorphism which commutes with the spin. This automorphism becomes manifest when we use the following basis for the fermionic algebra

$$q^{\ell}_{\alpha} = a^{\ell}_{\alpha} \qquad \text{e} \qquad q^{\ell+N}_{\alpha} = i(\sigma_2 (a^{\ell})^{\dagger})_{\alpha} = \epsilon_{\alpha\beta}(a^{\ell}_{\beta})^{\dagger} \qquad \forall\ \ell = 1, \dots, N; \tag{3.106}$$

This operators close the algebra

$$\{q^L_{\alpha}, q^M_{\beta}\} = C^{LM}\epsilon_{\alpha\beta}, \tag{3.107}$$

and satisfy the reality condition $(q^L)^{\dagger} = C^{LM} i\sigma_2 q^M$, where the matrix C is given by

$$(C^{LM}) = \begin{pmatrix} O & -\mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & O, \end{pmatrix}. \tag{3.108}$$

A redefinition $q^L \mapsto U^L{}_M q^M$ preserves the form of the algebra (3.107) if $UCU^T = C$ and the reality condition if $U^* = -CUC$. These two conditions imply

$$UU^{\dagger} = U(U^*)^T = -UCU^T C = -C^2 = \mathbb{1} \quad \Rightarrow \quad U \text{ is a } 2N \times 2N \text{ unitary matrix }. \tag{3.109}$$

The $2N \times 2N$ unitary matrices preserving the quadratic form C form the group $USp(2N)$, namely the unitary symplectic group. This is the relevant $R-$symmetry. The fact that this automorphism commutes with the spatial rotations becomes even more manifest if we write the generators R^{LM} of this $\mathcal{R}-$symmetry in terms of the fermionic operator q_L. We find $R^{LM} = \frac{1}{2}\epsilon^{\alpha\beta}[q^L_{\alpha}, q^M_{\beta}]$, *i.e.* they are built out of singlets of $SU(2)$.
Therefore in a massive supermultiplet all the state of the same spin can be arranged in a representation of the group $USp(2N)$. Now, we shall illustrate this in some examples with $N = 1$ and $N = 2$ supersymmetries.

$\mathbf{N = 1}$ Case: Consider a Fock vacuum of spin zero and invariant under $R-$simmetry. The states of this multiplet are

$$|\Omega\rangle \qquad a^{\dagger}_1 a^{\dagger}_2|\Omega\rangle \qquad\qquad a^{\dagger}_{\beta}|\Omega\rangle. \tag{3.110}$$

The first two states have spin 0 while the third has spin 1/2. In order to see how these states can be arranged in multiplets of $USp(2) \sim SU(2)$, let us write explicitly the generator of the $\mathcal{R}-$symmetry

$$\begin{gathered} J_+ = \frac{1}{2}\epsilon^{\alpha\beta}[q^2_{\alpha}, q^2_{\beta}] = 2(a_1)^{\dagger}(a_2)^{\dagger} \qquad J_- = \frac{1}{2}\epsilon^{\alpha\beta}[q^1_{\alpha}, q^1_{\beta}] = 2a_1 a_2 \\ J_3 = \frac{1}{2}\epsilon^{\alpha\beta}[q^1_{\alpha}, q^2_{\beta}] = (a_1)^{\dagger}a_1 + (a_2)^{\dagger}a_2 - \mathbb{1}. \end{gathered} \tag{3.111}$$

It is immediate to see that $a^{\dagger}_{\beta}|\Omega\rangle$ is annihilated by all these generators and thus it is a singlet. Moreover $J_+|\Omega\rangle = 2a^{\dagger}_1 a^{\dagger}_2|\Omega\rangle$ and $J_- a^{\dagger}_1 a^{\dagger}_2|\Omega\rangle = -2|\Omega\rangle$, then the two spin 0 states are a doublet of $USp(2)$. The subgroup associated with the $U_R(1)$ is generated by J_3.
We can summarize the above result in the following table

	2 states of spin 0		2 states of spin 1/2
	$\vert\Omega\rangle$	$(a_1)^\dagger(a_2)^\dagger\vert\Omega\rangle$	$\{a_1^\dagger\vert\Omega\rangle, a_2^\dagger\vert\Omega\rangle\}$
$U_R(1)$	-1	1	0
$USp(2)$	doublet		singlet

N = 2 Case: Next we consider the case of the supersymmetry $N = 2$. We shall consider again a vacuum of spin zero and invariant under $\mathcal{R}-$symmetry. Starting from this scalar vacuum, we can build the following states by acting with the supersymmetry charges:
(a) 4 states with one supercharges: **spin 1/2** : $\psi^I_\alpha = (a^I_\alpha)^\dagger|\Omega\rangle$
(b) 6 states with two supersymmetry charges: $(a^I_\alpha)^\dagger(a^J_\beta)^\dagger|\Omega\rangle$. This states decompose in

$$\begin{aligned} &\textbf{spin 0}: \quad T^{IJ} = \frac{1}{2}\epsilon^{\alpha\beta}(a^I_\alpha)^\dagger(a^J_\beta)^\dagger|\Omega\rangle \quad T^{IJ} = T^{JI}, \\ &\textbf{spin 1}: \quad V_{\alpha\beta} = \frac{1}{2}\epsilon_{IJ}(a^I_\alpha)^\dagger(a^J_\beta)^\dagger|\Omega\rangle \quad V_{\alpha\beta} = V_{\beta\alpha}. \end{aligned} \tag{3.112}$$

(c) 4 states with three supercharges: **spin 1/2** : $\chi^K_\rho = \frac{1}{4}\epsilon_{IJ}\epsilon^{\alpha\beta}(a^K_\alpha)^\dagger(a^I_\beta)^\dagger(a^J_\rho)^\dagger|\Omega\rangle$
(d) 1 state with four supercharges: **spin 0** : $\phi = \epsilon^{\alpha\beta}\epsilon^{\rho\sigma}\epsilon_{IK}\epsilon_{JL}(a^I_\alpha)^\dagger(a^J_\beta)^\dagger(a^K_\rho)^\dagger(a^L_\sigma)^\dagger|\Omega\rangle$.

The $\mathcal{R}-$symmetry group $USp(4)$ acting on this states is generated by

$$H^{IJ} = \frac{1}{2}([(a^\dagger)^I_1, a^J_1] + [(a^\dagger)^I_2, a^J_2]) \quad F^{IJ} = \frac{1}{2}\epsilon^{\alpha\beta}[a^I_\alpha, a^J_\beta] \qquad G^{IJ} = \frac{1}{2}\epsilon^{\alpha\beta}[(a^I_\alpha)^\dagger, (a^J_\beta)^\dagger]. \tag{3.113}$$

The operators H^{IJ} generates the subalgebra $U(4)$ and in particular the subgroup $U_R(1)$ can be associated to

$$S = \sum_{i=1} 2H^{II} = ((a^1_1)^\dagger a^1_1 + (a^1_2)^\dagger a^1_2 + (a^2_1)^\dagger a^2_1 + (a^2_2)^\dagger a^2_2 - 2). \tag{3.114}$$

With these choice the $R-$charges of the different states under the $U_R(1)$ are

$$\textbf{spin 0}: \quad \overset{-2}{\eta} \quad \overset{0}{T^{IJ}} \quad \overset{2}{\phi} \quad \Big| \quad \textbf{spin 1/2}: \quad \overset{-1}{\psi^I_\alpha} \quad \overset{1}{\chi^I_\alpha} \quad \Big| \quad \textbf{spin 1}: \quad \overset{0}{V_{\alpha\beta}}.$$

The property of transformation under the $SU_R(2)$ are instead given by

$$\textbf{spin 0}: \quad \overset{sing.}{\eta} \quad \overset{tripl.}{T^{IJ}} \quad \overset{sing.}{\phi} \quad \Big| \quad \textbf{spin 1/2}: \quad \overset{doubl.}{\psi^I_\alpha} \quad \overset{doubl.}{\chi^I_\alpha} \quad \Big| \quad \textbf{spin 1}: \quad \overset{sing.}{V_{\alpha\beta}}.$$

Exploiting the explicit form of the generators of $USp(4)$ is not difficult to show that the state with the same spin carry an irreducible representation of $\mathcal{R}-$symmetry group: *spin 0* **5**, *spin 1/2* **4**, *spin 1* **1**.
We can summarize the results in the following table

	5 stati di spin 0			8 stati di spin 1/2	3 stati di spin 1
	η	T^{IJ}	ϕ	$\{\chi^I_\alpha, \psi^I_\alpha\}$	$V_{\alpha\beta}$
$U_R(1)$	-2	0	2	$(1,-1)$	0
$USp(2)$	**5**			**4**	**1**

The case with N supersymmetry charges is more involved and we shall not discuss it here in detail. However one can show that the following result holds:
If our Clifford vacuum is a scalar under the spin group and the $\mathcal{R}$-symmetry group, then the irreducible massive representation of supersymmetry has the following content

$$2^{2N} = [N/2,[0]] \oplus [(N-1)/2,[1]] \oplus [(N-2)/2,[2]] \oplus \dots [(N-k)/2,[k]] \dots \oplus [0,[N]], \quad (3.115)$$

where the first entry in the bracket denotes the spin and the last entry, say $[k]$ denotes which $k^{th}-$fold antisymmetric traceless irreducible representation of USp(2N) this spin belongs to.

3.3 Representation with central charges

In this section we shall briefly analyze the question of how to construct the representation of the supersymmetry algebra in the presence of central charges. We fix as reference momentum $p_m = (-M,0,0,0)$ ($P^2 = M^2$). The only modification with respect to the case considered in the previous section occurs in the fermionic algebra, which now is given by

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \quad (3.116)$$

$$\{Q^M_\alpha, Q^N_\beta\} = \epsilon_{\alpha\beta}Z^{MN} \qquad \{\bar{Q}^M_{\dot{\alpha}}, \bar{Q}^N_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}}(Z^*)^{MN} \qquad \{Z^{LM}, Q^J_\alpha\} = \{Z^{LM}, \bar{Q}^J_{\dot{\alpha}}\} = 0. \quad (3.117)$$

Here Z^{MN} is a complex matrix, antisymmetric in the indices (M,N).
Given any unitary matrix U, the transformations $Q^I \mapsto U^I{}_J Q^J$, $\bar{Q}^I \mapsto U^{*I}{}_J \bar{Q}^J$, $Z'^{MN} = U^M_R U^N_S Z^{RS}$ and $Z'^{*MN} = U^{*M}_R U^{*N}_S Z^{*RS}$ leave the form of the fermionic algebra unaltered and they can be used to simplify the form of the matrix Z. The lemma 4 in appendix D states that the matrix U can be chosen so that

$$Z' = \epsilon \otimes D, \quad (3.118)$$

for an even number of supersymmetries or

$$Z' = \begin{pmatrix} \epsilon \otimes D & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.119)$$

for an odd number of supersymmetries. Here $D = \mathrm{diag}(z_1, \dots, z_k, \dots)$ and ϵ is the 2×2 antisymmetric matrix $i\sigma_2$.
In the following we shall focus on the case of even N; the case of odd N can be investigate in a similar manner.

The form of the matrix Z suggests to arrange the indices (M, N) in a different way. We replace M and N with two pairs of indices: $M \mapsto (a, m)$ and $N \mapsto (b, n)$. The lowercase roman indices (a, b) run from 1 to 2, while the indices (m, n) run from 1 to $N/2$ and distinguish the different blocks of the matrix Z'. In this basis the anticommutators of the charges have the following form

$$\begin{aligned} \{Q^{am}_\alpha, (Q^{bn}_\alpha)^\dagger\} &= 2\delta^{mn}\delta_{ab}\delta^\alpha_\beta M \qquad & \{Q^{am}_\alpha, Q^{bn}_\beta\} &= \epsilon^{ab}\delta^{mn}Z^n\epsilon_{\alpha\beta} \\ \{(Q^{am}_\alpha)^\dagger, (Q^{bn}_\beta)^\dagger\} &= \epsilon^{ab}\delta^{mn}\epsilon_{\alpha\beta}Z^n \qquad & \{Z^n, Q\} &= \{Z^n, Q^\dagger\} = 0. \end{aligned} \tag{3.120}$$

Let us now define the operators

$$a^m_\alpha = \frac{1}{\sqrt{2}}[Q^{1m}_\alpha + \epsilon_{\alpha\beta}(Q^{2m}_\beta)^\dagger] \qquad b^m_\alpha = \frac{1}{\sqrt{2}}[Q^{1m}_\alpha - \epsilon_{\alpha\beta}(Q^{2m}_\beta)^\dagger] \tag{3.121}$$

These operators close the algebra

$$\begin{gathered} \{a^m_\alpha, a^{n\dagger}_\beta\} = \delta^{mn}\delta_{\alpha\beta}(2M + Z_n) \quad \{b^m_\alpha, b^{n\dagger}_\beta\} = \delta^{mn}\delta_{\alpha\beta}(2M - Z_n) \\ \{a, b\} = \{a^\dagger, b\} = \{a, b^\dagger\} = \{a, a\} = \{b, b\} = \{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\} = 0 \end{gathered} \tag{3.122}$$

Since the $\{a, a^\dagger\}$ and $\{b, b^\dagger\}$ are positive objects, consistency requires that

$$Z_n \geq -2M \qquad \text{e} \qquad Z_n \leq 2M \quad \forall\, n \quad \Rightarrow \qquad |Z_n| \leq 2M \quad \forall\, n \tag{3.123}$$

When this bound strictly holds for all the Z_n, this algebra is isomorphic to the one considered in the massive case up to a total rescaling. Therefore all the results of the massive case apply. A new phenomenon appears when the bound is saturated. Let us assume, for example, that there is a value k such that $Z_k = 2M$. Then the anticommutator $\{b^k, b^{k\dagger}\}$, which is a positive definite quantity, vanishes identically and the operators b^k and $b^{k\dagger}$ must be represented as the null operator. In this way, we have effectively lost one of the supercharges. If the bound is saturated by q central charges, we will loose q supercharges and only $N - q$ supercharge will be realized non trivially. In particular this means that the multiplets do not have dimension 2^{2N} but they are shorter. In fact their dimensions will be $2^{2(N-q)}$. These short multiplets are known as $BPS-$multiplets.

add something on massive hypermultiplets

4 The Basics of superspace

In the following we shall address the problem of realizing the representations of the supersymmetry discussed in the previous section in terms of local fields. There are many approaches to this question, that however can be more conveniently handled using the so-called *superspace*. In these lectures we shall describe the construction of the $N = 1$ superspace in some details, but we shall not consider the extensions to higher supersymmetries (a part from some remarks on the $N = 2$ superspace).

The notion of superspace was firstly introduced by Salam and Stradhee and the mathematical idea behind its construction is the concept of *coset*. Let us recall what a coset is. Consider a group G and a subgroup H of G, then an equivalence relation can be defined between the elements of G

$$g_1 \sim g_2 \;\text{ if there is } h \in H \text{ such that } g_1 = g_2 h. \tag{4.124}$$

Such relation separates G into equivalence classes. The set of all equivalence classes is called (*left*) *coset* and it is denoted with G/H. [Analogously one can define the right coset: $g_1 \sim g_2$ if there is $h \in H$ such that $g_1 = h g_2$.] Since an element of the coset is an equivalence class, we shall denote it by choosing one of its elements, *e.g.* g, and we shall write $[g]$. On the elements of the coset, it is naturally defined a right action of the group G: for any $k \in G$

$$k[g] = [kg]. \tag{4.125}$$

It is trivial to check that this action does not depend on the representative g.

If G and H are topological groups the coset G/H is called an homogeneous space. Summarizing, given a group G we have constructed a space where an action of this group is naturally defined. Let us illustrate this abstract procedure with a pedagogical example: the construction of the Minkowski space starting from the Poincarè group.

Consider the quotient of the Poincarè group with respect to the Lorentz group. Since any element of the Poincarè group can be decomposed uniquely as the product of a translation and a Lorentz transformation,

$$T(\omega, x) = \exp(ix^m P_m) \exp\left(\frac{i}{2}\omega^{mn} M_{mn}\right), \tag{4.126}$$

the coset is the set of equivalence classes $[\exp(ix^m P_m)]$. Namely each *point* in the coset is defined by four real coordinates x^m. Now, let us compute the action of the Poincarè transformations on these coordinates

Translations:

$$\exp(ia^m P_m) \exp(ix^m P_m) = \exp(i(x^m + a^m)P_m) \qquad x^m \mapsto x'^m = x^m + a^m. \tag{4.127}$$

Lorentz Transformations: $\Lambda = \exp\left(\frac{i}{2}\omega^{mn}M_{mn}\right)$

$$
\begin{aligned}
&\exp\left(\frac{i}{2}\omega^{ab}M_{ab}\right)\exp(ix^m P_m) = \\
&= \exp\left(\frac{i}{2}\omega^{ab}M_{ab}\right)\exp(ix^m P_m)\exp\left(-\frac{i}{2}\omega^{ab}M_{ab}\right)\exp\left(\frac{i}{2}\omega^{ab}M_{ab}\right) \sim \\
&= \exp\left(ix^m \exp\left(\frac{i}{2}\omega^{ab}M_{ab}\right) P_m \exp\left(-\frac{i}{2}\omega^{ab}M_{ab}\right)\right) = \\
&= \exp\left(ix^m \Lambda^n{}_m P_n\right) \qquad x^m \mapsto x'^m = \Lambda^m{}_n x^n,
\end{aligned}
\tag{4.128}
$$

where we used that $\exp\left(\frac{i}{2}\omega^{ab}M_{ab}\right) P_m \exp\left(-\frac{i}{2}\omega^{ab}M_{ab}\right) = \Lambda^m{}_n P_m$. Therefore this procedure has produced a four-dimensional space isomorphic to $\mathbb{R}^4$ where the Poincarè transformations act in the usual way. We can safely say that this is the Minkowski space.

To construct the superspace we shall proceed similarly, defining the the $N = 1$ superspace to be the following coset

$$
\text{Superspace}_{\text{N=1}} = \frac{N = 1 \text{ Poincaré supergroup}}{\text{LorentzGroup}}. \tag{4.129}
$$

For us the $N = 1$ Poincarè supergroup will be simply defined as the exponential of the superalgebra constructed in the previous lectures. To exponentiate the fermionic sector, we have to introduce a set of grassmannian coordinates, which play the role of the infinitesimal parameters of the transformation. In $N = 1$ we have a supercharge Q_α and its hermitian conjugate $\bar{Q}_{\dot{\alpha}}$, thus we shall introduce the fermionic coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$. This will allows us to form the hermitian bosonic combination

$$
\theta Q + \bar{\theta}\bar{Q} = \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}, \tag{4.130}
$$

which we can exponentiate to yield a supersymmetry transformation. Then, any element of the supergroup can be parametrized as follows

$$
\exp\left(-ix^\mu P_\mu + i(\theta Q + \bar{\theta}\bar{Q})\right)\exp\left(\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right). \tag{4.131}
$$

Therefore the elements of the coset (4.129) are given by

$$
\exp\left(-ix^\mu P_\mu + i(\theta Q + \bar{\theta}\bar{Q})\right). \tag{4.132}
$$

Namely, the $N = 1$ superspace is defined by four bosonic coordinates which span an $\mathbb{R}^4$, and by a pair of fermionic coordinates $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ which are related by complex conjugation.

The action of a supersymmetry transformation on these coordinates is easily computed as follows

$$\begin{aligned}
&\exp\left(i(\epsilon Q + \bar{\epsilon}\bar{Q})\right)\exp\left(-ix^m P_m + i(\theta Q + \bar{\theta}\bar{Q})\right) = \\
&= \exp\left(-ix^m P_m + i((\theta+\epsilon)Q + (\bar{\theta}+\bar{\epsilon})\bar{Q}) - \frac{1}{2}[(\epsilon Q + \bar{\epsilon}\bar{Q}), (\theta Q + \bar{\theta}\bar{Q})]\right) = \\
&= \exp\left(-ix^m P_m + i((\theta+\epsilon)Q + (\bar{\theta}+\bar{\epsilon})\bar{Q}) - \frac{1}{2}\epsilon^\alpha\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}\bar{\theta}^{\dot{\alpha}} + \frac{1}{2}\theta^\alpha\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}\bar{\epsilon}^{\dot{\alpha}}\right) = \\
&= \exp\left(-i(x^m + i\theta\sigma^m\bar{\epsilon} - i\epsilon\sigma^m\bar{\theta})P_m + i((\theta+\epsilon)Q + (\bar{\theta}+\bar{\epsilon})\bar{Q})\right).
\end{aligned} \tag{4.133}$$

We have thus obtained

$$x^m \mapsto x'^m = x^m + i\theta\sigma^m\bar{\epsilon} - i\epsilon\sigma^m\bar{\theta} \tag{4.134a}$$

$$\theta \mapsto \theta' = \theta + \epsilon \qquad \bar{\theta} \mapsto \bar{\theta}' = \bar{\theta} + \bar{\epsilon} \tag{4.134b}$$

The action of translations and Lorentz transformations can be computed in a similar manner and one obtain

Translations

$$x^m \mapsto x'^m = x^m - a^m \quad \theta \mapsto \theta' = \theta \qquad \bar{\theta} \mapsto \bar{\theta}' = \bar{\theta} \tag{4.135}$$

Lorentz Transformations

$$x^m \mapsto x'^m = \Lambda^m{}_n x^n \qquad \theta^\alpha \mapsto \theta'^\alpha = \theta^\beta (\Lambda^{-1})_\beta{}^\alpha \qquad \bar{\theta}^{\dot{\alpha}} \mapsto \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\beta}} (\Lambda^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}} \tag{4.136}$$

Notice that all the transformations are *in agreement* with the indices carried by the coordinates. Together with the usual bosonic derivatives ∂_m, we have two graded (right-)derivatives acting on the Grassmann coordinates as follows

$$\frac{\partial}{\partial\theta^\beta}\theta^\alpha = \delta^\alpha_\beta \qquad \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\bar{\theta}^{\dot{\alpha}} = \delta^{\dot{\alpha}}_{\dot{\beta}} \tag{4.137}$$

Here 'graded' means that these derivatives obey the anti-Leibnitz rule, *e.g.*

$$\frac{\partial}{\partial\theta^\beta}(\phi_1\phi_2) = \frac{\partial\phi_1}{\partial\theta^\beta}(\phi_2) + (-1)^{\mathrm{grad}(\phi_1)}\phi_1\frac{\partial\phi_2}{\partial\theta^\beta}. \tag{4.138}$$

4.1 Superfields

The standard field can be seen as function over the Minkowski space. We can now define the superfields in a similar way: they are functions over the supespace, *i.e.*

$$\Phi_\alpha(x, \theta, \bar{\theta}). \tag{4.139}$$

The index α is a Lorentz group index. In general a superfield can carry a representation of the Lorentz group. Since θ and $\bar{\theta}$ are anticommuting coordinates, the superfield will be a polynomial of finite degree in these variables. The coefficient of this polynomial are standard functions over the Minkowski space and they can be identified with the standard fields.
In the following, to avoid useless complications we shall drop the Minkowski index and consider a scalar superfield. This kind of superfield will be sufficient for our goals.
We define the action of supersymmetry transformations on a superfield as follows

$$e^{ia^m P_m + i\epsilon Q + i\bar{\epsilon}\bar{Q}} \Phi(x,\theta,\bar{\theta}) e^{-ia^m P_m - i\epsilon Q - i\bar{\epsilon}\bar{Q}} = \Phi(x - a + i\theta\sigma\bar{\epsilon} - i\epsilon\sigma\bar{\theta}, \theta+\epsilon, \bar{\theta}+\bar{\epsilon}). \tag{4.140}$$

This definition mimics the analogous definition of translations on standard fields: it just acts on the coordinates of the superfield.
At the infinitesimal level the supersymmetry transformations (4.140) are given by

$$\begin{aligned}
\delta_\epsilon \Phi &= -i[\Phi, \epsilon Q] = \Phi(x - i\epsilon\sigma\bar{\theta}, \theta+\epsilon, \bar{\theta}) - \Phi(x,\theta,\bar{\theta})\big|_{\text{lin. in } \epsilon} = \\
&= \epsilon^\alpha \frac{\partial}{\partial\theta^\alpha}\Phi(x,\theta,\bar{\theta}) - i\epsilon\sigma^m\bar{\theta}\partial_m\Phi(x,\theta,\bar{\theta}) = \epsilon^\alpha\left(\frac{\partial}{\partial\theta^\alpha} - i(\sigma^m\bar{\theta})_\alpha\partial_m\right)\Phi(x,\theta,\bar{\theta}) \equiv \\
&\equiv \epsilon^\alpha Q_\alpha \Phi(x,\theta,\bar{\theta}) \qquad \Rightarrow \qquad Q_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} - i(\sigma^m\bar{\theta})_\alpha\partial_m
\end{aligned} \tag{4.141a}$$

$$\begin{aligned}
\delta_{\bar{\epsilon}} \Phi &= -i[\Phi, \bar{\epsilon}\bar{Q}] = \Phi(x + i\theta\sigma\bar{\epsilon}, \theta, \bar{\theta}+\bar{\epsilon}) - \Phi(x,\theta,\bar{\theta})\big|_{\text{lin. in } \bar{\epsilon}} = \\
&= \bar{\epsilon}_{\dot{\alpha}} \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}\Phi(x,\theta,\bar{\theta}) + i\theta\sigma^m\bar{\epsilon}\partial_m\Phi(x,\theta,\bar{\theta}) = \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^m)_{\dot{\alpha}}\partial_m\right)\bar{\epsilon}^{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) \equiv \\
&\equiv \bar{Q}\bar{\epsilon}\Phi(x,\theta,\bar{\theta}) \qquad \Rightarrow \qquad \bar{Q}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^m)_{\dot{\alpha}}\partial_m
\end{aligned} \tag{4.141b}$$

The action of translations is instead simply given by

$$\delta_a\Phi = -i[\Phi, a^m P_m] = \Phi(x-a,\theta,\bar{\theta}) - \Phi(x,\theta,\bar{\theta})\big|_{\text{lin. in } a} = -a^m\partial_m\Phi(x,\theta,\bar{\theta}) \tag{4.141c}$$

We can check the consistency of this approach by computing in two different ways the commutator $[\delta_{\bar{\epsilon}}, \delta_\epsilon]$: from the algebra

$$[\delta_{\bar{\epsilon}}, \delta_\epsilon]\Phi(x,\theta,\bar{\theta}) = -[\Phi, [\epsilon Q, \bar{\epsilon}\bar{Q}]] = -\epsilon^\alpha[\Phi, \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}]\bar{\epsilon}^{\dot{\alpha}} = -2\epsilon\sigma^m\bar{\epsilon}[\Phi, P_m] = -2i\epsilon\sigma^m\bar{\epsilon}\partial_m\Phi\ , \tag{4.142}$$

and from the actual representations (4.141a) and (4.141b)

$$[\delta_{\bar{\epsilon}}, \delta_\epsilon]\Phi(x,\theta,\bar{\theta}) = (\bar{\epsilon}\bar{Q}\epsilon Q - \epsilon Q\bar{\epsilon}\bar{Q})\Phi = -\epsilon^\alpha(\bar{Q}_{\dot{\alpha}}Q_\alpha + Q_\alpha\bar{Q}_{\dot{\alpha}})\bar{\epsilon}^{\dot{\alpha}}\Phi = -2i\epsilon\sigma^m\bar{\epsilon}\partial_m\Phi. \tag{4.143}$$

This check also shows that the differential operator representing the supercharges satisfies the anticommutator

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma^m_{\alpha\dot{\alpha}}\partial_m. \tag{4.144}$$

4.2 (Covariant) Derivative on the superspace

The derivative introduced at the end of the introduction does not commute with the supercharges. This means that these derivatives break the *covariance* under supersymmetry transformations. To develop a formalism which is manifestly invariant under supersymmetry transformations, we must define a new derivation D (and $\bar{D}$) such that

$$[\delta_\epsilon, D] = [\delta_{\bar{\epsilon}}, D] = 0. \tag{4.145}$$

A brute force computation shows that all the graded derivations with this property are obtained by taking linear combinations of

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^m\bar{\theta})_\alpha\partial_m \quad \text{and} \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i(\theta\sigma^m)^{\dot{\alpha}}\partial_m. \tag{4.146}$$

These derivations close the following algebra

$$\{Q_\alpha, D_\alpha\} = \{Q_\alpha, \bar{D}_{\dot{\alpha}}\} = \{\bar{Q}_{\dot{\alpha}}, D_\alpha\} = \{\bar{Q}_{\dot{\alpha}}, \bar{D}_{\dot{\alpha}}\} = 0. \tag{4.147}$$

and

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^m_{\alpha\dot{\alpha}}\partial_m \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0. \tag{4.148}$$

The origin of these two graded derivations can be understood at the level of group theory. In our construction of the superspace we have used the left coset and consequently we have defined the left action of the group. However we could have equivalently defined the superspace through the right coset and used the right action to realize the supersymmetry transformations. We would have obtained the same superspace with a different form of the supersymmetry charges given by $(D_\alpha, \bar{D}_{\dot{\alpha}})$. Since the left and right action commute by definition, we must conclude that $(Q_\alpha, \bar{Q}_{\dot{\alpha}})$ and $(D_\alpha, \bar{D}_{\dot{\alpha}})$ commute as well.

4.2.1 Integration

Given the Grassmann algebra generated by the N anticommuting variable θ^I, we can define the integral as follows

$$\int \prod_{I=1}^{N} d\theta^I f(\theta) = \frac{\partial}{\partial\theta^1}\cdots\frac{\partial}{\partial\theta^N} f(\theta). \tag{4.149}$$

In other words the result of the integral is proportional to the coefficient of the monomial of degree N in the expansion of the function $f(\theta)$.

Therefore we shall define the integral of a superfield over the entire superspace as follows

$$I = \int d^4x d^2\theta d^2\bar{\theta}\Phi(x,\theta,\bar{\theta}) = \int d^4x \frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\bar{\theta}^{\dot{1}}}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\Phi(x,\theta,\bar{\theta}). \tag{4.150}$$

The above definition can be rewritten in terms of the covariant derivatives D_α and $\bar{D}_{\dot{\alpha}}$. For example if we replace $\frac{\partial}{\partial\bar{\theta}^{\dot{1}}}$ with D_1, we find the following expression for I

$$I = \int d^4x D_1 \frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\bar{\theta}^{\dot{1}}}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\Phi(x,\theta,\bar{\theta}) - i\int d^4x\partial_m\left[(\sigma^m\theta)_1\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\bar{\theta}^{\dot{1}}}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\Phi(x,\theta,\bar{\theta})\right]. \tag{4.151}$$

The second term is a total divergence and it can be neglected if the fields vanish at infinity. Therefore

$$\int d^4x D_1 \frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\bar{\theta}^{\dot{1}}}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\Phi(x,\theta,\bar{\theta}) = \int d^4x \frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\bar{\theta}^{\dot{1}}}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\Phi(x,\theta,\bar{\theta}). \tag{4.152}$$

If we repeat this procedure for all the odd coordinates we find

$$I = \int d^4x D_1 D_2 \bar{D}_{\dot{1}}\bar{D}_{\dot{2}}\Phi(x,\theta,\bar{\theta}), \tag{4.153}$$

or, in manifestly Lorentz invariant form,

$$I = -\frac{1}{4}\int d^4x D^2\bar{D}^2\Phi(x,\theta,\bar{\theta}). \tag{4.154}$$

This definition in terms of the covariant derivative seems to be strongly dependent on the order of the derivatives. In fact, D_α and $\bar{D}_\alpha$ do not commute. This dependence is however harmless: different orderings disagree for terms in the Lagrangian which are total divergences. Namely, the action does not depend on the ordering.
The measure of integration above is also invariant under supersymmetry transformation. In fact

$$\begin{aligned}\delta_\epsilon I &= -\frac{1}{4}\int d^4x D^2\bar{D}^2\delta_\epsilon\Phi(x,\theta,\bar{\theta}) = -\frac{1}{4}\int d^4x D^2\bar{D}^2\epsilon Q\Phi(x,\theta,\bar{\theta}) = \\ &= -\frac{1}{4}\int d^4x\epsilon Q[D^2\bar{D}^2\Phi(x,\theta,\bar{\theta})] = -\frac{1}{4}\int d^4x(\epsilon D - 2i\epsilon\sigma^m\bar{\theta}\partial_m)[D^2\bar{D}^2\Phi(x,\theta,\bar{\theta})] = \\ &= \frac{i}{2}\int d^4x\partial_m\left[\epsilon\sigma^m\bar{\theta}D^2\bar{D}^2\Phi(x,\theta,\bar{\theta})\right].\end{aligned} \tag{4.155}$$

5 Scalar superfield

Consider the scalar superfield, namely a spin zero superfield. We can expand it as a polynomial in the Grassmann variables θ^α and $\bar{\theta}^{\dot{\alpha}}$. Taking into account that all the independent monomials built out of the θ and $\bar{\theta}$ are

$$\underset{1}{\mathbb{1}},\quad \underset{2}{\theta^\alpha},\quad \underset{2}{\bar{\theta}^{\dot{\alpha}}},\quad \underset{1}{\theta^2}(\equiv\theta^\alpha\theta_\alpha),\quad \underset{1}{\bar{\theta}^2}(\equiv\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}),\quad \underset{4}{\theta\sigma^m\bar{\theta}},\quad \underset{2}{\bar{\theta}^2\theta^\alpha},\quad \underset{2}{\theta^2\bar{\theta}^{\dot{\alpha}}},\quad \underset{1}{\bar{\theta}^2\theta^2}, \tag{5.156}$$

the expansion of the most general scalar superfield is given by

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = \phi(x) + \theta\chi(x) + \bar{\psi}(x)\bar{\theta} + F(x)\theta^2 + G(x)\bar{\theta}^2 + V_m(x)\theta\sigma^m\bar{\theta} + \\ + \bar{\tau}(x)\bar{\theta}\theta^2 + \bar{\theta}^2\theta\lambda(x) + \theta^2\bar{\theta}^2 D(x).\end{aligned} \tag{5.157}$$

It contains 8 complex bosonic fieds

$$\underbrace{\phi(x),\ F(x),\ G(x),\ D(x),}_{4\ \text{scalars}}\ \underbrace{V_m(x)}_{1\ \text{vector}} \tag{5.158}$$

and 8 c fermionic fields

$$\underbrace{\chi(x),\ \lambda(x),\ \bar{\psi}(x),\ \bar{\tau}(x).}_{4\ \text{Weyl spinors}} \tag{5.159}$$

It is immediate to recognize that this superfield provides a reducible representation of the supersymmetry. In order to select the irreducible components we must impose some constraints on the superfield. In the following we shall show two possible ways of reducing this representation: the former produces the *chiral superfield*, while the latter yields the *vector superfield*.

6 On-shell v.s. Off-shell representations

The representations of the supersymmetry discussed at the group theory level are on-shell. This means that the states satisfy the constraint $p^2 = M^2$, which is equivalent to the equations of motion. However, when we try to realize them in terms of local fields governed by a local Lagrangian, we are forced to relax this assumption in order to develop a manifestly covariant formalism. This is not peculiar of supersymmetry, but also occurs for the representations of the Poincarè group. Consider, for example, a parity invariant massless particle of spin 1, its representation only contains two helicity states. Instead, in terms of local fields, it has to be described by a vector $V_m(x)$, which possesses 4 degrees of freedom. The correct counting is only restored when the field satisfies the equation of motion and the gauge invariance is used. From a group theoretical point of view, $V_m(x)$ is an enlarged (thus reducible) representation of the Poincarè group, the equations of motion and the gauge invariance work as projectors that throw away the unwanted d.o.f.

Therefore, in order to write a supersymmetric action we must enlarge the on-shell multiplet considered in the previous lecture. We will have two possibilities:

- The multiplet is enlarged to a reducible representation of the supersymmetry. In this case the supersymmetry is realized also **off-shell** and the formalism is manifestly invariant under supersymmetry at all stages.
- The multiplet is enlarged only to have a manifest invariance under the Poincarè group. The supersymmetry is recovered only when the equations of motion are imposed (**on-shell supersymmetry**), while is broken off-shell. In this setting the number of bosonic and fermionic d.o.f. does not match unless the e.o.m are satisfied.

The approach in terms of $N = 1$ superfields will provide us with an off-shell realization of the $N = 1$ supersymmetry and a manifestly covariant formalism. However this is a lucky situation. For extended supersymmetry and supergravities the off-shell representations are not known in general and only the second possibility is available.

7 Chiral superfield

Given the set of scalar superfields Φ, the subset defined by the condition

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0 \tag{7.160}$$

is un invariant subset under supersymmetry transformations. In fact, if a superfield Φ solves the constraint (7.160) also $\delta_\epsilon\Phi$ is a solution

$$\bar{D}_{\dot{\alpha}}\delta_{\epsilon,\bar{\epsilon}}\Phi(x,\theta,\bar{\theta}) = \bar{D}_{\dot{\alpha}}(\epsilon Q + \bar{\epsilon}\bar{Q})\Phi(x,\theta,\bar{\theta}) = (\epsilon Q + \bar{\epsilon}\bar{Q})\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0. \tag{7.161}$$

The general solution of the constraint (7.160) can be determined by introducing the variable $y^m = x^m + i\theta\sigma^m\bar{\theta}$ for which $\bar{D}_{\dot{\alpha}}y^m = 0$. If we consider the superfield as a function of y, θ and $\bar{\theta}$, the constraint (7.160) becomes

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\Phi(y,\theta,\bar{\theta}) = 0. \tag{7.162}$$

In other words the superfield depends on y and θ, but not $\bar{\theta}$: $\Phi(y,\theta)$. This superfield is now a *chiral superfield* and it can be expanded either in a polynomial of the Grassmannian variable θ

$$\Phi(y,\theta) = A(y) + \sqrt{2}\chi(y)\theta + \theta^2 F(y), \tag{7.163}$$

or in terms of the original 4 Grassmannian variables

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) =& A(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) - \frac{1}{2}\theta\sigma^m\bar{\theta}\theta\sigma^n\bar{\theta}\partial_m\partial_n A(x) + \\ &+ \sqrt{2}\chi(x)\theta + i\sqrt{2}\theta\sigma^m\bar{\theta}\partial_m\chi(x)\theta + \theta^2 F(x) = \\ =& A(x) + \sqrt{2}\chi(x)\theta + \theta^2 F(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) - \frac{i}{\sqrt{2}}\theta^2\partial_m\chi\sigma^m\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x).\end{aligned} \tag{7.164}$$

This superfield describes two complex scalar fields $A(x)$, $F(x)$ (4 real bosonic fiels) and a Weyl spinor $\chi(x)$ (4 real fermionic components): it contains twice the number of components that are necessary to describe the chiral multiplet. However we cannot further reduce the degrees of freedom by imposing a non-dynamical constraint. The unwanted fields will disappear from the game when we shall impose that the fields satisfy the equations of motion. For this reason the chiral superfield is said to provide the *off-shell chiral multiplets.*

Let us consider the action of the supersymmetry transformations on the chiral superfield. This action is given by

$$\delta_{\epsilon,\bar{\epsilon}}\Phi = (\epsilon Q + \bar{\epsilon}\bar{Q})\Phi. \tag{7.165}$$

Note that the supersymmetry charges for a chiral superfield can be written in a very simple form because of its particular dependence on the variables x, θ and $\bar{\theta}$. We have

$$Q_\alpha = \left.\frac{\partial}{\partial\theta^\alpha}\right|_{y,\theta,\bar{\theta}} \qquad \text{and} \qquad \bar{Q}_{\dot{\alpha}} = 2i(\theta\sigma^m)_{\dot{\alpha}}\left.\frac{\partial}{\partial y^m}\right|_{y,\theta,\bar{\theta}}. \tag{7.166}$$

Thus

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\Phi &= \left(\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} + 2i(\theta\sigma^m\bar{\epsilon})\frac{\partial}{\partial y^m}\right)\Phi(y,\theta) = \sqrt{2}\epsilon\chi(y) + 2\epsilon\theta F(y) + 2i(\theta\sigma^m\bar{\epsilon})\partial_m A(y) + \\ &\quad + 2i\sqrt{2}(\theta\sigma^m\bar{\epsilon})\partial_m\chi(y)\theta = \sqrt{2}\epsilon\chi(y) + 2\epsilon\theta F(y) + 2i(\theta\sigma^m\bar{\epsilon})\partial_m A(y) + \\ &\quad - i\sqrt{2}\theta^2\partial_m\chi\sigma^m\bar{\epsilon}.\end{aligned} \tag{7.167}$$

In components, the above transformations read

$$\delta_{\epsilon,\bar{\epsilon}}A(x) = \sqrt{2}\epsilon\chi(x) \tag{7.168a}$$

$$\delta_{\epsilon,\bar{\epsilon}}\chi_\alpha(x) = i\sqrt{2}(\sigma^m\bar{\epsilon})_\alpha\partial_m A(x) + \sqrt{2}\epsilon_\alpha F(x) \tag{7.168b}$$

$$\delta_{\epsilon,\bar{\epsilon}}F(x) = -\sqrt{2}i\partial_m\chi\sigma^m\bar{\epsilon} = \sqrt{2}i\bar{\epsilon}\bar{\sigma}^m\partial_m\chi. \tag{7.168c}$$

When the spinor field χ is on-shell, *i.e.* it satisfies the equation of motion $\bar{\sigma}^m\partial_m\chi = 0$, the scalar field $F(x)$ becomes invariant under supersymmetry $\delta_{\epsilon,\bar{\epsilon}}F(x) = 0$. Consistency requires that

$$\begin{aligned}0 = \delta_{\zeta,\bar{\zeta}}\delta_{\epsilon,\bar{\epsilon}}F(x) &= \sqrt{2}i\bar{\epsilon}\bar{\sigma}^m\partial_m\delta_{\zeta\bar{\zeta}}\chi = -2(\bar{\epsilon}\bar{\sigma}^m)^\alpha\partial_m((\sigma^n\bar{\zeta})_\alpha\partial_n A(x) + \zeta_\alpha F(x)) = \\ &= -2(\bar{\epsilon}\bar{\sigma}^m\sigma^n\bar{\zeta}\partial_m\partial_n A(x) + \bar{\epsilon}\bar{\sigma}^m\zeta\partial_m F(x)) = -2(-\bar{\epsilon}\bar{\zeta}\Box A(x) + \bar{\epsilon}\bar{\sigma}^m\zeta\partial_m F(x)),\end{aligned} \tag{7.169}$$

namely

$$\Box A(x) = 0 \qquad \partial_m F(x) = 0. \tag{7.170}$$

The field $A(x)$ is on-shell as well, and F is a constant non propagating field. Summarizing, A and χ are the only degrees of freedom which survive on-shell: this is exactly the content of the chiral multiplet when discussed at the level of representation theory.
The set of all chiral superfields is closed under multiplication. In fact

$$\bar{D}_{\dot{\alpha}}(\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_{n-1}}\Phi_{i_n}) = 0, \tag{7.171}$$

since $\bar{D}_{\dot{\alpha}}$ is a (graded) derivation and it respects the (graded) Leibnitz rule. Moreover, any polynomial or function in the chiral fields is still a chiral field.

The chiral superfield $\Phi(x,\theta,\bar\theta)$ is a complex object and thus we can define its hermitian conjugate $\Phi^\dagger(x,\theta,\bar\theta)$. This superfield is no longer chiral, because $y \mapsto \bar y^n = x^n - i\theta\sigma^n\bar\theta$ and $\theta \mapsto \bar\theta$ under hermitian conjugation. However, it satisfies the constraint

$$D_\alpha \Phi^\dagger(x,\theta,\bar\theta) = 0, \tag{7.172}$$

which is the hermitian conjugate of the constraint (7.160). A generic superfield $\bar\Phi$ satisfying the condition $D_\alpha\bar\Phi = 0$ is called *anti-chiral superfield.* Its expansion is given by

$$\bar\Phi(\bar y,\bar\theta) = B(\bar y) + \sqrt{2}\ \bar\lambda(\bar y)\bar\theta + \bar\theta^2 H(\bar y), \tag{7.173}$$

or, in terms of the original variables,

$$\begin{aligned}\bar\Phi(x,\theta,\bar\theta) =&B(x) - i\theta\sigma^n\bar\theta\partial_n B(x) - \frac{1}{2}\theta\sigma^m\bar\theta\theta\sigma^n\bar\theta\partial_m\partial_n B(x) + \sqrt{2}\bar\lambda(x)\bar\theta - \\ &- \sqrt{2}i\theta\sigma^m\bar\theta\partial_m\bar\lambda(x)\bar\theta + \bar\theta^2 H(x) = \\ =&B(x) + \sqrt{2}\bar\lambda(x)\bar\theta - i\theta\sigma^n\bar\theta\partial_n B(x) + \bar\theta^2 H(x) + \frac{1}{4}\theta^2\bar\theta^2\Box B(x) + \frac{i}{\sqrt{2}}\bar\theta^2\theta\sigma^n\partial_n\bar\lambda.\end{aligned} \tag{7.174}$$

The supersymmetry charges for the antichiral superfield then take the simplified form

$$Q_\alpha = -2i(\sigma^m\bar\theta)_\alpha \frac{\partial}{\partial \bar y^m}\bigg|_{\bar y,\theta,\bar\theta} \qquad \text{and} \qquad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\bigg|_{\bar y,\theta,\bar\theta}. \tag{7.175}$$

Consequently, we can easily compute the supersymmetry transformations for an antichiral superfield and find

$$\delta_{\epsilon,\bar\epsilon} B(x) = \sqrt{2}\bar\lambda(x)\bar\epsilon \tag{7.176a}$$

$$\delta_{\epsilon,\bar\epsilon}\bar\lambda_\alpha(x) = i\sqrt{2}(\epsilon\sigma^m)_{\dot\alpha}\partial_m B(x) + \sqrt{2}\bar\epsilon_{\dot\alpha}H(x) \tag{7.176b}$$

$$\delta_{\epsilon,\bar\epsilon}H(x) = \sqrt{2}i\epsilon\sigma^m\partial_m\bar\lambda(x). \tag{7.176c}$$

7.1 An action for the chiral fields

The next step is to write an invariant action for the chiral multiplets. This action must be an integral over the whole superspace of (*super-*)Lagrangian density, which depends on the chiral super-fields Φ^I (I is an index running over all the possible chiral superfields appearing in our model). This Lagrangian, however, cannot be simply a function of the chiral superfields Φ^I. In fact

$$\int d^4x d^2\bar\theta d^2\theta \mathcal{F}(\Phi^I) = -\frac{1}{4}\int d^4x D^2\bar D^2\mathcal{F}(\Phi^I) = -\frac{1}{4}\int d^4x D^2\bar D_{\dot\alpha}\left(\frac{\partial\mathcal{F}(\Phi^I)}{\partial\Phi^A}\bar D^{\dot\alpha}\Phi^A\right) = 0. \tag{7.177}$$

In order to obtain a non-vanishing result, the Lagrangian density must depend on both chiral and anti-chiral super-fields, namely $\mathcal{L} = \mathcal{K}(\Phi^{I\dagger},\Phi^I)$. For the moment we shall consider the simplest choice for a function of this type: $\mathcal{L} = -\frac{1}{4}\Phi^{I\dagger}\Phi^I$, where Φ^I with $I = 1, \ldots, N$ represent N chiral superfields and $\Phi^{I\dagger}$ their hermitian conjugates. Then the action is given by

$$
\begin{aligned}
-\frac{1}{4}\int d^4x d^2\bar{\theta} d^2\theta \left(\Phi^I \Phi^{I\dagger}\right) &= \frac{1}{16}\int d^4x \bar{D}^2 D^2 (\Phi^I \Phi^{I\dagger}) = \frac{1}{16}\int d^4x \bar{D}^2[(D^2\Phi^I)\Phi^{I\dagger}] = \\
&= \frac{1}{16}\int d^4x [([\bar{D}^2, D^2]\Phi^I)\Phi^{I\dagger} + 2([\bar{D}_{\dot{\alpha}}, D^2]\Phi^I)\bar{D}^{\dot{\alpha}}\Phi^{I\dagger} + D^2\Phi^I \bar{D}^2\Phi^{I\dagger}].
\end{aligned}
\tag{7.178}
$$

The graded commutators appearing in the above expansion can be easily evaluated and we find

$$
[\bar{D}_{\dot{\alpha}}, D^2] = 4i(D\sigma^m)_{\dot{\alpha}}\partial_m \quad \text{and} \quad [\bar{D}^2, D^2] = 16\Box + 8i(D\sigma^m\bar{D})\partial_m. \tag{7.179}
$$

Thus, the action reads

$$
-\frac{1}{4}\int d^4x d^2\bar{\theta} d^2\theta \left(\Phi^I \Phi^{I\dagger}\right) = \frac{1}{16}\int d^4x [16\Phi^{I\dagger}\Box\Phi^I + 8i(D\sigma^m)_{\dot{\alpha}}\partial_m \Phi^I \bar{D}^{\dot{\alpha}}\Phi^{I\dagger} + D^2\Phi^I \bar{D}^2\Phi^{I\dagger}] - \tag{7.180}
$$

It can be rewritten in terms of the usual fields by means of the following identities

$$
\begin{array}{ll}
\Phi^I\big|_{\theta=\bar{\theta}=0} = A^I(x) & \Phi^{I\dagger}\big|_{\theta=\bar{\theta}=0} = A^{I\dagger}(x) \\
D_\alpha \Phi^I\big|_{\theta=\bar{\theta}=0} = \sqrt{2}\chi^I_\alpha(x) & \bar{D}_{\dot{\alpha}}\Phi^{I\dagger}\big|_{\theta=\bar{\theta}=0} = -\sqrt{2}\bar{\chi}^I_{\dot{\alpha}}(x) \\
D^2\Phi^I\big|_{\theta=\bar{\theta}=0} = -4F^I(x) & \bar{D}^2\Phi^{I\dagger}\big|_{\theta=\bar{\theta}=0} = -4F^{I\dagger}(x)
\end{array}
\tag{7.181}
$$

and one finds

$$
\begin{aligned}
-\frac{1}{4}\int d^4x d^2\bar{\theta} d^2\theta \left(\Phi^I \Phi^{I\dagger}\right) &= \int d^4x [A^{I\dagger}\Box A^I - i(\chi^I \sigma^m \partial_m \bar{\chi}^I) + F^I F^{I\dagger}] = \\
&= \int d^4x [A^{I\dagger}\Box A^I - i(\bar{\chi}^I \bar{\sigma}^m \partial_m \chi^I) + F^I F^{I\dagger}].
\end{aligned}
\tag{7.182}
$$

This is the correct free action for N complex scalar fields and N Weyl fermions. The N complex scalar fields F^I do not propagate and they identically vanish on the equations of motion. They can be dropped if we require that the supersymmetry is realized just on shell.
The next issue is how to introduce interactions such as *scalar potentials* and *Yukawa couplings* and to preserve supersymmetry. This cannot be done by simply adding more complicate functions of Φ^I and $\Phi^{I\dagger}$. These kind of terms will lead to derivative interactions, which are, moreover, generically non renormalizable. This type of interactions can be instead obtained by integrating a function of the chiral super-fields (but not of the anti-chiral ones) $\mathcal{F}(\Phi^I)$ over half of the superspace

$$
\begin{aligned}
\int d^4x d^2\theta \mathcal{F}(\Phi^I) &\equiv \int d^4y \frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2}\mathcal{F}(\Phi^I) = \\
&= \frac{1}{2}\epsilon^{\alpha\beta}\int d^4y D_\alpha D_\beta \mathcal{F}(\Phi^I) = -\frac{1}{2}\int d^4y D^2\mathcal{F}(\Phi^I).
\end{aligned}
\tag{7.183}
$$

This term is manifestly Lorentz invariant. Its invariance under supersymmetry transformations is a little more subtle. The transformations generated by ϵQ preserve (7.183), since they corresponds to take an additional derivative with respect to θ. The transformations of the form $\bar{\epsilon}\bar{Q}$ correspond instead to take a total divergence of the integrand and so the action is again invariant.

The analogous integration can be defined on the functions of the anti-chiral superfields

$$\int d^4x d^2\bar{\theta}\bar{\mathcal{F}}(\bar{\Phi}^I) \equiv -\int d^4y \frac{\partial}{\partial\bar{\theta}^{\dot{1}}}\frac{\partial}{\partial\bar{\theta}^{\dot{2}}}\bar{\mathcal{F}}(\bar{\Phi}^I) = -\frac{1}{2}\int d^4y \bar{D}^2\bar{\mathcal{F}}(\bar{\Phi}^I). \tag{7.184}$$

Since the Lagrangian must be real, a candidate interaction term for the chiral superfields is

$$\int d^4x d^2\theta \mathcal{F}(\Phi^I) + \text{c.c.} = \int d^4x d^2\theta \mathcal{F}(\Phi^I) + \int d^4x d^2\bar{\theta}\mathcal{F}^*(\Phi^{\dagger I}) \tag{7.185}$$

The first and the second integral, when written in terms of the standard fields, yield

$$\int d^4x d^2\theta \mathcal{F}(\Phi^K) = -\int d^4x \left(\frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I\partial\Phi^J}\chi^J\chi^I - 2\frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I}F^I\right), \tag{7.186a}$$

$$\int d^4x d^2\bar{\theta}\mathcal{F}^*(\Phi^{\dagger K}) = -\int d^4x \left(\frac{\partial\mathcal{F}^*(A^{\dagger K})}{\partial\Phi^{\dagger I}\partial\Phi^{\dagger J}}\bar{\chi}^J\bar{\chi}^I - 2\frac{\partial\mathcal{F}^*(A^{\dagger K})}{\partial\Phi^{\dagger I}}F^{\dagger I}\right), \tag{7.186b}$$

Therefore the supersymmetric action for a system of N chiral multiplets is given by

$$\begin{aligned} S = &-\frac{1}{4}\int d^4x d^2\bar{\theta}d^2\theta \left(\Phi^I\Phi^{I\dagger}\right) + \frac{1}{2}\int d^4x d^2\theta\mathcal{F}(\Phi^I) + \frac{1}{2}\int d^4x d^2\bar{\theta}\mathcal{F}^*(\Phi^{\dagger I}) = \\ &= \int d^4x \left[A^{I\dagger}\Box A^I - i(\bar{\chi}^I\bar{\sigma}^m\partial_m\chi^I) + F^I F^{I\dagger} - \frac{1}{2}\frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I\partial\Phi^J}\chi^J\chi^I + \frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I}F^I - \right. \\ &\left. - \frac{1}{2}\frac{\partial\mathcal{F}^*(A^{\dagger K})}{\partial\Phi^{\dagger I}\partial\Phi^{\dagger J}}\bar{\chi}^J\bar{\chi}^I + \frac{\partial\mathcal{F}^*(A^{\dagger K})}{\partial\Phi^{\dagger I}}F^{\dagger I}\right]. \end{aligned} \tag{7.187}$$

The introduction of the interactions has preserved the property that the fields F^I and their conjugates are not dynamical. We can eliminate them by means of their equations of motion, which are solved by

$$F^I = -\frac{\partial\mathcal{F}^*(A^{\dagger K})}{\partial\Phi^{\dagger I}} \qquad \text{and} \qquad F^{I\dagger} = -\frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I}. \tag{7.188}$$

Then the action takes the following form

$$S = \int d^4x \left[A^{I\dagger}\Box A^I - i(\bar{\chi}^I\bar{\sigma}^m\partial_m\chi^I) - \frac{1}{2}\frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I\partial\Phi^J}\chi^J\chi^I - \frac{1}{2}\frac{\partial\mathcal{F}^*(A^{\dagger K})}{\partial\Phi^{\dagger I}\partial\Phi^{\dagger J}}\bar{\chi}^J\bar{\chi}^I - \underbrace{\left|\frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I}\right|^2}_{\text{F-terms}}\right]. \tag{7.189}$$

and it is invariant under the supersymmetry transformations

$$\delta_{\epsilon,\bar{\epsilon}}A^I(x) = \sqrt{2}\epsilon\chi^I(x) \tag{7.190a}$$

$$\delta_{\epsilon,\bar{\epsilon}}\chi^I_\alpha(x) = i\sqrt{2}(\sigma^m\bar{\epsilon})_\alpha\partial_m A^I(x) - \sqrt{2}\epsilon_\alpha\frac{\partial\mathcal{F}(A^K)}{\partial\Phi^I}. \tag{7.190b}$$

The potential for the scalar fields A^I is manifestly positive definite. It is the sum of the square absolute values. This property follows directly from the supersymmetry algebra, as shown in (3.69). The absolute minima of this potential (*vacua*) are therefore determined by the system of equations

$$F^I = \frac{\partial \mathcal{F}(A^K)}{\partial \Phi^I} = 0. \tag{7.191}$$

If this system of equations does not admit a solution the supersymmetry is spontaneously broken. If we limit ourself to renormalizable interactions, the most general super-potential is a polynomial at most cubic in the chiral superfiedls

$$F(\Phi) = \lambda_I \Phi^I + \frac{1}{2} m_{IJ} \Phi^I \Phi^J + \frac{1}{3} g_{IJK} \Phi^I \Phi^J \Phi^K. \tag{7.192}$$

Then the most general supersymmatric action with particles of spin less than 1 and with renormalizable interaction is

$$\begin{aligned} S = \int d^4x \Big[& A^{I\dagger} \Box A^I - i(\bar{\chi}^I \bar{\sigma}^m \partial_m \chi^I) - \frac{1}{2}(m_{IJ} + 2g_{IJK} A^K)\chi^I \chi^J - \\ & - \frac{1}{2}(m^*_{IJ} + 2g^*_{IJK} A^{\dagger K})\bar{\chi}^I \bar{\chi}^J - \sum_{I=1}^{N} \left| \lambda_I + m_{IJ} A^J + g_{IJK} A^J A^K \right|^2 \Big]. \end{aligned} \tag{7.193}$$

7.2 Non-linear sigma model

As long as we want to formulate a fundamental quantum field theory, *i.e.* valid at all scales, renormalizability is a guiding principle. Then the most general Lagrangian containing only chiral superfields has the form discussed in the previous section. It contains a kinetic term given by $-\frac{1}{4}\phi^{I\dagger}\phi^I$, and a superpotential which is at most cubic in the chiral superfields. Instead, if we consider our supersymmetric theory as an *effective model* the constrains imposed by renormalizability must be relaxed and we can write a more general action

$$S = \int d^4x \int d^2\theta d^2\bar{\theta} \mathcal{K}(\phi^{I\dagger}, \phi^I) + \int d^4x \int d^2\theta \mathcal{F}(\phi^I) + \int d^4x \int d^2\bar{\theta} \mathcal{F}^*(\phi^{I\dagger}). \tag{7.194}$$

The function $\mathcal{K}(\phi^{I\dagger}, \phi^I)$ must define a real superfield, and this will be the case if $K^\dagger(z^{I\dagger}, z^I) = K(z^I, z^{I\dagger})$ and $\mathcal{F}$ is an arbitrary function.

The expansion of this action in terms of component fields requires a lengthy and tedious analysis. Firstly, we shall consider the case of vanishing superpotenzial F, and we shall evaluate the integral over the Grassmann coordinates of the kinetic term $\mathcal{K}(\phi^{I\dagger}, \phi^I)$. We find

$$\begin{aligned}
&\int d^2\theta d^2\bar{\theta}\mathcal{K}(\phi^{I\dagger}, \phi^I) = -\frac{1}{4}\bar{D}^2 D^2(\mathcal{K}) = -\frac{1}{4}D^\alpha \bar{D}^{\dot{\alpha}}(\bar{D}_{\dot{\alpha}} D_\alpha(\mathcal{K})) + \text{total div.} = \\
&= -\frac{1}{4}(\partial_L\partial_I\bar{\partial}_J\bar{\partial}_K\mathcal{K}D^\alpha\Phi^L\bar{D}^{\dot{\alpha}}\Phi^{K\dagger}\bar{D}_{\dot{\alpha}}\phi^{J\dagger}D_\alpha\Phi^I - 4i\partial_I\bar{\partial}_J\bar{\partial}_K\mathcal{K}\partial_m\Phi^{K\dagger}\bar{D}\phi^{J\dagger}\bar{\sigma}^m D\Phi^I - \\
&+ \partial_I\bar{\partial}_J\bar{\partial}_K\mathcal{K}\bar{D}^{\dot{\alpha}}\Phi^{K\dagger}\bar{D}_{\dot{\alpha}}\phi^{J\dagger}D^2\Phi^I - \partial_I\bar{\partial}_J\partial_K\mathcal{K}D^\alpha\Phi^K\bar{D}^2\phi^{J\dagger}D_\alpha\Phi^I + \\
&+ 4i\partial_I\bar{\partial}_J\mathcal{K}D\Phi^I\sigma^m\partial_m\bar{D}\phi^{J\dagger} - \partial_I\bar{\partial}_J\mathcal{K}\bar{D}^2\phi^{J\dagger}D^2\Phi^I - 4i\partial_I\bar{\partial}_J\partial_K\mathcal{K}D\Phi^K\sigma^m\bar{D}\Phi^{\dagger J}\partial_m\Phi^I - \\
&- 8\mathrm{Tr}(\sigma^m\bar{\sigma}^n)\partial_I\bar{\partial}_J\mathcal{K}\partial_n\Phi^{\dagger J}\partial_m\Phi^I - 4i\partial_I\bar{\partial}_J\mathcal{K}\partial_m D\Phi^I\sigma^m\bar{D}\Phi^{\dagger J}) + \text{total div..}
\end{aligned} \tag{7.195}$$

It is convenient to introduce the following notation: $\bar{\partial}_I \mapsto \partial_{\bar{I}}$, $\Phi^{I\dagger} \mapsto \bar{\Phi}^{\bar{I}}$, $\partial_I\bar{\partial}_J\mathcal{K} \mapsto \mathcal{K}_{I\bar{J}}$, $\partial_I\partial_J\bar{\partial}_K\mathcal{K} \mapsto \mathcal{K}_{IJ\bar{K}}$, $\bar{\partial}_I\bar{\partial}_J\partial_K\mathcal{K} \mapsto \mathcal{K}_{\bar{I}\bar{J}K}$ and $\partial_I\partial_J\bar{\partial}_K\bar{\partial}_L\mathcal{K} \mapsto \mathcal{K}_{IJ\bar{K}\bar{L}}$. Then

$$\begin{aligned}
&\int d^2\theta d^2\bar{\theta}\mathcal{K}(\bar{\Phi}^{\bar{I}}, \phi^I) = \\
&= -\frac{1}{4}(\mathcal{K}_{IL\bar{J}\bar{K}}D^\alpha\Phi^L\bar{D}^{\dot{\alpha}}\bar{\Phi}^{\bar{K}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{J}}D_\alpha\Phi^I - 4i\mathcal{K}_{I\bar{J}\bar{K}}\partial_m\bar{\Phi}^{\bar{K}}\bar{D}\bar{\Phi}^{\bar{J}}\bar{\sigma}^m D\Phi^I - \\
&+ \mathcal{K}_{I\bar{J}\bar{K}}\bar{D}^{\dot{\alpha}}\bar{\Phi}^{\bar{K}}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{J}}D^2\Phi^I - \mathcal{K}_{IK\bar{J}}D^\alpha\Phi^K\bar{D}^2\bar{\Phi}^{\bar{J}}D_\alpha\Phi^I + \\
&+ 4i\mathcal{K}_{I\bar{J}}D\Phi^I\sigma^m\partial_m\bar{D}\bar{\Phi}^{\bar{J}} - \mathcal{K}_{I\bar{J}}\bar{D}^2\bar{\Phi}^{\bar{J}}D^2\Phi^I - 4i\mathcal{K}_{IK\bar{J}}D\Phi^K\sigma^m\bar{D}\bar{\Phi}^{\bar{J}}\partial_m\Phi^I - \\
&- 8\mathrm{Tr}(\sigma^m\bar{\sigma}^n)\mathcal{K}_{I\bar{J}}\partial_n\bar{\Phi}^{\bar{J}}\partial_m\Phi^I - 4i\mathcal{K}_{I\bar{J}}\partial_m D\Phi^I\sigma^m\bar{D}\bar{\Phi}^{\bar{J}}) + \text{total div..}
\end{aligned} \tag{7.196}$$

This expression, by construction, contains only terms of grading 0. Therefore can be written in terms of the component fields by means of the following table

$$\begin{aligned}
&\Phi^I\big|_{\theta=\bar{\theta}=0} = A^I(x) && \bar{\Phi}^{\bar{I}}\Big|_{\theta=\bar{\theta}=0} = \bar{A}^{\bar{I}}(x) \\
&D_\alpha\Phi^I\big|_{\theta=\bar{\theta}=0} = \sqrt{2}\chi^I_\alpha(x) && \bar{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{I}}\Big|_{\theta=\bar{\theta}=0} = -\sqrt{2}\bar{\chi}^{\bar{I}}_{\dot{\alpha}}(x) \ , \\
&D^2\Phi^I\big|_{\theta=\bar{\theta}=0} = -4F^I(x) && \bar{D}^2\bar{\Phi}^{\bar{I}}\Big|_{\theta=\bar{\theta}=0} = -4\bar{F}^{\bar{I}}(x)
\end{aligned} \tag{7.197}$$

where the same notations used for the superfields were also applied to the component fields.

$$\begin{aligned}
&\int d^2\theta d^2\bar{\theta}\mathcal{K}(\bar{\Phi}^{\bar{I}}, \phi^I) = \\
&= -\frac{1}{4}(-4\mathcal{K}_{IL\bar{J}\bar{K}}\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} + 8i\mathcal{K}_{I\bar{J}\bar{K}}\partial_m\bar{A}^{\bar{K}}\bar{\chi}^{\bar{J}}\bar{\sigma}^m\chi^I + \\
&+ 8\mathcal{K}_{I\bar{J}\bar{K}}\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}}F^I + 8\mathcal{K}_{IK\bar{J}}\chi^K\chi^I\bar{F}^{\bar{J}} + \\
&- 8i\mathcal{K}_{I\bar{J}}\chi^I\sigma^m\partial_m\bar{\chi}^{\bar{J}} - 16\mathcal{K}_{I\bar{J}}F^I\bar{F}^{\bar{J}} + 8i\mathcal{K}_{IK\bar{J}}\chi^K\sigma^m\bar{\chi}^{\bar{J}}\partial_m A^I + \\
&+ 16\mathcal{K}_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^n A^I + 8i\mathcal{K}_{I\bar{J}}\partial_m\chi^I\sigma^m\bar{\chi}^{\bar{J}}) + \text{total div..}
\end{aligned} \tag{7.198}$$

It is convenient to eliminate the auxiliary fields F^I and its complex conjugate $\bar{F}^{\bar{I}}$ by means of the equations of motion

$$\begin{aligned}
&-16\mathcal{K}_{I\bar{J}}\bar{F}^{\bar{J}} + 8\mathcal{K}_{I\bar{J}\bar{K}}\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} = 0 && -16\mathcal{K}_{I\bar{J}}F^I + 8\mathcal{K}_{IK\bar{J}}\chi^K\chi^I = 0 \\
&\Rightarrow \bar{F}^{\bar{I}} = \frac{1}{2}K^{\bar{I}I}\mathcal{K}_{I\bar{J}\bar{K}}\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} && F^I = \frac{1}{2}\mathcal{K}^{I\bar{I}}\mathcal{K}_{\bar{I}JK}\chi^J\chi^K,
\end{aligned} \tag{7.199}$$

where $\mathcal{K}^{I\bar{I}} = \mathcal{K}^{I\bar{I}}$ are the inverse of $\mathcal{K}_{I\bar{I}}$. Then we find

$$\begin{aligned}
&\int d^2\theta d^2\bar{\theta}\mathcal{K}(\bar{\Phi}^{\bar{I}}, \phi^I) = \\
=&\Big[(\mathcal{K}_{IL\bar{J}\bar{K}} - K^{A\bar{A}}K_{A,\bar{J}\bar{K}}K_{\bar{A},IL})\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} - 2i\mathcal{K}_{I\bar{J}\bar{K}}\partial_m\bar{A}^{\bar{K}}\bar{\chi}^{\bar{J}}\bar{\sigma}^m\chi^I + 2i\mathcal{K}_{I\bar{J}}\chi^I\sigma^m\partial_m\bar{\chi}^{\bar{J}} - \\
&- 2i\mathcal{K}_{IK\bar{J}}\chi^K\sigma^m\bar{\chi}^{\bar{J}}\partial_m A^I - 4\mathcal{K}_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^n A^I - 2i\mathcal{K}_{I\bar{J}}\partial_m\chi^I\sigma^m\bar{\chi}^{\bar{J}}\Big] + \text{total div.} = \\
=&\Big[(\mathcal{K}_{IL\bar{J}\bar{K}} - K^{A\bar{A}}K_{A,\bar{J}\bar{K}}K_{\bar{A},IL})\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} - 2i\mathcal{K}_{I\bar{J}\bar{K}}\partial_m\bar{A}^{\bar{K}}\bar{\chi}^{\bar{J}}\bar{\sigma}^m\chi^I + 2i\mathcal{K}_{I\bar{J}}\chi^I\sigma^m\partial_m\bar{\chi}^{\bar{J}} - \\
&- 2i\mathcal{K}_{IK\bar{J}}\chi^K\sigma^m\bar{\chi}^{\bar{J}}\partial_m A^I - 4\mathcal{K}_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^n A^I + \\
&+ 2i\mathcal{K}_{I\bar{J}}\chi^I\sigma^m\bar{\partial}_m\chi^{\bar{J}} + 2i\mathcal{K}_{IL\bar{J}}\partial_m A^L\chi^I\sigma^m\bar{\chi}^{\bar{J}} + 2i\mathcal{K}_{I\bar{L}\bar{J}}\partial_m\bar{A}^{\bar{L}}\chi^I\sigma^m\bar{\chi}^{\bar{J}}\Big] + \text{total div.} = \\
=&\Big[(\mathcal{K}_{IL\bar{J}\bar{K}} - K^{A\bar{A}}K_{A,\bar{J}\bar{K}}K_{\bar{A},IL})\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} - 2i\mathcal{K}_{I\bar{J}\bar{K}}\partial_m\bar{A}^{\bar{K}}\bar{\chi}^{\bar{J}}\bar{\sigma}^m\chi^I + 4i\mathcal{K}_{I\bar{J}}\chi^I\sigma^m\partial_m\bar{\chi}^{\bar{J}} - \\
&- 4\mathcal{K}_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^n A^I + 2i\mathcal{K}_{I\bar{L}\bar{J}}\partial_m\bar{A}^{\bar{L}}\chi^I\sigma^m\bar{\chi}^{\bar{J}}\Big] + \text{total div.} = \\
=&\Big[(\mathcal{K}_{IL\bar{J}\bar{K}} - K^{A\bar{A}}K_{A,\bar{J}\bar{K}}K_{\bar{A},IL})\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} + 4i\mathcal{K}_{I\bar{J}\bar{K}}\partial_m\bar{A}^{\bar{K}}\chi^I\bar{\sigma}^m\bar{\chi}^{\bar{J}} + 4i\mathcal{K}_{I\bar{J}}\chi^I\sigma^m\partial_m\bar{\chi}^{\bar{J}} - \\
&- 4\mathcal{K}_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^n A^I\Big] + \text{total div.} = \\
=&\Big[(\mathcal{K}_{IL\bar{J}\bar{K}} - K^{A\bar{A}}K_{A,\bar{J}\bar{K}}K_{\bar{A},IL})\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}} + 4i\mathcal{K}_{I\bar{J}}\chi^I\sigma^m(\partial_m\bar{\chi}^{\bar{J}} + K^{\bar{J}R}\mathcal{K}_{R\bar{S}\bar{K}}\partial_m\bar{A}^{\bar{K}}\bar{\chi}^{\bar{S}}) \\
&- 4\mathcal{K}_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^n A^I\Big] + \text{total div.}.
\end{aligned} \tag{7.200}$$

This Lagrangian can be written in a manifestly covariant form if we interpret the function $\mathcal{K}$ as the Kaehler potential for the complex manifold spanned by the scalars $(A^I, \bar{A}^{\bar{I}})$. We have then the following identifications

(a) the complex metric is given by

$$G_{I\bar{J}} = \partial_I\bar{\partial}_J\mathcal{K} \quad G_{IJ} = G_{\bar{I}\bar{J}} = 0. \tag{7.201}$$

(b) the corresponding Christoffel symbol are

$$\begin{aligned}
&\Gamma_{I,JK} = 0 \quad \Gamma_{\bar{I},\bar{J}\bar{K}} = 0 \quad \Gamma_{\bar{I},JK} = \frac{1}{2}(\partial_J G_{\bar{I}K} + \partial_K G_{\bar{I}J} - \bar{\partial}_I G_{JK}) = \partial_J\partial_K\partial_{\bar{I}}\mathcal{K} \\
&\Gamma_{I,\bar{J}K} = \frac{1}{2}(\bar{\partial}_J G_{IK} + \partial_K G_{I\bar{J}} - \partial_I G_{\bar{J}K}) = 0 \quad \Gamma_{\bar{I},\bar{J}K} = \frac{1}{2}(\bar{\partial}_J G_{\bar{I}K} + \partial_K G_{\bar{I}\bar{J}} - \bar{\partial}_I G_{\bar{J}K}) = 0 \\
&\Gamma_{I,\bar{J}\bar{K}} = \frac{1}{2}(\bar{\partial}_J G_{I\bar{K}} + \bar{\partial}_K G_{I\bar{J}} - \partial_I G_{\bar{J}\bar{K}}) = \bar{\partial}_J\bar{\partial}_K\partial_I\mathcal{K}.
\end{aligned}$$

(c) We shall also need the following component of the curvature

$$\begin{aligned}
R_{I\bar{L},M\bar{N}} =&\partial_M\Gamma_{I\bar{L}\bar{N}} - \partial_{\bar{N}}\Gamma_{I\bar{L}M} + g^{A\bar{A}}\Gamma_{A,\bar{L}M}\Gamma_{\bar{A},\bar{N}I} + g^{A\bar{A}}\Gamma_{\bar{A},\bar{L}M}\Gamma_{A,\bar{N}I} - g^{A\bar{A}}\Gamma_{A,\bar{L}\bar{N}}\Gamma_{\bar{A},MI} + \\
&+ g^{\bar{A}A}\Gamma_{\bar{A},\bar{L}\bar{N}}\Gamma_{A,MI} = \partial_M\Gamma_{I\bar{L}\bar{N}} - g^{A\bar{A}}\Gamma_{A,\bar{L}\bar{N}}\Gamma_{\bar{A},MI} = \\
=&\partial_I\partial_M\bar{\partial}_L\bar{\partial}_N K - g^{A\bar{A}}\Gamma_{A,\bar{L}\bar{N}}\Gamma_{\bar{A},MI}.
\end{aligned} \tag{7.202}$$

Then (7.200) takes the following form

$$\begin{aligned}&\int d^2\theta d^2\bar{\theta}\mathcal{K}(\bar{\Phi}^{\bar{I}},\phi^I)=\\=&R_{L\bar{J}I\bar{K}}\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}}+4iG_{I\bar{J}}\chi^I\sigma^m(\partial_m\bar{\chi}^{\bar{J}}+\Gamma^{\bar{J}}_{\bar{R}\bar{K}}\partial_m\bar{A}^{\bar{K}}\bar{\chi}^{\bar{R}})-4G_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^nA^I+\text{total div..}\end{aligned} \tag{7.203}$$

We can then write the following action as

$$\begin{aligned}&\frac{1}{4}\int d^4x\int d^2\theta d^2\bar{\theta}\mathcal{K}(\bar{\Phi}^{\bar{I}},\phi^I)=\\=&\int d^4x\left[\frac{1}{4}R_{L\bar{J}I\bar{K}}\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}}+iG_{I\bar{J}}\chi^I\sigma^mD_m\bar{\chi}^{\bar{J}}-G_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^nA^I\right].\end{aligned} \tag{7.204}$$

If there is also a superpotential, the additional contributions are

$$\frac{1}{2}\int d^4xd^2\theta\mathcal{F}(\Phi^K)=-\frac{1}{2}\int d^4x\left(\mathcal{F}_{IJ}\chi^J\chi^I-2\mathcal{F}_IF^I\right), \tag{7.205a}$$

$$\frac{1}{2}\int d^4xd^2\bar{\theta}\bar{\mathcal{F}}(\bar{\Phi}^{\bar{K}})=-\frac{1}{2}\int d^4x\left(\bar{\mathcal{F}}_{\bar{I}\bar{J}}\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{I}}-2\bar{\mathcal{F}}_{\bar{I}}\bar{F}^{\bar{I}}\right). \tag{7.205b}$$

These terms modify the equation of motion for F^I and $\bar{F}^{\bar{I}}$

$$\begin{aligned}&\mathcal{K}_{I\bar{J}}\bar{F}^{\bar{J}}-\frac{1}{2}\mathcal{K}_{I\bar{J}\bar{K}}\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}}+\mathcal{F}_I=0 \qquad \mathcal{K}_{I\bar{J}}F^I-\frac{1}{2}\mathcal{K}_{IK\bar{J}}\chi^K\chi^I+\bar{\mathcal{F}}_{\bar{I}}=0\\&\Rightarrow\bar{F}^{\bar{I}}=\frac{1}{2}\Gamma^{\bar{I}}_{\bar{J}\bar{K}}\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}}-G^{\bar{I}J}\mathcal{F}_J \qquad F^I=\frac{1}{2}\Gamma^I_{JK}\chi^J\chi^K-G^{I\bar{J}}\bar{\mathcal{F}}_{\bar{J}},\end{aligned} \tag{7.205c}$$

Then the action in the presence of a superpotential is

$$\begin{aligned}&\frac{1}{4}\int d^4x\int d^2\theta d^2\bar{\theta}\underbrace{\mathcal{K}(\bar{\Phi}^{\bar{I}},\phi^I)}_{\text{Kaheler potential}}+\frac{1}{2}\int d^4xd^2\theta\underbrace{\mathcal{F}(\Phi^K)}_{\text{superpotential}}+\frac{1}{2}\int d^4xd^2\bar{\theta}\underbrace{\mathcal{F}^*(\Phi^{\dagger K})}_{\text{superpotential}}=\\=&\int d^4x\left[\frac{1}{4}R_{L\bar{J}I\bar{K}}\chi^L\chi^I\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{K}}+iG_{I\bar{J}}\chi^I\sigma^mD_m\bar{\chi}^{\bar{J}}-G_{I\bar{J}}\partial_n\bar{A}^{\bar{J}}\partial^nA^I-\frac{1}{2}\nabla_{\bar{I}}\nabla_{\bar{J}}\bar{\mathcal{F}}\bar{\chi}^{\bar{J}}\bar{\chi}^{\bar{I}}-\right.\\&\left.-\frac{1}{2}\nabla_I\nabla_J\mathcal{F}\chi^J\chi^I-G^{I\bar{J}}\nabla_I\mathcal{F}\nabla_{\bar{J}}\bar{\mathcal{F}}\right].\end{aligned} \tag{7.205d}$$

Exercise: *Prove eq.* (7.195).

SOLUTION: If we expand the derivatives acting on the Kinetic term, we find

$$
\begin{aligned}
\bar{D}^2 D^2(\mathcal{K}) =& D^\alpha \bar{D}^{\dot\alpha}(\bar{D}_{\dot\alpha} D_\alpha(\mathcal{K})) + \text{total div.} = \\
=& D^\alpha \bar{D}^{\dot\alpha}(\partial_I \bar\partial_J \mathcal{K} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I + \partial_I \mathcal{K} \bar{D}_{\dot\alpha} D_\alpha \Phi^I) + \text{total div.} = \\
=& D^\alpha \bar{D}^{\dot\alpha}(\partial_I \bar\partial_J \mathcal{K} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I - 2i\partial_I \mathcal{K} \sigma^m_{\alpha\dot\alpha}\partial_m \Phi^I) + \text{total div.} = \\
=& D^\alpha(\partial_I \bar\partial_J \bar\partial_K \mathcal{K} \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I - \partial_I \bar\partial_J \mathcal{K} \bar{D}^2 \phi^{J\dagger} D_\alpha \Phi^I - \\
& - \partial_I \bar\partial_J \mathcal{K} \bar{D}_{\dot\alpha}\phi^{J\dagger} \bar{D}^{\dot\alpha} D_\alpha \Phi^I - 2i\partial_I \bar\partial_J \mathcal{K} \sigma^m_{\alpha\dot\alpha} \bar{D}^{\dot\alpha}\Phi^{\dagger J}\partial_m \Phi^I) + \text{total div.} = \\
=& D^\alpha(\partial_I \bar\partial_J \bar\partial_K \mathcal{K} \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I - \partial_I \bar\partial_J \mathcal{K} \bar{D}^2 \phi^{J\dagger} D_\alpha \Phi^I + \\
& + \partial_I \bar\partial_J \mathcal{K} \bar{D}^{\dot\alpha}\phi^{J\dagger} \bar{D}_{\dot\alpha} D_\alpha \Phi^I - 2i\partial_I \bar\partial_J \mathcal{K} \sigma^m_{\alpha\dot\alpha} \bar{D}^{\dot\alpha}\Phi^{\dagger J}\partial_m \Phi^I) + \text{total div.} = \\
=& D^\alpha(\partial_I \bar\partial_J \bar\partial_K \mathcal{K} \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I - \partial_I \bar\partial_J \mathcal{K} \bar{D}^2 \phi^{J\dagger} D_\alpha \Phi^I - \\
& - 4i\partial_I \bar\partial_J \mathcal{K} \sigma^m_{\alpha\dot\alpha} \bar{D}^{\dot\alpha}\Phi^{\dagger J}\partial_m \Phi^I) + \text{total div.} = \\
=& (\partial_L \partial_I \bar\partial_J \bar\partial_K \mathcal{K} D^\alpha \Phi^L \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I + \partial_I \bar\partial_J \bar\partial_K \mathcal{K} D^\alpha \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I - \\
& - \partial_I \bar\partial_J \bar\partial_K \mathcal{K} \bar{D}^{\dot\alpha}\Phi^{K\dagger} D^\alpha \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I + \partial_I \bar\partial_J \bar\partial_K \mathcal{K} \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D^2 \Phi^I - \\
& - \partial_I \bar\partial_J \partial_K \mathcal{K} D^\alpha \Phi^K \bar{D}^2 \phi^{J\dagger} D_\alpha \Phi^I - \partial_I \bar\partial_J \mathcal{K} D^\alpha \bar{D}^2 \phi^{J\dagger} D_\alpha \Phi^I - \partial_I \bar\partial_J \mathcal{K} \bar{D}^2 \phi^{J\dagger} D^2 \Phi^I - \\
& - 4i\partial_I \bar\partial_J \partial_K \mathcal{K} \sigma^m_{\alpha\dot\alpha} D^\alpha \Phi^K \bar{D}^{\dot\alpha}\Phi^{\dagger J}\partial_m \Phi^I - 4i\partial_I \bar\partial_J \mathcal{K} \sigma^m_{\alpha\dot\alpha} D^\alpha \bar{D}^{\dot\alpha}\Phi^{\dagger J}\partial_m \Phi^I + \\
& + 4i\partial_I \bar\partial_J \mathcal{K} \sigma^m_{\alpha\dot\alpha} \bar{D}^{\dot\alpha}\Phi^{\dagger J}\partial_m D^\alpha \Phi^I) + \text{total div.} = \\
=& (\partial_L \partial_I \bar\partial_J \bar\partial_K \mathcal{K} D^\alpha \Phi^L \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D_\alpha \Phi^I - 4i\partial_I \bar\partial_J \bar\partial_K \mathcal{K} \partial_m \Phi^{K\dagger} \bar{D}\phi^{J\dagger} \bar\sigma^m D\Phi^I - \\
& + \partial_I \bar\partial_J \bar\partial_K \mathcal{K} \bar{D}^{\dot\alpha}\Phi^{K\dagger} \bar{D}_{\dot\alpha}\phi^{J\dagger} D^2 \Phi^I - \partial_I \bar\partial_J \partial_K \mathcal{K} D^\alpha \Phi^K \bar{D}^2 \phi^{J\dagger} D_\alpha \Phi^I + \\
& + 4i\partial_I \bar\partial_J \mathcal{K} D\Phi^I \sigma^m \partial_m \bar{D}\phi^{J\dagger} - \partial_I \bar\partial_J \mathcal{K} \bar{D}^2 \phi^{J\dagger} D^2 \Phi^I - \\
& - 4i\partial_I \bar\partial_J \partial_K \mathcal{K} D\Phi^K \sigma^m \bar{D}\Phi^{\dagger J}\partial_m \Phi^I - 8\mathrm{Tr}(\sigma^m \bar\sigma^n)\partial_I \bar\partial_J \mathcal{K} \partial_n \Phi^{\dagger J}\partial_m \Phi^I + \\
& - 4i\partial_I \bar\partial_J \mathcal{K} \partial_m D\Phi^I \sigma^m \bar{D}\Phi^{\dagger J}) + \text{total div.}.
\end{aligned}
$$

7.3 First implications of supersymmetry: SUSY Ward Identity

In the following we shall examine the first consequences of the $N = 1$ supersymmetry at the quantum level. In particular we shall show how supersymmetry can constrain the dependence on the space-time coordinates and on the couplings present in the theory. For our goals it is more natural to write the supersymmetry transformations (7.168) in operator language

$$[A^I(x), Q_\alpha] = i\sqrt{2}\chi^I_\alpha(x) \tag{7.206a}$$

$$[A^I(x), \bar{Q}_{\dot\alpha}] = 0 \tag{7.206b}$$

$$\{\chi^I_\beta(x), Q_\alpha\} = -i\sqrt{2}\epsilon_{\beta\alpha}F^I(x) \tag{7.206c}$$

$$\{\chi^I_\beta(x), \bar{Q}_{\dot\alpha}\} = -\sqrt{2}\sigma^m_{\beta\dot\alpha}\partial_m A^I(x) \tag{7.206d}$$

$$[F^I(x), Q_\alpha] = 0 \tag{7.206e}$$

$$[F^I(x), \bar{Q}_{\dot\alpha}] = \sqrt{2}(\partial_m \chi^I \sigma^m)_{\dot\alpha}. \tag{7.206f}$$

Here $A^I(x)$, $\chi^I(x)$ and $F^I(x)$ can be thought as elementary fields or they can be composite operators which span a chiral super-multiplet. Then let us consider the following Green function

$$\langle A^{I_1}(x_1)\cdots A^{I_n}(x_n)\rangle_0 \tag{7.207}$$

and let us the derivative ∂_m with respect to x_1 and contract with $-\sqrt{2}\sigma^m_{\alpha\dot\beta}$, we find

$$\begin{aligned}
&-\sqrt{2}\sigma^m_{\alpha\dot\beta}\partial^{x_1}_m\langle T(A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0 = -\sqrt{2}\sigma^m_{\alpha\dot\beta}\langle T(\partial_m A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0 + \text{eq. t. comm.} =\\
&= -\sqrt{2}\sigma^m_{\alpha\dot\beta}\langle T(\partial_m A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0 = \langle T(\{\chi^{I_1}_\alpha(x_1),\bar Q_{\dot\beta}\}A^{I_2}(x_2)\cdots A^{I_n}(x_n))\rangle_0 =\\
&= \langle 0|\bar Q_{\dot\beta}T(\chi^{I_1}_\alpha(x_1)A^{I_2}(x_2)\cdots A^{I_n}(x_n))|0\rangle + \langle 0|T(\chi^{I_1}_\alpha(x_1)A^{I_2}(x_2)\cdots A^{I_n}(x_n))\bar Q_{\dot\beta}|0\rangle = 0.
\end{aligned} \tag{7.208}$$

This follows immediately from the fact the $\bar Q_{\dot\alpha}$ annihilates the vacuum and from the vanishing of the commutator $[A^I,\bar Q]$ and of the extra equal time commutators arising from ∂_m acting on the $\theta-$functions of the time-ordering. Since σ^m are set of independent matrices, we have

$$\langle \partial_m A^{I_1}(x_1)\cdots A^{I_n}(x_n)\rangle_0 = 0, \tag{7.209}$$

i.e. the correlation function does not depend on x_1. In the same way one can show that this correlator does not depend on any of the coordinates x_i. Taking the limit of large separation among the fields, we can apply the cluster property and we can conclude

$$\langle A^{I_1}(x_1)\cdots A^{I_n}(x_n)\rangle_0 = \langle A^{I_1}\rangle_0\langle A^{I_2}\rangle_0\cdots\langle A^{I_n}\rangle_0. \tag{7.210}$$

The next step is to see how the supersymmetry constrains the dependence of the above correlation functions on the couplings appearing in the superpotential. Suppose that our superpotential contains a term of the form

$$\lambda\int d^4x d^2\theta\Phi_0 + h.c. \tag{7.211}$$

where Φ_0 is a composite chiral superfield. We want to analyze the depence on $\bar\Lambda$ of the above correlator

$$\begin{aligned}
&\frac{\partial}{\partial\bar\lambda}\langle T(A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0 = \langle T(\int d^4x d^2\bar\theta\bar\Phi_0 A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0 =\\
&= \int d^4x\langle T(\mathcal{F}_0(x)A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0
\end{aligned} \tag{7.212}$$

Here F_0 is the highest component of the antichiral supermultiplet Φ_0.
Since $\bar F_0(x) = -\frac{i}{2\sqrt{2}}\{\bar Q_{\dot\alpha},\bar\chi^{\dot\alpha}\}$

$$\frac{\partial}{\partial\bar\lambda}\langle T(A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0 = -\frac{i}{2\sqrt{2}}\int d^4x\langle T(\{\bar Q_{\dot\alpha},\bar\chi^{\dot\alpha}\}A^{I_1}(x_1)\cdots A^{I_n}(x_n))\rangle_0 = 0 \tag{7.213}$$

For the same argument given in the other case. Therefore this correlation function can depend only holomorphically on the coupling appearing in the (microscopic) superpotential. Exploiting also the other WI we can conclude that given any composite chiral superfield Φ its expectation value on the vacuum $\langle\Phi\rangle_0$ is an holomorphic function of the couplings.

7.4 Renormalization properties of the WZ and NLS model: Non-renormalization theorems

In this section we shall try to investigate the property of WZ and NLS models under renormalization. We shall look at this question from a modern point of view, namely in terms of *low energy (or Wilsonian) effective action.* For a given physical system, it is a *local* [6] action, but potentially with an infinite number of couplings, which is suitable to descrive the relevant degrees of freedom below a certain scale of energy given by a *cut-off* μ.
In high energy the typical example of this situation is realized by the *the chiral effective action* for QCD, theories describing the strong interactions at energy below Λ_{QCD} in terms of pions. This example also illustrates the common feature in the effective theories that low energy degrees of freedom (pions) are very different from the degrees of freedom of the fundamental theory.
Given the theory at scale μ_0, the *effective theory* at lower scale μ is obtained by integrating out all the fluctuation in the range of energy $\mu < E < \mu_0$. The resulting action can be expanded as a (potentially) infinite sum of local operator

$$S_\mu = \int d^4x \sum_i g_i(\mu)\mathcal{O}_i(x). \tag{7.214}$$

This local expansion is meaningful on length scale of the order $1/\mu$ and it describes the processes in a unitary way up to energy less than μ. The effect of integrating out the modes the modes between $\mu - d\mu$ and μ can be described by an infinite set of differential equations governing the couplings in (7.214)

$$\mu\frac{dg_i}{d\mu} = \beta_i(g_k,\mu). \tag{7.215}$$

If there is a point g_k^0 in the space of coupling such that $\beta_i(g_k^0,\mu) = 0$, we shall call this point *fixed point.* With this value of the couplings the theory does not change with the scale. Such a theory is naturally called *scale invariant theory.* Suppose now to expand the above equation around the fixed point g_k^0. At the lowest order, we find

$$\mu\frac{dg_i(\mu)}{d\mu} = \frac{\partial\beta_i}{\partial g_j}(g_k^0,\mu)(g_j(\mu) - g_k^0). \tag{7.216}$$

[6] We shall explain below in which sense we are using the adjective local

We can always redefine our basis of local operator so that the matrix $\beta_{ij}(g^0,\mu) = \frac{\partial \beta_i}{\partial g_j}(g^0_k,\mu)$ is diagonal $\beta_{ij}(g^0,\mu) = \lambda_i \delta_{ij}$, then the above system is easily solved

$$\mu \frac{dg_i(\mu)}{d\mu} = \lambda_i (g_i(\mu) - g_i^0) \quad \Rightarrow \quad g_i(\mu) = g_i^0 + \left(\frac{\mu}{\mu_0}\right)^{\lambda_i} c_i \tag{7.217}$$

when when we decrease μ, if $\lambda_i > 0$, $g_i(\mu)$ is driven to g_i^0, then the eigenvalue is said to be IRRELEVANT in the IR; if $\lambda_i < 0$ $g_i(\mu)$ is driven away from g_0^i, and λ_i is called RELEVANT; finally if $\lambda_i = 0$, $g_i(\mu)$ is fixed at g_i^0 and λ_i is MARGINAL.

Consider the case when the fixed point in the IR is a free theory (gaussian fixed point). This, for example, occurs for any theory containing only scalars, spinors and U(1) fields if the the interactions are sufficiently small (Coleman-Gross Theorem). A free theory is scale invariant when we choose the following scaling for the fields

$$\phi \mapsto \left(\frac{\mu}{\mu_0}\right) \phi, \quad \psi \mapsto \left(\frac{\mu}{\mu_0}\right)^{3/2} \psi, \quad V_\mu \mapsto \left(\frac{\mu}{\mu_0}\right) V_\mu \tag{7.218}$$

Then any operator built out of these fields will scale with its mass dimension Δ_i, *i.e.* $\mathcal{O}_i(x) \mapsto s_i^{\Delta_i} \mathcal{O}_i(x)$. Therefore the scaling of the interaction associated to this operator $\mathcal{O}_i(x)$ in the effective action is

$$\int d^4x \mathcal{O}_i(x) \mapsto \left(\frac{\mu}{\mu_0}\right)^{\Delta_i - 4} \int d^4x \mathcal{O}_i(x). \tag{7.219}$$

We immediately see that in the infrared the operator with $\Delta_i > 4$ are irrelevant, those with $\Delta_i < 4$ are relevant while $\Delta_i = 4$ are marginal. [Strictly speaking, we are assuming that the quantum fluctuations are not not so large to destroy the free scaling. Namely, they are small enough not to alter the qualitative picture implied by (7.218) and (7.219). However the marginal operators are in delicate situation. It is sufficient a small perturbation to change their status: they might become relevant or irrelevant. An operator which is still marginal after the inclusion of the quantum correction is said exactly marginal.]

The above analysis shows us that the physics in the infrared is dominated by relevant and marginal operator, while the contribution of irrelevant operator can be consistently neglected. The fortune and the power of effective action is that the number of relevant and marginal operators that we can write for a given problem is in general limited. This means that a good description of our physical system below a certain scale μ will only require the inclusion of limited number of terms in S_μ.

In the case of a free fixed point in the infrared we shall choose to parameterize our effective theory as follows:

$$S_\mu = S_{free} + \sum_i \int d^4x \mu^{\Delta_i - 4} g_i(\mu) \mathcal{O}_i(x). \tag{7.220}$$

Here we have factored out the mass dimension of the operator $\mathcal{O}_i(x)$ and the coupling constants $g_i(\mu)$ get correction from loops of virtual particles with energy in the range $\mu < E < \mu_0$. Since the integration region is limited the quantum correction does not suffer neither UV or IR divergences, even in the presence of massless particles. This should be contrasted with the usual $1-$PI action which is strongly affected by these divergences.
For the moment we have just considered the case when the far IR is described by a free field theory, but one can encounter other possibilities:

- Consider a theory where all excitations are massive. When our scale μ is below the mass of the lightest excitation, there is no propagating degree of freedom and the system is frozen. We have a **trivial** effective action: there is no propagation to be described.

- There are surviving massless degrees of freedom in the infrared which interact non-trivially. The effective action in the IR is given by **an interacting conformal field theory**.

Constraining $\mathbf{S}_\mu$: holomorphicity and symmetries.

holomorphicity: We have just discussed how to construct an action that describes a physical system below a certain scale μ starting from a microscopic theory valid at higher scale μ_0. If the microscopic theory is generic and it does not possess any particular property, the resulting Wilsonian action S_μ will be a mess containing all sort of terms. There is no systematic way to predict the structure of S_μ.
In this respect supersymmetric field theories are quite special. If we assume that the supersymmetry is not spontaneously broken when μ flows in the infrared, the form of S_μ must obey to very strict constraints. Consider, for example, a supersymmetric NLSM which describe the physics at certain scale μ_0 (microscopic theory), we want to follow its flow when μ_0 is lowered to μ.
To begin with, we shall assume that the physics at the scale μ *is still described by a NLS model* (macroscopic theory) with a specified set of light chiral, which is not necessarily a simple subset of those of the microscopic action at the scale μ_0. We have no derivation of this assumption. We can only check if it gives a self-consistent prediction.
Now we want to compare the structure of the superpotentials present in the two actions. On one side there is $\mathcal{F}_{\mu_0}(g_i, \Phi^I)$, on the other side we have the macroscopic superpotential $\mathcal{F}_\mu$, which potentially depends on certain superfields $\hat{\Phi}^I$, describing the light degrees of freedom, on the couplings g_i and $\bar{g}_i$ and on the scale μ.
Property 1. [**Holomorphicity**] *The superpotential $\mathcal{F}_\mu$ does not depend on $\bar{g}_i$, namely it is an holomorphic function of the couplings appearing in the microscopic superpotential.*

In order to prove this, we shall use a trick let us promote the couplings g_i to chiral superfields G_i, then $\mathcal{F}_{\mu_0}(g_i, \Phi^I) \mapsto \mathcal{F}_{\mu_0}(G_i, \Phi^I)$. The original theory is recovered when G_i is chosen to be constant. Since supersymmetry imposes that a chiral superfield can appear only holomorphically in a superpotential, $\mathcal{F}_\mu$ can depend on G_i and not on $\bar{G}_i$. Setting G_i to be a constant we find the property 1.
This trick might seems unusual and somewhat strange. However it is just the supersymmetric version of the familiar technique used in QM to get selection rules. Consider the case of the Stark effect, where we have rotationally invariant system, the Hydrogen atom, subject to a non-invariant perturbation $\propto \vec{E}\cdot\vec{x}$. The dependence of the energy splitting on the background electric field cannot be arbitrary. Indeed the simple remark that $\vec{E}$ can be considered a vector and not simply three constant in the Hamiltonian produces selection rules for the possible contribution of $\vec{E}$ (Wigner-Eckart theorem).

symmetry: R-SYMMETRY. Further constraint on the form of the superpotential $\mathcal{F}_\mu$ can come from the the bosonic symmetries of the microscopic theory. In supersymmetric theories, a special role is played by the bosonic symmetries that commutes with the Poincarè generators, but they do not with the supersymmetry charges:

$$[B_\ell, Q^I_\alpha] = (S_\ell)^I{}_L Q^L_\alpha \qquad [B_\ell, \bar{Q}^I_{\dot\alpha}] = -(S^*_\ell)^I{}_L \bar{Q}^L_{\dot\alpha}, \tag{7.221}$$

where S_l is an hermitian matrix. In a theory with just one supersymmetry, the above commutation relation reduces to

$$[B_\ell, Q_\alpha] = S_\ell Q_\alpha \qquad [B_\ell, \bar{Q}_{\dot\alpha}] = -S_\ell \bar{Q}^L_{\dot\alpha}, \tag{7.222}$$

with S_ℓ a real number. The B_ℓ for which S_ℓ does not vanish must generate abelian $U(1)$ symmetries. In fact

$$[B_r, [B_s, Q_\alpha]] + [B_s, [Q_\alpha, B_r]] + [Q_\alpha, [B_r, B_s]] = 0 \quad \Rightarrow \quad f_{rs}{}^k S_k = 0. \tag{7.223}$$

This, in turn, implies that the vector $\{S_k\}$ belongs to the kernel of the Killing form and thus its entries can be different from zero only in the abelian sector of the bosonic internal symmetry. Then we can define at most a single (independent) $U(1)$ generator with the following properties

$$[R, Q_\alpha] = -Q_\alpha \qquad [R, \bar{Q}_{\dot\alpha}] = \bar{Q}^L_{\dot\alpha} \tag{7.224}$$

The generator R is given by $R = -\frac{\sum_\ell S_\ell B_\ell}{\sum S_\ell S_\ell}$. This particular $U(1)$ is called **R-symmetry**. Determining the action of this symmetry on the chiral superfields requires a small generalization of our previous approach. In the presence of an $R-$symmetry the superspace is defined by

$$\text{Superspace}_{\text{N=1}} = \frac{N = 1 \text{ Poincaré supergroup}}{\text{Lorentz Group} \times \text{R-symmetry}}. \tag{7.225}$$

Consequently the action of the $R-$symmetry charges is defined by

$$\begin{aligned} &\exp(i\alpha R)\exp\left(-ix^m P_m + i(\theta Q + \bar{\theta}\bar{Q})\right) \sim \\ &\exp(i\alpha R)\exp\left(-ix^m P_m + i(\theta Q + \bar{\theta}\bar{Q})\right)\exp(-i\alpha R) = \\ =&\exp\left(-ix^m P_m + i(e^{-i\alpha}\theta Q + e^{i\alpha}\bar{\theta}\bar{Q})\right), \end{aligned} \tag{7.226}$$

and on a chiral superfield it is given by

$$\Phi \mapsto \Phi'(\theta', y') = \Phi(e^{-i\alpha}\theta, y)e^{in\alpha}, \tag{7.227}$$

where we have assumed that the superfield possesses a global supercharge n. This means that the component fields transforms as follows

$$\phi(x) \mapsto \phi'(x') = e^{n\alpha}\phi(x) \quad \chi(x) \mapsto \chi'(x') = e^{(n-1)\alpha}\chi(x) \quad F(x) \mapsto F'(x') = e^{(n-2)\alpha}F(x) \tag{7.228}$$

These transformations are consistent with the following assignment for the $R-$symmetry charges of the superspace coordinates

$$R(\theta) = 1, \quad R(\bar{\theta}) = -1, \quad R(d\theta) = -1, \quad R(d\bar{\theta}) = 1. \tag{7.229}$$

Recall that $d\theta \sim \frac{\partial}{\partial\theta}$. Thus, in order to have an action that is invariant under $R-$symmetry the superpotential term must carry a $+2$ $R-$charge

$$\int d^4y \int \underset{-2}{d^2\theta} \underset{2}{\mathcal{F}} \quad \Rightarrow \quad R(\mathcal{F}) = 2. \tag{7.230}$$

Other $U(1)$ charges and the cubic superpotential in WZ model. Consider the case of one chiral superfield and the standard superpotential of the Wess-Zumino model defined at scale μ_0

$$\mathcal{F}_{\mu_0} = \frac{1}{2}m\mu_0\phi^2 + \frac{\lambda}{3}\phi^3 \tag{7.231}$$

We shall promote the mass m to a chiral superfield M and the coupling constant to a chiral superfield Λ. Then

$$\mathcal{F}_{\mu_0} = \frac{1}{2}M\mu_0\phi^2 + \frac{\Lambda}{3}\phi^3 \tag{7.232}$$

We shall assume that $\Lambda \mapsto \Lambda' = e^{iq_\Lambda\alpha}\Lambda$ and $M \mapsto M' = e^{iq_M\alpha}M$ and $\Phi \mapsto \Phi' = e^{in\alpha}\Phi$, then

$$\begin{aligned} &\int d^2\theta\mathcal{F}_{\mu_0} \mapsto \int d^2\theta'\mathcal{F}'_{\mu_0} = \int d^2\theta'(\frac{1}{2}M'\mu_0\phi'^2 + \frac{\Lambda'}{3}\phi'^3) = \\ &= \int d^2\theta\left[e^{i(-2+q_M+2n)\alpha}\frac{1}{2}M\mu_0\phi^2 + e^{i(-2+q_\Lambda+3n)\alpha}\frac{\Lambda}{3}\phi^3\right] \end{aligned} \tag{7.233}$$

Therefore, in order to have an action, which is invariant under $R-$symmetry we have to impose that $q_\Lambda = 2 - 3n$ and $q_M = 2 - 2n$.

Notice that is possible to define an additional $U(1)$ under which the coordinate of the superspace do not transform and the superpotential is uncharged. It is sufficient to choose $q_\Lambda = -3n$ and $q_M = -2n$. Then we have the following set of charges

	$U_R(1)$	$U(1)$
Φ	n	n
M	$2-2n$	$-2n$
Λ	$2-3n$	$-3n$

Can we fix a sensible value of n? We can select n by imposing the $U_R(1)$ does not possess neither gravitational nor gauge anomaly

$$\begin{aligned} &\textbf{gravitational}: \quad n+2-2n+2-3n = 4-4n \quad \Rightarrow \quad n=1 \\ &\textbf{gauge}: \qquad\qquad n^3+(2-2n)^3+(3-2n)^3 = 0, \quad \text{which vanishes for } n=1. \end{aligned} \tag{7.234}$$

With this choice we have the standard table

	$U_R(1)$	$U(1)$
Φ	1	1
M	0	-2
Λ	-1	-3

We want to find the superpotential at a lower scale μ. We shall assume that the theory is described by the same chiral superfield. The superpotential must obey the $U_R(1)$ symmetry and the additional $U(1)$. Then w hat are the possible invariant monomials that we can construct with the fields? The invariance of $\Lambda^\alpha M^\beta \Phi^\gamma$ imposes

$$\gamma n-(2n-2)\beta-\alpha(3n-2)=0 \;\text{ and }\; \gamma n-2n\beta-3n\alpha=0 \quad \Rightarrow \alpha=\gamma \;\text{ and }\; \beta=-\gamma. \tag{7.235}$$

Thus all the monomials of the form are $\left(\frac{\Lambda\Phi}{M}\right)^\gamma$ are invariant. This, in turn, implies that any function $f\left(\frac{\Lambda\Phi}{M}\right)$ will be unaltered by the above transformation.

Since the super-potential must have $R-$charge 2, let us also find all the monomials with this property. The constraints are

$$\gamma n-(2n-2)\beta-\alpha(3n-2)=2 \;\text{ and }\; \gamma n-2n\beta-3n\alpha=0 \quad \Rightarrow \alpha=-2+\gamma \;\text{ and }\; \beta=3-\gamma. \tag{7.236}$$

Therefore all the monomial with the right $R-$charge are $\Lambda^{\gamma-2}M^{3-\gamma}\Phi^\gamma = M\Phi^2\left(\frac{\Lambda\Phi}{M}\right)^{\gamma-2} \sim M\Phi^2$. The most general superpotential obeying the above symmetries is then

$$\mathcal{F}_\mu = M\mu\Phi^2 f\left(\frac{\Lambda\Phi}{M\mu}\right) \qquad \Rightarrow \qquad \mathcal{F}_\mu = m\mu\Phi^2 f\left(\frac{\lambda\Phi}{m\mu}\right) \tag{7.237}$$

Since $\mathcal{F}_{\mu_0}$ is an holomorphic we can expand it in a Laurent-series

$$\mathcal{F}_\mu = \sum_{k=-\infty}^{\infty} a_k (m\mu)^{1-k}\lambda^k\Phi^{2+k} \tag{7.238}$$

The explicit form of the function f can be restricted if we made the following assumptions

Assumption 1: *Smoothness of the limit* $\lambda \to 0$. This assumption imposes $k \geq 0$

Assumption 2: *Smoothness of the limit* $m \to 0$. More correctly we are taking λ and m to 0 with the additional requirement that $\frac{m}{\lambda} \mapsto 0$.The Wilsonian action is regular in this limit and thus this requires $k \leq 1$.
With this restriction the above superpotential (7.237) reduces to

$$\mathcal{F}_\mu = a_0(\mu) m\mu\Phi^2 + a_1(\mu)\lambda\Phi^3. \tag{7.239}$$

Then we have the following effective action

$$S_\mu = \frac{1}{2}\int d^4x \int d^2\theta d^2\bar{\theta} Z\ [\Phi^\dagger\Phi + \cdots] + \int d^4x \left[\int d^2\theta (a_0(\mu)m\mu\Phi^2 + a_1(\mu)\lambda\Phi^3) + c.c.\right] \tag{7.240}$$

In order to determine $a_0(\mu)$ and $a_1(\mu)$, let us compare the prediction of the two theories. If we consider the limit $\lambda \to 0$, we reach the free theory then the mass in the microscopic and macroscopic theory have to be the same, this implies

$$a_0(\mu) = \frac{1}{2}\frac{\mu_0}{\mu} Z. \tag{7.241}$$

[Recall that we have to normalize canonically the kinetic term to obtain the mass.] But since we are approaching a free theory $Z = \mu/\mu_=0$ and consequently. $a_0 = 1/2$. To obtain a_1, we shall impose that at tree level the two theories must give the same prediction. This immediately fixes $a_1 = 1/3$. Summarizing

$$\mathcal{F}_\mu = \frac{1}{2} m\mu\Phi^2 + \frac{1}{3}\lambda\Phi^3. \tag{7.242}$$

PROPERTY 2. [**Holomorphicity**] *In the WZ model the superpotential* $\mathcal{F}_\mu$ *is not renormalized.*
EXTENSION OF THE PROOF TO ANY SUPERPOTENTIAL. Consider now the case of a generic superpotential in the microscopic theory $\mathcal{F}(\Phi_i, \mu_0)$. A useful trick it is to replace the above superpotential with $Y\mathcal{F}(\Phi_i, \mu_0)$, where Y is a chiral superfield. The original theory is then recovered for $Y = 1$. The enlarged theory possesses an $U_R(1)$ symmetry with the following assignments for the charges

$$R[Y] = 2 \text{ and } R[\Phi_i] = 0. \tag{7.243}$$

Assume now that the superpotential depends on the same set of chiral superfield at the scale μ as well. Then holomorphicity and $R-$symmetry implies that the superpotential must have the following form

$$Y\mathcal{W}(\Phi_i, \mu) \tag{7.244}$$

As $Y \mapsto 0$, we approach a free theory and the UV and IR action must match in perturbation theory. This immediately implies that

$$\mathcal{W}(\Phi_i, \mu) = \mathcal{F}(\Phi_i, \mu). \tag{7.245}$$

Notice that the hypothesis the the IR and the UV theory are described by the same degree of freedom appear to be self-consistent. This also is an agreement with the Coleman-Gross theorem that the scalar/spinor theory are infrared free.

THE KAHLER POTENTIAL. We have seen that the renormalization of the superpotential is severely constrained, one might wonder whether something special occurs for the Kahler potential $\mathcal{K}(\Phi, \Phi^\dagger)$. Unfortunately the answer is not. There is no particular constraint for dependence of the renormalized $\mathcal{K}(\Phi, \Phi^\dagger)$ on the coupling of the microscopic theory. Both g_i and $\bar{g}_i$ can appear. For example a simple one-loop computation shows that the wave function renormalization in the WZ model yields

$$Z = 1 + \#|g|^2 \log\left(\frac{\mu_0^2}{\mu^2}\right). \tag{7.246}$$

7.5 Integrating out (and in)

For the moment we have just examined cases where the IR degrees of freedom coincide with the UV ones. Consider now the following superpotential with two chiral superfields

$$W = \frac{M}{2}\Phi_H^2 + \frac{g}{2}\Phi_H\Phi_0^2 \tag{7.247}$$

We want to integrate out all the modes down to $\mu < M$. At this energy the superfield Φ_H is no longer dynamical and it can be integrated out.

INTEGRATING USING SYMMETRIES. First of all consider the case of $U_R(1)$, the invariance of the action imposes the following constrains for the charges

$$q_M + 2q_H - 2 = 0 \quad \text{and} \quad q_g + q_H + 2q_0 - 2 = 0. \tag{7.248}$$

For a generic $U(1)$ we have the following constrain for the charges

$$\hat{q}_M + 2\hat{q}_H = 0 \quad \text{and} \quad \hat{q}_g + \hat{q}_H + 2\hat{q}_0 = 0. \tag{7.249}$$

Therefore apart from the $U_1(R)$, we can define two additional $U(1)$: $U_A(1)$ and $U_B(1)$. The solution of the constrains (7.249) depends on two free parameters. The above constraints can be, for example, by the following assignments of charges (a different choice will not affect the final result)

	$U_R(1)$	$U_A(1)$	$U_B(1)$
Φ_H	1	1	0
Φ_0	1/2	0	1
M	0	-2	0
g	0	-1	-2

Consider now the invariant monomial we can construct out of M, g and Φ_0. The invariance under $U_B(1)$ selects: $M^\alpha(g\Phi_0^2)^\beta$; that under $U_A(1)$ imposes $\left(\frac{g\Phi_0^2}{\sqrt{M}}\right)^\beta$. Finally, the $U_R(1)$ fixes $\beta = 2$. Therefore the only possible superpotential compatible with the symmeetries is

$$a_0(\mu)\frac{g^2}{M}\Phi_0^4. \tag{7.250}$$

Comparing the three level perturbation theory we have $a_0(\mu) = -1/8$. This result can be checked by integrating out the field Φ_H through its equation of motion

$$\frac{\partial\mathcal{F}}{\partial\Phi_H} = M\Phi_H + \frac{g}{2}\Phi_0^2 = 0 \quad \Rightarrow \quad \Phi_H = -\frac{g}{2M}\Phi_0^2 \quad \Rightarrow \quad \mathcal{F} = -\frac{g^2}{8M}\Phi_0^4. \tag{7.251}$$

Making the example more interesting. Let us consider what happens if we add a cubic term in Φ_H^3

$$\mathcal{F} = \frac{M}{2}\Phi_H^2 + \frac{g}{2}\Phi_H\Phi_0^2 + \frac{y}{6}\Phi_H^3 \tag{7.252}$$

We want to integrate out all the modes down to $\mu < M$. It appears natural to eliminate Φ_H from the dynamical degrees of freedom and write a superpotential only for Φ_0. The previous symmetry extends to this case with this assignment for the charges

	$U_R(1)$	$U_A(1)$	$U_B(1)$
Φ_H	1	1	0
Φ_0	1/2	0	1
M	0	-2	0
g	0	-1	-2
y	-1	-3	0

Consider now the invariant monomial we can construct out of M, g, y and Φ_0. The invariance under $U_B(1)$ selects: $M^\alpha y^\beta(g\Phi_0^2)^\gamma$; that under $U_A(1)$ imposes $M^{-\frac{\gamma}{2}-\frac{3\beta}{2}}y^\beta(g\Phi_0^2)^\gamma$. Finally, the $U_R(1)$ fixes $\beta = \gamma - 2$: $M^{-2\gamma+3}y^{\gamma-2}(g\Phi_0^2)^\gamma = \frac{M^3}{y^2}\left(\frac{gy\Phi_0^2}{M^2}\right)^\gamma$.Thus the most general superpotential compatible with the symmeetries is

$$\mathcal{F} = \frac{M^3}{y^2}f\left(\frac{gy\Phi_0^2}{M^2}\right). \tag{7.253}$$

The function cannot be determined by symmetry arguments. However it can be obtained by eliminating the field Φ_H through its equation of motion

$$\frac{\partial\mathcal{F}}{\partial\Phi_H} = M\Phi_H + \frac{g}{2}\Phi_0^2 + \frac{y}{2}\Phi_H^2 = 0 \quad \Rightarrow \quad \Phi_H = -\frac{M}{y}\left(1 \pm \sqrt{1 - \frac{gy\phi_0^2}{M^2}}\right). \tag{7.254}$$

Then we obtain

$$\mathcal{F} = \frac{M^3}{3y^2}\left[1 - \frac{3gy\phi_0^2}{2M^2} \pm \left(1 - \frac{gy\phi_0^2}{M^2}\right)\sqrt{1 - \frac{gy\phi_0^2}{M^2}}\right] \tag{7.255}$$

The dependence is the one suggested by the symmetries, but there is an unexpected singularity. The superpotential $\mathcal{F}$ has a brunch cut at $\phi_0^2 = \frac{M^2}{gy}$. What is the meaning of this singularity? Let us compute the mass of Φ_H

$$M_H^2 = \frac{\partial^2 \mathcal{F}}{\partial \phi_H^2} = M + y\Phi_H = \mp M\sqrt{1 - \frac{gy\phi_0^2}{M^2}} \tag{7.256}$$

At $\phi_0^2 = \frac{M^2}{gy}$, the superfield Φ_H becomes massless and it should not be integrated out. This is a lecture that we have to keep in mind: *the presence of such a singularity in the effective superpotential denote the appearance of massless modes around the singularity.*

7.6 Spontaneous supersymmetry breaking in WZ and NLS model

We have already stressed that we have a spontaneous breaking of the supersymmetry if and only if the energy of the vacuum is different from zero. In fact the susy algebra allosw us to write the hamiltonian as follows

$$H = P^0 = \frac{1}{4}[\{Q_1^I, \bar{Q}_{\dot{1}}^I\} + \{Q_2^I, \bar{Q}_{\dot{2}}^I\}] = \frac{1}{4}[\{Q_1^I, (Q_1^I)^\dagger\} + \{Q_2^I, (Q_2^I)^\dagger\}], \tag{7.257}$$

where we used that $(Q_{1,2}^I)^\dagger = \bar{Q}_{\dot{1},\dot{2}}^I$. Consequently the Hamiltonian is positive definite

$$\langle\psi|H|\psi\rangle = \frac{1}{4}\left[\big|\big|Q_1^I|\psi\rangle\big|\big|^2 + \big|\big|\bar{Q}_{\dot{1}}^I|\psi\rangle\big|\big|^2 + \big|\big|Q_2^I|\psi\rangle\big|\big|^2 + \big|\big|\bar{Q}_{\dot{2}}^I|\psi\rangle\big|\big|^2\right] \geq 0. \tag{7.258}$$

Let $|\Omega\rangle$ be the vacuum of a supersymmetric theory. If supersymmetry is spontaneously broken, there is at least one Q, which does not annihilate the vacuum, then

$$\langle\Omega|H|\Omega\rangle = \frac{1}{4}\left[\big|\big|Q_1^I|\Omega\rangle\big|\big|^2 + \big|\big|\bar{Q}_{\dot{1}}^I|\Omega\rangle\big|\big|^2 + \big|\big|Q_2^I|\Omega\rangle\big|\big|^2 + \big|\big|\bar{Q}_{\dot{2}}^I|\Omega\rangle\big|\big|^2\right] > 0. \tag{7.259}$$

Vice versa if the vacuum energy is different from zero, the above equation implies that there is at least one supersymmetric charge, which does not annihilate the vacuum. Namely, the supersymmetry is spontaneously broken.

This condition of spontaneous breaking can be stated by saying that the vacuum expectation value of the supersymmetry transformation of one of the field is different from zero:

$$\langle\Omega|\delta(\text{Field})|\Omega\rangle = \langle\Omega|\{\epsilon Q, \text{Field}\}|\Omega\rangle \neq 0. \tag{7.260}$$

It would be zero if the vacuum is invariant. The field enjoying this property cannot be a boson since its variation are fermions and a vev of fermions field will break Lorentz invariance. Then it must be a fermion. For a theory containing only chiral multiplets, then we must have

$$\langle\Omega|\delta_{\epsilon,\bar{\epsilon}}\chi_\alpha^I(x)|\Omega\rangle = \langle\Omega|i\sqrt{2}(\sigma^m\bar{\epsilon})_\alpha\partial_m A^I(x) + \sqrt{2}\epsilon_\alpha F^I(x)|\Omega\rangle \neq 0. \tag{7.261}$$

The first contribution $\partial_m A^I(x)$ vanishes for Poincarè invariance and we are left with the following vev $\langle\Omega|F^I(x)|\Omega\rangle \neq 0$. Therefore the supersimmetric is broken if and only if

$$\langle\Omega|F^I(x)|\Omega\rangle \neq 0. \tag{7.262}$$

This condition is equivalent to the requirement that the scalar potential,

$$V = \sum_I |F_I|^2, \tag{7.263}$$

does not possess a minimum of vanishing energy.
Similarly, the supersymmetry is not spontaneously broken if and only there exists a solution of the system of equations

$$F_I = 0 \qquad \text{admits a solution.} \tag{7.264}$$

Let us see investigate the generic spectrum of the theory in the presence of the breaking. We must write the mass matrices. For the fermions, we find

$$M_{1/2} = (F_{IJ}) = \left(\frac{\partial\mathcal{F}}{\partial A_I \partial A_J}\right), \tag{7.265}$$

while for the scalars we get

$$M_0^2 = \begin{pmatrix} \frac{\partial^2 V}{\partial A_I \partial \bar{A}_J} & \frac{\partial^2 V}{\partial A_I \partial A_J} \\ \frac{\partial^2 V}{\partial \bar{A}_I \partial \bar{A}_J} & \frac{\partial^2 V}{\partial \bar{A}_I \partial A_J} \end{pmatrix} = \begin{pmatrix} \sum_K \bar{F}_{KJ} F_{KI} & \sum_K \bar{F}_K F_{KIJ} \\ \sum_K F_K \bar{F}_{KIJ} & \sum_K \bar{F}_{KI} F_{KJ} \end{pmatrix}. \tag{7.266}$$

A non-supersymmetric vacuum solves

$$\frac{\partial V}{\partial A_I} = \sum_K \mathcal{F}_{IK} \bar{\mathcal{F}}_K = 0 \qquad \text{but not} \;\; \mathcal{F}_I = 0. \tag{7.267}$$

This means that $\bar{\mathcal{F}}_K$ is not vanishing and it is a non-trivial element of the kernel of the matrix F_{IJ}. In other words the fermion $\psi = \sum \bar{F}_I \chi_I$ is massless and take the name of *goldstino*. This is the hallmark of spontaneous supersymmetric breaking as the goldstone boson is for the usual bosonic symmetries.
However the spontaneous symmetry breaking cannot produce an arbitrary pattern for the masses. In fact supersymmetries imposes the following constraint on theory containing scalars and spinors

$$\mathrm{Tr}\left(M_0^2\right) - 2\mathrm{Tr}\left(M_{1/2}^2\right) = 2\sum_{IJ} \mathcal{F}_{IJ}\bar{\mathcal{F}}_{IJ} - 2\sum_{IJ} \mathcal{F}_{IJ}\bar{\mathcal{F}}_{IJ} = 0. \tag{7.268}$$

This mass formula in an abstract way can be rewritten as

$$\mathrm{STr}(\mathcal{M}^2) \equiv \sum_s (-1)^{2s}(2s+1)\mathrm{Tr}\left(M_s^2\right) = 0, \tag{7.269}$$

where STr is called supertrace and $\mathcal{M}^2$ denote the mass matrix of all degrees of freedom of the theory bosonic and fermionic. Note that the boson states contribute with a plus sign while fermion states with the minus sign, but all the states are weighted with the spin degeneracy.
The above supertrace formula also allows us to illustrate the generic pattern of the mass splitting in the spontaneous breaking. Let us apply the formula to a chiral multiplet. First of all, (7.269) implies that the two real components (X, Y) of the complex scalar A are no longer degenerate, otherwise all the states are still degenerate and there is no breaking. Then, if we parameterize the spectrum as follows $m_X^2 = m_\chi^2 + \Delta_1$, $m_Y^2 = m_\chi^2 + \Delta_2$, where m_χ^2 is the square mass of the fermion, we find

$$0 = m_X^2 + m_Y^2 - 2m_\chi^2 = \Delta_1 + \Delta_2 \quad \Rightarrow \Delta_1 = -\Delta_2 = \Delta, \tag{7.270}$$

see fig. below One of the scalar is always lighter than the fermion: which one depends on the

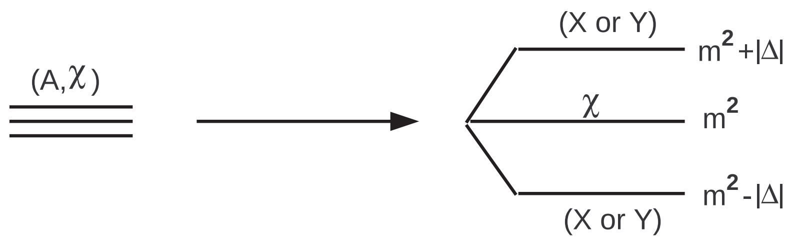

Figure 1: mass splitting in spontaneous supersymmetry breaking

sign of Δ. This is bad new for a phenomenological applictation: the selectron cannot be lighter than the electron.
A remark on the possibility of spontaneously breaking supersymmetry (SSB) is in order. Supersymmetry is unbroken if and only if

$$\mathcal{F}_I(A) = \frac{\partial \mathcal{F}}{\partial \Phi_I}(A) = 0, \quad \text{with} \;\; I = 1, \ldots, N. \tag{7.271}$$

These are N complex equation in N complex unknowns and so there will be *generically* a solution. *Generically* means that by making an arbitrary small change in the couplings we move from a theory with SSB to a theory with unbroken supersymmetry.
Restricting the form of the superpotential by imposing global symmetries improves the situations only partially. If the global symmetries commute with the supersymmetry charges, the presence of supersymmetric vacua is still a *generic* feature. The situation changes if we consider a potential invariant under $R-$symmetry, which spontaneously breaks $U_R(1)$. In fact if $U_R(1)$ is spontaneously broken, there is a charged scalar field, which takes an expectation value different

from zero on the vacuum. For simplicity, we shall assume that carries $R-$charge equal to 1 and we shall denoted with A_1. Since the superpotential is $U_R(1)$ invariant we can write it as

$$\mathcal{F}(A_1, A_2, \ldots, A_N) = A_1^2 f\left(\frac{A_2}{A_1^{q_2}}, \cdots, \frac{A_N}{A_1^{q_N}}\right) \equiv A_1^2 f(U_2, \ldots, U_N). \tag{7.272}$$

The conditions for having a supersymmetric minimum are

$$f(U_2, \ldots, U_N) = 0 \qquad \frac{\partial f}{\partial U_i}(U_2, \cdots, U_N) = 0, \tag{7.273}$$

where we have used $A_1 \neq 0$. These are N equations in $N-1$ unknowns. No solution *generically* exists. Therefore we have the following net result:

if the superpotential is a generic function constrained only by global symmetries supersymmetry is spontaneously broken if and only if there is a spontaneous breaking of the $R-$symmetry.

A FAMOUS EXAMPLE: THE O' RAIFEARTAIGH MODEL. Consider the WZ model containing three chiral superfield Φ_0, Φ_1 and Φ_2 with a superpotential of the following form

$$\mathcal{F} = \mu\Phi_0 + m\Phi_1\Phi_2 + g\Phi_0\Phi_1^2. \tag{7.274}$$

Among the renormalizable superpotentials, this superpotential is completely specified by the $\mathbb{Z}_2$ symmetry

$$P(\Phi_0) = \Phi_0 \quad P(\Phi_1) = -\Phi_1 \quad P(\Phi_2) = -\Phi_2 \tag{7.275}$$

and the by the following assignment of the supercharges

$$R(\Phi_0) = 2 \quad R(\Phi_1) = 0 \quad R(\Phi_2) = 2 \tag{7.276}$$

The supersymmetric vacua of the theory must solve the following three equations

$$\mathcal{F}_0 = \mu + gA_1^2 = 0 \tag{7.277}$$
$$\mathcal{F}_1 = mA_2 + 2gA_0A_1 = 0 \tag{7.278}$$
$$\mathcal{F}_2 = mA_1 = 0 \tag{7.279}$$

The last equation implies that A_1 must vanish. Then $\mathcal{F}_0 = \mu$ and $\mathcal{F}_1 = mA_2$, thus no supersymmetric solution exists. To find the actual vacuum, we have to minimize the potential $V = |\mathcal{F}_0|^2 + |\mathcal{F}_1|^2 + |\mathcal{F}_2|^2$

$$\frac{\partial V}{\partial A_0} = \bar{\mathcal{F}}_1 2gA_1 = 0 \tag{7.280}$$
$$\frac{\partial V}{\partial A_1} = \bar{\mathcal{F}}_0 2gA_1 + \bar{\mathcal{F}}_1 2gA_1 + \bar{\mathcal{F}}_2 m = 0 \tag{7.281}$$
$$\frac{\partial V}{\partial A_2} = \bar{\mathcal{F}}_1 m = 0 \tag{7.282}$$

Notice tha $A_1 = 0$ solve the first and the second equation since $\mathcal{F}_2 \propto A_1$. We have just left with $\bar{F}_1 = 0 \Rightarrow A_2 = 0$. Therefore there is a set of non-supersymmetric vacua with $(A_0, A_1, A_2) = (v_0, 0, 0)$ parameterized by the vev of A_0. Let us compute the spectrum of this model around these vacua. The mass matrix for the fermions is

$$M_f = \left(\frac{\partial \mathcal{F}}{\partial A_I \partial A_J}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2gv_0 & m \\ 0 & m & 0 \end{pmatrix} \tag{7.283}$$

whose eigenvalues are $\left\{0, gv_0 - \sqrt{m^2 + g^2 v_0{}^2}, gv_0 + \sqrt{m^2 + g^2 v_0{}^2}\right\}$. Thus the square of the fermionic masses are $\{0, m_{F1}^2, m_{F2}^2\} = \left\{0, \left(gv_0 - \sqrt{m^2 + g^2 v_0{}^2}\right)^2, \left(gv_0 + \sqrt{m^2 + g^2 v_0{}^2}\right)^2\right\}$. Reconstructing the mass of the scalars is a little bit more involved. We have the following mass matrix

$$M_s^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m^2 + 4g^2 v_0^2 & 2gmv_0 & 0 & 2g\mu & 0 \\ 0 & 2gmv_0 & m^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2g\mu & 0 & 0 & m^2 + 4g^2 v_0^2 & 2gmv_0 \\ 0 & 0 & 0 & 0 & 2gmv_0 & m^2 \end{pmatrix}. \tag{7.284}$$

If we diagonalize we obtain the following pattern for the scalar masses

$$\begin{aligned} &\left\{0, 0, \left(gv_0 - \sqrt{\frac{\mu^2}{4v_0^2} + m^2 + g^2 v_0^2 + g\mu}\right)^2 - \frac{\mu^2}{4v_0^2}, \left(gv_0 - \sqrt{\frac{\mu^2}{4v_0^2} + m^2 + g^2 v_0^2 - g\mu}\right)^2 - \frac{\mu^2}{4v_0^2},\right. \\ &\left.\left(gv_0 + \sqrt{\frac{\mu^2}{4v_0^2} + m^2 + g^2 v_0^2 - g\mu}\right)^2 - \frac{\mu^2}{4v_0^2}, \left(gv_0 + \sqrt{\frac{\mu^2}{4v_0^2} + m^2 + g^2 v_0^2 + g\mu}\right)^2 - \frac{\mu^2}{4v_0^2}\right\} \end{aligned}.$$

We have two real massless scalars and 4 real massive real scalar. To understand better the pattern of the breaking, consider, for example, the small μ limit, then the above expressions simplify to

$$\left\{0, 0, \left(m_{F1} \pm \frac{g\mu}{\sqrt{m^2 + g^2 v_0^2}}\right)^2, \left(m_{F2} \pm \frac{g\mu}{\sqrt{m^2 + g^2 v_0^2}}\right)^2\right\}.$$

and the general pattern becomes manifest. The two massless scalars correspond to the goldston mode associate to the $R-$symmetry breaking.

8 Vector superfield

The name vector superfield is actually misleading. It does not refer to the properties of transformation of the superfield under the Lorentz group, but to the field of maximal spin present in it. A more correct, but less used name is real superfield.

To begin with, we shall again consider the scalar superfield and we shall try to reduce the representation carried by this superfield following a different path. The generator of supersymmetry transformation is an hermitian operator:

$$(\epsilon Q + \bar{\epsilon}\bar{Q}); \tag{8.285}$$

thus it cannot alter the reality properties of a superfield. Therefore a constraint of the form

$$V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta}) \tag{8.286}$$

defines another invariant subspace (under supersymmetry transformations). Recalling that $(\lambda\psi)^\dagger = \bar{\lambda}\bar{\psi}$ and $(\lambda\sigma^n\bar{\psi})^\dagger = \psi\sigma^n\bar{\lambda}$ it is a straightforward exercise to show that the most general *real* superfield has the following form

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& C(x) + i\theta\chi(x) - i\bar{\chi}(x)\bar{\theta} + \frac{i}{2}H(x)\theta^2 - \frac{i}{2}H^\dagger(x)\bar{\theta}^2 - V_m(x)\theta\sigma^m\bar{\theta} + \\ &+ i\bar{\lambda}(x)\bar{\theta}\theta^2 - i\bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x), \end{aligned} \tag{8.287}$$

where the fields V_m, C, D are real. The information carried by this super-field is still redundant. In fact within the set of all real superfields there is the invariant subspace

$$\Lambda(x,\theta,\bar{\theta}) + \Lambda^\dagger(x,\theta,\bar{\theta}) \tag{8.288}$$

where $\Lambda(x,\theta,\bar{\theta})$ is a chiral superfield, *i.e.* $\bar{D}_{\dot{\alpha}}\Lambda(x,\theta,\bar{\theta}) = 0$. The real super-fields (8.288) are somehow trivial: they simply carry a replica of the chiral super-fields already discussed. We can eliminate this redundancy by introducing the following equivalence relation among vector super-fields

$$V(x,\theta,\bar{\theta}) \sim \tilde{V}(x,\theta,\bar{\theta}) \qquad \text{iff} \qquad V(x,\theta,\bar{\theta}) - \tilde{V}(x,\theta,\bar{\theta}) = \Lambda(x,\theta,\bar{\theta}) + \Lambda^\dagger(x,\theta,\bar{\theta}), \tag{8.289}$$

with $\bar{D}_{\dot{\alpha}}\Lambda(x,\theta,\bar{\theta}) = 0$. In order to implement this equivalence relation it is convenient to change the parametrization used for the vector super-field and to write

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& C(x) + i\theta\chi(x) - i\bar{\chi}(x)\bar{\theta} + \frac{i}{2}H(x)\theta^2 - \frac{i}{2}H^\dagger(x)\bar{\theta}^2 - V_m(x)\theta\sigma^m\bar{\theta} + \\ &+ i\theta^2\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^m\partial_m\chi(x)\right] - i\bar{\theta}^2\theta\left[\lambda(x) + \frac{i}{2}\sigma^m\partial_m\bar{\chi}(x)\right] + \\ &+ \frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) + \frac{1}{2}\Box C(x)\right). \end{aligned} \tag{8.290}$$

In this form the presence of the hidden *real* chiral super-field is more manifest. Now if we recall the complete expansion for the chiral super-field $\Lambda(x,\theta,\bar{\theta})$

$$\Lambda(x,\theta,\bar{\theta}) = A(x) + \sqrt{2}\psi(x)\theta + \theta^2 F(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) - \frac{i}{\sqrt{2}}\theta^2\partial_m\psi\sigma^m\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x),$$

the above equivalence relation in components reads

$$\begin{array}{lll} C(x) \sim \tilde{C}(x) & \text{if} & \tilde{C}(x) = C(x) + A(x) + A^\dagger(x) \\ \chi(x) \sim \tilde{\chi}(x) & \text{if} & \tilde{\chi}(x) = \chi(x) - i\sqrt{2}\psi(x) \\ H(x) \sim \tilde{H}(x) & \text{if} & \tilde{H}(x) = H(x) - 2iF(x) \\ V_m(x) \sim \tilde{V}_m(x) & \text{if} & \tilde{V}_m(x) = V_m(x) - i\partial_m(A(x) - A^\dagger(x)) \\ \lambda(x) \sim \tilde{\lambda}(x) & \text{if} & \tilde{\lambda}(x) = \lambda(x) \\ D(x) \sim \tilde{D}(x) & \text{if} & \tilde{D}(x) = D(x). \end{array} \tag{8.291}$$

This equivalence relation among different vector superfields is reminiscent of the usual gauge invariance. In particular, in the case of field V_m, it has exactly the form of a $U(1)$ gauge transformation whose parameter is given by the imaginary part of $A(x)$.
This naturally suggests to use $V(x,\theta,\bar{\theta})$ to describe the supersymmetric $U(1)$ gauge multiplet. However the (on-shell) vector multiplet must only contain a vector and a Majorana (or Weyl) spinor, while the superfield (8.290) appears to accomodate additional scalar and spinor fields. These degrees of freedom are actually unphysical and they can be eliminated by means of the *the gauge transformation* generated by the chiral superfield

$$\begin{aligned} \tilde{\Lambda}(x,\theta,\bar{\theta}) =& \frac{1}{2}(-C(x) + if(x)) - i\chi(x)\theta - \frac{i}{2}\theta^2 H(x) + \frac{i}{2}\theta\sigma^m\bar{\theta}\partial_m(-C(x) + if(x)) - \\ & - \frac{i}{\sqrt{2}}\theta^2\partial_m\psi\sigma^m\bar{\theta} + \frac{1}{8}\theta^2\bar{\theta}^2\Box(-C(x) + if(x)). \end{aligned} \tag{8.292}$$

In fact we find that

$$\begin{aligned} \hat{V}(x,\theta,\bar{\theta}) &= V(x,\theta,\bar{\theta}) + \tilde{\Lambda}(x,\theta,\bar{\theta}) + \tilde{\Lambda}^\dagger(x,\theta,\bar{\theta}) = \\ &= -V_m(x)\theta\sigma^m\bar{\theta} + i\theta^2\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x). \end{aligned} \tag{8.293}$$

where

$$V_m(x) \sim \tilde{V}_m(x) \qquad \text{if} \qquad \tilde{V}_m(x) = V_m(x) + \partial_m f(x), \tag{8.294}$$

and λ and D are gauge-invariant. This choice for the equivalence-class representative is known as "*Wess-Zumino gauge*". It contains four bosonic fields, a $U(1)$ field V_m ($3 = 4 - 1$ off-shell d.o.f) and one real scalar D, and a Weyl spinor λ (4 off-shell d.o.f). We cannot further reduce the representation without using the equations of motion.
The "*Wess-Zumino gauge*" does not only break the gauge invariance, but also supersymmetry. In fact

$$\delta_{\epsilon,\bar{\epsilon}}\hat{V}(x,\theta,\bar{\theta}) = (\epsilon Q + \bar{\epsilon}Q)\hat{V}(x,\theta,\bar{\theta}) \notin \text{Wess-Zumino gauge} \tag{8.295}$$

or explicitly

$$
\begin{aligned}
\delta_{\epsilon,\bar\epsilon}\hat V(x,\theta,\bar\theta) =& - V_m(x)\epsilon\sigma^m\bar\theta - V_m(x)\theta\sigma^m\bar\epsilon - i\theta\sigma^m\bar\theta\epsilon\sigma_m\bar\lambda(x) + i\theta\sigma^m\bar\theta\lambda\sigma_m\bar\epsilon - i\bar\theta^2\epsilon\lambda(x) + i\theta^2\bar\epsilon\bar\lambda + \\
&+ \epsilon\theta\bar\theta^2 D(x) + \bar\epsilon\bar\theta\theta^2 D(x) + \frac{i}{2}\bar\theta^2(\epsilon\sigma^\ell\bar\sigma^m\theta)\partial_\ell V_m(x) - \frac{i}{2}\theta^2(\bar\theta\bar\sigma^m\sigma^\ell\bar\epsilon)\partial_\ell V_m(x) - \\
&- \frac{1}{2}\bar\theta^2\theta^2(\epsilon\sigma^m\partial_m\bar\lambda(x) + \partial_m\lambda(x)\sigma^m\bar\epsilon).
\end{aligned} \tag{8.296}
$$

Now, let us perform the gauge transformation generated by the chiral supefield

$$
\begin{aligned}
\Lambda(y,\theta) =& - \bar\epsilon\bar\sigma^m\theta V_m(y) - i\theta^2\bar\epsilon\bar\lambda(y) = -\bar\epsilon\bar\sigma^m\theta V_m(x) - i\bar\epsilon\bar\sigma^m\theta\theta\sigma^n\bar\theta\partial_n V_m - i\theta^2\bar\epsilon\bar\lambda(x) = \\
=&\theta\sigma^m\bar\epsilon V_m(x) + \frac{i}{2}\theta^2\bar\theta\bar\sigma^n\sigma^m\bar\epsilon\partial_n V_m - i\theta^2\bar\epsilon\bar\lambda(x)
\end{aligned}
$$

on the vector superfield $\hat V(x,\theta,\bar\theta) + \delta_{\epsilon,\bar\epsilon}\hat V(x,\theta,\bar\theta)$. We find

$$
\begin{aligned}
\hat V' =&\hat V(x,\theta,\bar\theta) + \delta_{\epsilon,\bar\epsilon}\hat V(x,\theta,\bar\theta) + \delta_{\Lambda,\Lambda^\dagger}(\hat V + \delta_{\epsilon,\bar\epsilon}\hat V(x,\theta,\bar\theta)) = \\
=& - (V_m(x) + i\bar\epsilon\bar\sigma_m\lambda - i\bar\lambda(x)\bar\sigma_m\epsilon)\theta\sigma^m\bar\theta + i\theta^2\bar\theta\left(\bar\lambda(x) - F_{lm}\bar\sigma^{m\ell}\bar\epsilon - i\bar\epsilon D(x)\right) - \\
&- i\bar\theta^2\left(\lambda(x) - \epsilon\sigma^{\ell m}F_{\ell m} + i\epsilon D(x)\right)\theta + \frac{1}{2}\theta^2\bar\theta^2\left(D(x) - \epsilon\sigma^m\partial_m\bar\lambda(x) - \partial_m\lambda(x)\sigma^m\bar\epsilon\right).
\end{aligned} \tag{8.297}
$$

This superfield is again in the Wess-Zumino gauge. Therefore we can define a combination of the supersymmetry and gauge transformations, which leaves the Wess-Zumino gauge invariant. It corresponds to the following transformations

$$\delta V_m(x) = i\bar\epsilon\bar\sigma_m\lambda + i\epsilon\sigma_m\bar\lambda(x) \tag{8.298a}$$
$$\delta\lambda(x) = \sigma^{\ell m}\epsilon F_{\ell m}(x) + i\epsilon D(x) \tag{8.298b}$$
$$\delta D(x) = \bar\epsilon\bar\sigma^m\partial_m\lambda(x) - \epsilon\sigma^m\partial_m\bar\lambda(x) \tag{8.298c}$$

These transformations do not close the standard super-symmetry algebra. For example

$$
\begin{aligned}
[\delta_\xi,\delta\epsilon]V_m =& i\bar\epsilon\bar\sigma_m\sigma^{rs}\xi F_{rs}(x) - \bar\epsilon\bar\sigma_m\xi D(x) + \text{c.c.} - (\epsilon\leftrightarrow\xi) = 2iF_{mr}(\bar\epsilon\bar\sigma^r\xi - \bar\xi\bar\sigma^r\epsilon) = \\
=& \underbrace{2i\partial_m[V_r(\bar\epsilon\bar\sigma^r\xi - \bar\xi\bar\sigma^r\epsilon)]}_{\text{gauge transformation}} \underbrace{- 2i(\bar\epsilon\bar\sigma^r\xi - \bar\xi\bar\sigma^r\epsilon)\partial_r V_m}_{\text{translations}}
\end{aligned} \tag{8.299}
$$

The commutator does not simply yield a translation, but also a gauge transformation. This is not in contradiction the the supersymmetry algebra. In fact (8.298) are the composition of a supersymmetry and a field dependent gauge transformation. This additional contribution is responsible for the new term in the commutator. In this framework local gauge transformations and supersymmetry merged in a unique giant supergroup and they cannot be disentangled. This is price to be paid if we want to throw out of the game the fields C, H and χ.

This analysis suggests a reduced framework for the off-shell description of the gauge multiplet. We forget about the vector superfield and we consider, as a starting point, a multiplet given by

(V_m, λ, D) endowed with the super-transformations (8.298) and with the gauge $V_m \mapsto V_m + \partial_m f$. Then one looks for an action that is invariant under (8.298) without any reference to superspace. A final technical, but very useful remark on the Wess Zumino gauge. In this gauge we cannot consider arbitrary powers of the vector field. In fact the following equalities hold

$$\begin{aligned}
\hat{V}(x, \theta, \bar{\theta}) &= -V_m(x)\theta\sigma^m\bar{\theta} + i\theta^2\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x) \\
\hat{V}^2(x, \theta, \bar{\theta}) &= V_m(x)\theta\sigma^m\bar{\theta}V_n(x)\theta\sigma^n\bar{\theta} = -\frac{1}{2}V_m V^m\theta^2\bar{\theta}^2 \\
\hat{V}^3(x, \theta, \bar{\theta}) &= 0.
\end{aligned} \tag{8.300}$$

8.1 The action for the abelian vector superfield

We have stressed that the vector superfield has the correct matter content to describe the supersymmetric version of a $U(1)$ gauge theory. In the following we shall show how to construct an action for this superfield. This action will describe a $U(1)$ gauge field and a Majorana spinor. The first step is to construct a superfield carrying only the gauge invariant part of V_m: *i.e.* the analog of the field strength. This can be obtained by taking a certain number of covariant derivatived of V. To begin with, let us consider the action of D_α on V, then

$$D_\alpha\tilde{V} = D_\alpha(V + \Lambda + \bar{\Lambda}^\dagger) = D_\alpha V + D_\alpha\Lambda. \tag{8.301}$$

This remove the dependence on the antichiral part of the gauge transformation. Next, let us take derivative $\bar{D}_{\dot{\beta}}$,

$$\bar{D}_{\dot{\beta}}D_\alpha\tilde{V} = \bar{D}_{\dot{\beta}}D_\alpha V + \bar{D}_{\dot{\beta}}D_\alpha\Lambda = \bar{D}_{\dot{\beta}}D_\alpha V + \{\bar{D}_{\dot{\beta}}, D_\alpha\}\Lambda = \bar{D}_{\dot{\beta}}D_\alpha V - 2i\sigma^m_{\alpha\dot{\beta}}\partial_m\Lambda. \tag{8.302}$$

It is clear from the above result, that the superfield

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V, \tag{8.303}$$

is gauge invariant. A similar analysis show that

$$\bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2\bar{D}_{\dot{\alpha}}V. \tag{8.304}$$

is invariant as well. By the definition, the superfielsd W_α and $\bar{W}_{\dot{\alpha}}$ are chiral and antichiral respectively. However they are not independent since they are related from the following constraint

$$\bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = D^\beta W_\beta, \tag{8.305}$$

which follows directly from their definition. Finding their explicit expression is a straightforward exercise and we find

$$\begin{aligned} W_\alpha =& - i\lambda_\alpha(y) + \left(\delta_\alpha^\beta D(y) - \frac{i}{2}(\sigma^m\bar\sigma^n)_\alpha{}^\beta F_{mn}(y)\right)\theta_\beta + \theta^2(\sigma^n\partial_n\bar\lambda(y))_\alpha, \\ \bar W_{\dot\alpha} =& i\bar\lambda_{\dot\alpha}(\bar y) + \bar\theta^{\dot\beta}\left(\epsilon_{\dot\alpha\dot\beta}D(\bar y) + \frac{i}{2}\epsilon_{\dot\alpha\dot\rho}(\bar\sigma^n\sigma^m)^{\dot\rho}{}_{\dot\beta}F_{nm}(\bar y)\right) + \bar\theta^2(\partial_n\lambda(\bar y)\sigma^n)_{\dot\alpha}. \end{aligned} \tag{8.306}$$

Exercise: *Show the expressions* (8.306) *for* W_α *and* $\bar W_{\dot\alpha}$.

SOLUTION: Since we are computing gauge invariant quantities we can use any gauge for the field V. We shall choose the Wess-Zumino gauge and we shall write the superfield in terms of the variable $y = x + i\theta\sigma\bar\theta$ or of the variable $\bar y = x - i\theta\sigma\bar\theta$

$$\begin{aligned} \hat V(x,\theta,\bar\theta) =& -V_m(y)\theta\sigma^m\bar\theta + i\theta^2\bar\theta\bar\lambda(y) - i\bar\theta^2\theta\lambda(y) + \frac{1}{2}\theta^2\bar\theta^2\left(D(y) - i\partial_n V^n(y)\right) = \\ =& -V_m(\bar y)\theta\sigma^m\bar\theta + i\theta^2\bar\theta\bar\lambda(\bar y) - i\bar\theta^2\theta\lambda(\bar y) + \frac{1}{2}\theta^2\bar\theta^2\left(D(\bar y) + i\partial_n V^n(\bar y)\right). \end{aligned}$$

Then

$$\begin{aligned} \hat V_\alpha(y,\theta,\bar\theta) \equiv D_\alpha V(y,\theta,\bar\theta) =& - V_m(y)(\sigma^m\bar\theta)_\alpha + 2i\theta_\alpha\bar\theta\bar\lambda(y) - i\bar\theta^2\lambda_\alpha(y) + \\ &+ \bar\theta^2\left(\delta_\alpha^\beta D(y) - \frac{i}{2}(\sigma^m\bar\sigma^n)_\alpha{}^\beta F_{mn}(y)\right)\theta_\beta + \bar\theta^2\theta^2(\sigma^n\partial_n\bar\lambda(y))_\alpha, \end{aligned}$$

from which we get

$$\begin{aligned} W_\alpha =& - \frac{1}{4}\bar D^2 D_\alpha V = -\frac{1}{4}\bar D^2 D_\alpha \hat V = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta^{\dot\beta}}\hat V_\alpha(y,\theta,\bar\theta) = \\ =& - i\lambda_\alpha(y) + \left(\delta_\alpha^\beta D(y) - \frac{i}{2}(\sigma^m\bar\sigma^n)_\alpha{}^\beta F_{mn}(y)\right)\theta_\beta + \theta^2(\sigma^n\partial_n\bar\lambda(y))_\alpha. \end{aligned}$$

In the same way one shows the second result.

It is quite easy to write an invariant action by means of these two chiral fields. Since they already contain the first derivatives of the fields, the action can be only a real Lorentz invariant quadratic polynomial in W and $\bar W$. The only possibility is then

$$S = -\frac{1}{8}\int d^2\theta d^4x W^2 - \frac{1}{8}\int d^2\bar\theta d^4x \bar W^2. \tag{8.307}$$

The two contributions are given respectively by

$$\begin{aligned} -\frac{1}{8}\int d^4x d^2\theta W^2 =& - \frac{1}{8}\int d^4x\left(W^\alpha D^2 W_\alpha - D^\beta W^\alpha(D_\beta W_\alpha)\right) = \\ =& \frac{1}{8}\int d^4x\left(-4i\lambda\sigma^n\partial_n\bar\lambda + 2D^2 - F^{ab}F_{ab} - \frac{i}{2}\epsilon^{abcd}F_{ab}F_{cd}\right) \end{aligned} \tag{8.308}$$

and

$$\begin{aligned} -\frac{1}{8}\int d^4x d^2\bar\theta W^2 =& - \frac{1}{8}\int d^4x\left(\bar W_{\dot\alpha}(\bar D^2\bar W^{\dot\alpha}) - \bar D_{\dot\beta}\bar W_{\dot\alpha}(\bar D^{\dot\beta}\bar W^{\dot\alpha})\right) = \\ =& \frac{1}{8}\int d^4x\left(4i\partial_n\lambda(x)\sigma^n\bar\lambda(x) + 2D^2(x) - F_{ab}F^{ab} + \frac{i}{2}\epsilon^{abcd}F_{ab}F_{cd}\right), \end{aligned} \tag{8.309}$$

where we have used that

$$W_\alpha|_{\bar\theta,\theta=0} = -i\lambda_\alpha(x) \qquad \bar W_{\dot\alpha}\big|_{\bar\theta,\theta=0} = i\bar\lambda_{\dot\alpha}$$
$$D_\beta W_\alpha|_{\bar\theta,\theta=0} = \epsilon_{\alpha\beta}D(y) - i(\sigma^{mn})_{\alpha\beta}F_{mn}(x) \qquad \bar D_{\dot\beta}\bar W_{\dot\alpha}\Big|_{\bar\theta,\theta=0} = \epsilon_{\dot\alpha\dot\beta}D(\bar y) + i(\bar\sigma^{nm})\ {}_{\dot\beta\dot\alpha}F_{nm}(x)$$
$$D^2 W_\alpha\big|_{\bar\theta,\theta=0} = -4(\sigma^n\partial_n\bar\lambda(x))_\alpha \qquad \bar D^2\bar W_{\dot\alpha}\big|_{\bar\theta,\theta=0} = -4(\partial_n\lambda(x)\sigma^n)_{\dot\alpha}.$$

Then the action for the vector field is

$$\begin{aligned} S = &-\frac{1}{8}\int d^2\theta d^4x W^2 - \frac{1}{8}\int d^2\bar\theta d^4x \bar W^2 = \\ &= \int d^4x\left(\frac{i}{2}\partial_n\lambda(x)\sigma^n\bar\lambda(x) - \frac{i}{2}\lambda(x)\sigma^n\partial_n\bar\lambda(x) + \frac{1}{2}D^2(x) - \frac{1}{4}F_{ab}(x)F^{ab}(x)\right) = \\ &= \int d^4x\left(-i\lambda(x)\sigma^n\partial_n\bar\lambda(x) + \frac{1}{2}D^2(x) - \frac{1}{4}F_{ab}(x)F^{ab}(x)\right). \end{aligned} \tag{8.310}$$

This action describe a $U(1)$ gauge particle and massless Majorana fermion. The field is not dynamical. It has an algebraic Lagrangian and it can be eliminated by setting through its equation of motion which gives $D = 0$.

The action described in (8.311) is not the most general supersymmetric $U(1)$ action. In fact we can consider

$$S = -\frac{1}{8}\int d^2\theta d^4x W^2 - \frac{1}{8}\int d^2\bar\theta d^4x \bar W^2 - \xi\int d^2\bar\theta d^2\theta d^4x V. \tag{8.311}$$

The additional contribution is known as the **Fayet-Ilioupulos term**. It produces a linear term in the D field. At the moment its role can appear pointless, since the D is not dynamical. However it can and it will have role in the spontaneous of the supersymmetry.

9 Matter couplings and Non-abelian gauge theories

In the following we shall discuss how to couple a gauge vector superfield to a multiplet of supersymmetric matter. Since there is no fundamental difference between the abelian and non-abelian case, we shall consider directly the latter one. The procedure will also suggest how to construct the generalization of the kinetic term to the non-abelian case.

To begin with, we shall consider a chiral superfield Φ, namely a superfield such that $\bar D_{\dot\alpha}\Phi = 0$, and we shall assume that each component field transforms in the unitary representation R of a compact group $\mathcal{G}$, *i.e.*

$$\Phi \mapsto \Phi' = U(\lambda)\Phi = e^{-i\lambda}\Phi \qquad \text{con} \qquad \lambda = \lambda_a T^a_R, \tag{9.312}$$

The kinetic term $-\frac{1}{4}\Phi^\dagger\Phi$ possess is obviously invariant under these global transformations

$$\Phi^\dagger\Phi \mapsto \Phi'^\dagger\Phi' = \Phi^\dagger e^{i\lambda^\dagger}e^{-i\lambda}\Phi = \Phi^\dagger e^{i\lambda}e^{-i\lambda}\Phi = \Phi^\dagger\Phi. \tag{9.313}$$

since $\lambda^\dagger = \lambda$. We want to promote this global symmetry to a local one. In the superspace language, the constant hermitian matrix λ can be thought as a superfield which is both chiral and antichiral. In fact for a chiral superfield $\Lambda(y,\theta)$ satisfying the antichiral condition $D_\alpha\Lambda(y,\theta)=0$ as well, we find

$$0=\bar{D}_{\dot\alpha}D_\alpha\Lambda(y,\theta)=\{\bar{D}_{\dot\alpha},D_\alpha\}\Lambda(y,\theta)=-2i\sigma^m_{\alpha\dot\alpha}\partial_m\Lambda(y,\theta)\quad\Rightarrow\quad\Lambda(y,\theta)=\Lambda(\theta).\tag{9.314}$$

Consequently the condition $D_\alpha\Lambda(y,\theta)=D_\alpha\Lambda(\theta)=0$ requires that Λ is independent of θ, *i.e.* it is constant.

Then, to have a local transformation we shall replace the constant matrix λ with a local superfield $\Lambda(x,\theta,\bar\theta)=\Lambda^a(x,\theta,\bar\theta)T^a_R$. Since a gauge transformation must map a chiral into a chiral superfield we must impose that

$$\bar{D}_{\dot\alpha}(e^{-iq\Lambda(x,\theta,\bar\theta)}\Phi(y,\theta))=e^{-iq\Lambda(x,\theta,\bar\theta)}\Phi(y,\theta)\bar{D}_{\dot\alpha}\Lambda(x,\theta,\bar\theta)=0,\tag{9.315}$$

namely $\bar{D}_{\dot\alpha}\Lambda(x,\theta,\bar\theta)=0$. The superfield $\Lambda(x,\theta,\bar\theta)$ is chiral. Thus

$$\Phi\mapsto\Phi'(y,\theta)=e^{-i\Lambda(y,\theta)}\Phi(y,\theta)\qquad\Phi^\dagger\mapsto\Phi'^\dagger(\bar y,\bar\theta)=\Phi^\dagger(\bar y,\bar\theta)e^{i\Lambda^\dagger(y,\theta)}.\tag{9.316}$$

The kinetic term is no longer invariant

$$\Phi^\dagger\Phi\mapsto\Phi'^\dagger\Phi'=\Phi^\dagger e^{i\Lambda^\dagger(y,\theta)}e^{-i\Lambda(y,\theta)}\Phi,\tag{9.317}$$

since $\Lambda^\dagger(y,\theta)\neq\Lambda(y,\theta)$. We can recover the invariance by exploiting the vector superfield. In the abelian case V transforms as follows[7]

$$V\mapsto V'=V+i(\Lambda(y,\theta)-\Lambda^\dagger(y,\theta)),\tag{9.318}$$

and if we write this transformation in exponential form, we find

$$e^V\mapsto e^{V'}=e^{V+i(\Lambda(y,\theta)-\Lambda^\dagger(y,\theta))}=e^{-i\Lambda^\dagger(y,\theta)}e^Ve^{i\Lambda(y,\theta)}.\tag{9.319}$$

This ensures that the following kinetic term is invariant in the abelian case

$$\mathcal{L}=\Phi^\dagger e^V\Phi.\tag{9.320}$$

However we can also extend this result to the non-abelian case in a very simple way. We shall consider a matrix vector superfield $V=V^aT^a_R$ and we shall impose that this field transforms as follows

$$e^V\mapsto e^{V'}=e^{-i\Lambda^\dagger}e^Ve^{i\Lambda}.\tag{9.321}$$

[7] We have change convention with respect to the previous section. We have used as chiral superfield $i\Lambda$ instead of Λ.

under the non-abelian gauge transformation defined by the chiral superfield $\Lambda(y,\theta) = \Lambda^a(y,\theta)T^a_R$. With this choice the kinetic term (9.320) becomes invariant also when considering non abelian transformation.

What is the relation between (9.321) and the usual gauge transformations for the component fields?

Let us expand the gauge transformation (9.321) at the linear order in Λ by means of the following result on BCH formula

$$\begin{aligned} e^A e^B =& e^{A+\mathfrak{L}_{A/2}[B+\coth(\mathfrak{L}_{A/2})B]+O(B^2)} \\ e^B e^A =& e^A e^{-A} e^B e^A = e^A e^{e^{-\mathfrak{L}_A}B} = e^{A+\mathfrak{L}_{A/2}[e^{-\mathfrak{L}_A}B+\coth(\mathfrak{L}_{A/2})e^{-\mathfrak{L}_A}B]+O(B^2)} = \\ =& e^{A+\mathfrak{L}_{A/2}[-B+\coth(\mathfrak{L}_{A/2})B]+O(B^2)} = \end{aligned} \tag{9.322}$$

where $\mathfrak{L}_X Y = [X,Y]$ and $e^{-x} + \coth(x/2)e^{-x} = -1 + \coth(x/2)$. We find

$$e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda} = e^{-i\Lambda^\dagger} e^{V+i\mathfrak{L}_{V/2}[\Lambda+\coth(\mathfrak{L}_{V/2})\Lambda]+O(\Lambda^2)} = e^{V+i\mathfrak{L}_{V/2}[(\Lambda+\Lambda^\dagger)+\coth(\mathfrak{L}_{V/2})(\Lambda-\Lambda^\dagger)]+O(\Lambda^2)}, \tag{9.323}$$

which in turn implies

$$\begin{aligned} \delta V =& i\mathfrak{L}_{V/2}[(\Lambda+\Lambda^\dagger) + \coth(\mathfrak{L}_{V/2})(\Lambda-\Lambda^\dagger)] + O(\Lambda^2) = \\ =& i(\Lambda-\Lambda^\dagger) + i\left[V, \frac{\Lambda+\Lambda^\dagger}{2}\right] + \cdots . \end{aligned} \tag{9.324}$$

At the lowest order the transformation is identical to the abelian one. This suggests that we can choose the Wess-Zumino gauge also in the non abelian case. Unlike the abelian case, the relationship between the component fields of $V(x,\theta,\bar\theta)$ and $\Lambda(y,\theta)$ in the Wess-Zumino gauge fixing is nonlinear, due to the complicated form of (9.326). However the end result is the same: $V_{WZ}(x,\theta,\bar\theta)$ is as given in (8.293). Furthermore, as in the abelian case, the Wess-Zumino decomposition does not fix the gauge freedom. It only constrains the difference $i(\Lambda-\Lambda^\dagger)$, while the sum $(\Lambda+\Lambda^\dagger)$ is still an arbitrary quantity that can be used.

Let us analyze how this residual gauge transformation acts on the a vector super-field in the Wess-Zumino gauge. To preserve the $WZ-$gauge the only non vanishing component in $\Lambda-\Lambda^\dagger$ must be

$$i(\Lambda-\Lambda^\dagger) = \theta\sigma^m\bar\theta\partial_m f. \tag{9.325}$$

Then $\coth(\mathfrak{L}_{V_{WZ}/2})(\Lambda-\Lambda^\dagger) = (\Lambda-\Lambda^\dagger)$ because all higher terms in the Taylor vanishes since they are proportional to θ^3 or higher powers. Therefore in the WZ-gauge the residual gauge transformation linearize also in the superfield

$$\delta V_{WZ} = i(\Lambda-\Lambda^\dagger) + i\left[V_{WZ}, \frac{\Lambda+\Lambda^\dagger}{2}\right] + O(\Lambda^2) \tag{9.326}$$

Expanding this transformation, we find that V_m is a non abelian connection, while D and λ transform in the adjoint representation. Therefore the known and usual rules of transformation will become manifest only in the Wess-Zumino gauge. For a generic vector superfield, the gauge transformation are realized in a higly non-linear way.
This analysis exhausts the discussion of the coupling with chiral superfields. But we are still missing an action for the non abelian gauge superfield. There are many ways to construct this action, but we find very instructive to follow as much as possible the pattern used in the non supersymmetric case.
The fist step is to construct gauge covariant derivatives ∇_A with $A = (\alpha, \dot{\alpha}, m)$. They are defined by the property

$$\nabla_A(e^{-i\Lambda(y,\theta)}\Phi) = e^{-i\Lambda(y,\theta)}\nabla_A(\Phi) \tag{9.327}$$

Since the gauge transformation are realized by chiral field we have the immediate identification

$$\nabla_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}. \tag{9.328}$$

We cannot identify ∇_α with D_α. In fact

$$D_\alpha(e^{-i\Lambda(y,\theta)}\Phi) = D_\alpha(e^{-i\Lambda(y,\theta)})\Phi + e^{\Lambda(y,\theta)}D_\alpha\Phi \neq e^{-i\Lambda(y,\theta)}D_\alpha(\Phi). \tag{9.329}$$

However this mismatch can be easily resolved by defining the covariant derivative as follows

$$\nabla_\alpha = e^{-V}D_\alpha e^V. \tag{9.330}$$

In fact

$$\nabla'_\alpha\Phi' = e^{-V'}D_\alpha e^{V'}\Phi' = e^{-i\Lambda}e^{-V}e^{i\bar{\Lambda}}D_\alpha e^{-i\bar{\Lambda}}e^V e^{i\Lambda}e^{-i\Lambda}\Phi = e^{-i\Lambda}e^{-V}D_\alpha e^V\Phi = e^{-i\Lambda}\nabla_\alpha\Phi. \tag{9.331}$$

We shall define ∇_ℓ by setting

$$\{\nabla_\alpha, \nabla_{\dot{\alpha}}\} = -2i\sigma^l_{\alpha\dot{\alpha}}\nabla_\ell. \tag{9.332}$$

The super-connection is then defined by

$$\mathcal{A}_A = i(\nabla_A - D_A). \tag{9.333}$$

We have

$$\mathcal{A}_{\dot{\alpha}} = i(\nabla_{\dot{\alpha}} - D_{\dot{\alpha}}) = 0 \tag{9.334a}$$

$$\mathcal{A}_\alpha = i(\nabla_\alpha - D_\alpha) = i(e^{-V}(D_\alpha e^V)) = i\left(D_\alpha V - \frac{1}{2}[V, D_\alpha V] + \frac{1}{3!}[V,[V,D_\alpha V]] + \cdots\right) \tag{9.334b}$$

$$\mathcal{A}_\ell = \frac{1}{4}(\bar{\sigma}_\ell^{\dot{\alpha}\alpha}\{\nabla_\alpha, \nabla_{\dot{\alpha}}\} - \partial_\ell) = \frac{1}{4}\bar{\sigma}_\ell^{\dot{\alpha}\alpha}\bar{D}_{\dot{\alpha}}\mathcal{A}_\alpha \tag{9.334c}$$

Given the covariant derivatives, it is straightforward to define the curvature in the usual way

$$F_{AB} = i[\nabla_A, \nabla_B\} - iT^C_{AB}\nabla_C \tag{9.335}$$

T^C_{AB} is called supertorsion tensor. Its presence is due to the fact that the standard derivatives $(D_\alpha, \bar{D}_{\dot\alpha}, \partial_\ell)$ do not commute. The only non-vanishing contribution is $T^\ell_{\alpha\dot\alpha} = -2i\sigma^l_{\alpha\dot\alpha}$. Then

$$\begin{aligned}
F_{\dot\alpha\dot\beta} =& i\{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0\\
F_{\alpha\beta} =& i\{e^{-V}D_\alpha e^V, e^{-V}D_\beta e^V\} = e^{-V}\{D_\alpha, D_\beta\}e^V = 0\\
F_{\alpha\dot\beta} =& i\{\nabla_\alpha, \nabla_{\dot\beta}\} - 2\sigma^m_{\alpha\dot\beta}\nabla_m = 0 \quad \text{(due to the definition of } \nabla_m)\\
F_{\ell\dot\alpha} =& i[\nabla_\ell, \nabla_{\dot\alpha}] = i\left[\left(\partial_l - \frac{i}{4}\bar\sigma^{\dot\beta\beta}_\ell \bar{D}_{\dot\beta}\mathcal{A}_\beta\right), \bar{D}_{\dot\alpha}\right] = -\frac{1}{4}\bar\sigma^{\dot\beta\beta}_\ell \bar{D}_{\dot\alpha}\bar{D}_{\dot\beta}\mathcal{A}_\beta = \\
=& \frac{1}{8}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\dot\beta\beta}_\ell \bar{D}^2\mathcal{A}_\beta = \frac{1}{8}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\dot\beta\beta}_\ell \bar{D}^2(e^{-V}D_\beta e^V) \equiv -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\dot\beta\beta}_\ell \mathcal{W}_\beta\\
F_{\ell\alpha} =& i[\nabla_\ell, \nabla_\alpha] = (F_{\ell\dot\alpha})^\dagger = \left(-\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\sigma^{\dot\beta\beta}_\ell \mathcal{W}_\beta\right)^\dagger = -\frac{1}{2}\sigma_{\ell\alpha\dot\beta}\epsilon^{\dot\beta\dot\alpha}\bar{\mathcal{W}}_{\dot\alpha}
\end{aligned} \tag{9.336}$$

Therefore there are only two non vanishing component. Their content can be given in terms of two spinor chiral superfields

$$\mathcal{W}_\alpha = -\frac{1}{4}\bar{D}^2(e^{-V}D_\alpha e^V) \qquad \bar{\mathcal{W}}_{\dot\alpha} = -\frac{1}{4}D^2(e^{-V}\bar{D}_{\dot\alpha}e^V), \tag{9.337}$$

which transform covariantly under gauge transformations

$$W_\alpha \mapsto W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda} \qquad \bar{W}_{\dot\alpha} \mapsto \bar{W}'_{\dot\alpha} = e^{-i\bar\Lambda}\bar{W}_{\dot\alpha}e^{i\bar\Lambda} \tag{9.338}$$

These two quantities reduce to their abelian analog in the abelian limit.
It is now straightforward to write an action for the non abelian case. It is formally identical to the abelian case

$$S = -\frac{1}{32}\int d^2\theta d^4x \mathrm{Tr}(W^2) - \frac{1}{32}\int d^2\bar\theta d^4x \mathrm{Tr}(\bar{W}^2), \tag{9.339}$$

where the chiral and antichiral W_α and $\bar{W}_{\dot\alpha}$ are given by (9.337). Moreover the trace is taken with the following normalization $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.
Let us write this action in terms of the component fields present in the vector super-field. In order to have an action with a finite number of terms, we shall work in Wess-Zumino gauge, where $e^V = 1 + V + \frac{1}{2}V^2$

$$\mathcal{W}_\alpha = -\frac{1}{4}\bar{D}^2(D_\alpha V - \frac{1}{2}[V, D_\alpha V]) = -\frac{1}{4}\bar{D}^2D_\alpha V + \frac{1}{8}\bar{D}^2[V, D_\alpha V] \tag{9.340}$$

The first contribution is the same of the abelian case, while the second one contains the non abelian corrections. Therefore it is sufficient compute only the latter. Since

$$VD_\alpha V - (D_\alpha V)V = \frac{1}{2}[V_m, V_n](y)(\sigma^n\bar\sigma^m\theta)_\alpha\bar\theta^2 - i[V_m, (\sigma^m\bar\lambda)_\alpha](y)\theta^2\bar\theta^2, \tag{9.341}$$

we find

$$\frac{1}{8}\bar{D}^2([V,D_\alpha V])=\frac{1}{8}\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta^{\dot\beta}}([V,D_\alpha V])=-\frac{1}{4}[V_m,V_n](y)(\sigma^n\bar\sigma^m\theta)_\alpha+\frac{i}{2}[V_m,(\sigma^m\bar\lambda)_\alpha](y)\theta^2. \tag{9.342}$$

When we add these two contribution to the abelian part, we find the natural non abelian generalization

$$W_\alpha=-i\lambda_\alpha(y)+\left[\delta_\alpha^\beta D(y)-\frac{i}{2}(\sigma^m\bar\sigma^n)_\alpha{}^\beta F_{mn}(y)\right]\theta_\beta+\theta^2\left(\sigma^n\mathcal{D}_n\bar\lambda\right)_\alpha, \tag{9.343}$$

where

$$\mathcal{D}_n=\partial_n+\frac{i}{2}[V_n,\cdot]\qquad\text{and}\qquad F_{mn}=\partial_mV_n+\partial_nV_m+\frac{i}{2}[V_m,V_n]. \tag{9.344}$$

Then the action

$$\begin{aligned}
S=&-\frac{1}{32}\int d^4x\int d^2\theta\mathrm{Tr}(W^\alpha W_\alpha)+\text{c.c.}=\\
=&-\frac{1}{32}\int d^4x\int d^2\theta\mathrm{Tr}\left([2\theta^\alpha D(y)-i(\sigma^r\bar\sigma^s\theta)^\alpha F_{rs}(y)]\left[2\theta_\alpha D(y)-i(\sigma^m\bar\sigma^n\theta)_\alpha F_{mn}(y)\right]-\right.\\
&\left.-4i\theta^2\lambda\sigma^n\mathcal{D}_n\bar\lambda\right)+\text{c.c.}=\\
=&-\frac{1}{32}\int d^4x\int d^2\theta\theta^2\mathrm{Tr}\left(-2F_{mn}F^{mn}-2i\epsilon^{rsmn}F_{rs}F_{mn}+4D^2-4i\lambda\sigma^n\mathcal{D}_n\bar\lambda\right)+\text{c.c.}=\\
=&-\frac{1}{16}\int d^4x\mathrm{Tr}\left(F_{mn}F^{mn}+i\epsilon^{rsmn}F_{rs}F_{mn}-2D^2+2i\lambda\sigma^n\mathcal{D}_n\bar\lambda\right).+\text{c.c.}=\\
=&\int d^4x\left[-\frac{1}{8}\mathrm{Tr}(F_{mn}F^{mn})+\frac{1}{4}\mathrm{Tr}(D^2)-\frac{i}{4}\mathrm{Tr}(\lambda\sigma^n\mathcal{D}_n\bar\lambda)\right]
\end{aligned} \tag{9.345}$$

The action contains a non abelian gauge field, a Majorana spinor transforming in the adjoint representation coupled to the gauge field and finally a non dynamical field D, that can be eliminated through its equation of motion. For a pure supersymmetric gauge theory we have simply $D=0$.
Note that the action with the correct normalization is obtain after the rescaling $V\mapsto 2V$. Then

$$S=\int d^4x\left[-\frac{1}{2}\mathrm{Tr}(F_{mn}F^{mn})+\mathrm{Tr}(D^2)-i\mathrm{Tr}(\lambda\sigma^n\mathcal{D}_n\bar\lambda)\right] \tag{9.346}$$

Couplings to Matter: Let us investigate now the form of the coupling for the chiral superfield, namely we want to expand the matter action

$$\begin{aligned}
S=&-\frac{1}{4}\int d^4xd^2\theta d^2\bar\theta[\Phi^\dagger e^V\Phi]=\frac{1}{16}\int d^4xd^2\bar{D}^2D^2[\Phi^\dagger e^V\Phi]=\\
=&\frac{1}{16}\int d^4xd^2\bar{D}^2D^2[\Phi^\dagger\Phi]+\frac{1}{16}\int d^4xd^2\bar{D}^2D^2[\Phi^\dagger V\Phi]+\frac{1}{16}\int d^4xd^2\bar{D}^2D^2[\Phi^\dagger V^2\Phi]
\end{aligned} \tag{9.347}$$

The complete expansion of this action is quite tedious. The first term is the free one and it has been already discussed when we have considered the chiral superfield. The second term is quite lengthy, however it is sufficient to keep track of the terms whose final contribution will be proportional to $\theta^2\bar{\theta}^2$. We find

$$\begin{aligned}\Phi^\dagger(x,\theta,\bar{\theta})V(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta}) =& \frac{1}{2}\theta^2\bar{\theta}^2\Big(i[A^\dagger V^m(x)\partial_m A(x) - \partial_n A^\dagger V^n A] + \bar{\chi}\bar{\sigma}^m V_m\chi + \\ &+ \sqrt{2}iA^\dagger\lambda\chi + A^\dagger DA - \sqrt{2}i\bar{\chi}\bar{\lambda}A\Big) + \text{lower terms.}\end{aligned} \tag{9.348}$$

The second term is quite easy to be expanded

$$\frac{1}{2}\Phi^\dagger(x,\theta,\bar{\theta})V^2(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta}) = -\frac{1}{4}A^\dagger V_m V^m A\theta^2\bar{\theta}^2 + \text{lower terms.} \tag{9.349}$$

All the contribution can be collected together to find the action

$$\begin{aligned}\int d^4x \Big[&-\left(\partial_m A^\dagger - i\frac{1}{2}A^\dagger V_m\right)\left(\partial_m A + i\frac{1}{2}V_m A\right) - i\bar{\chi}\bar{\sigma}^m\left(\partial_m + i\frac{1}{2}V_m\right)\chi + \\ &+ \frac{i}{\sqrt{2}}(A^\dagger\lambda\chi - \bar{\lambda}\bar{\chi}A) + \frac{1}{2}A^\dagger DA\Big].\end{aligned} \tag{9.350}$$

Again the correct normalization for this action are restored when $V \mapsto 2V$, *i.e.*

$$\begin{aligned}\int d^4x \Big[&-\left(\partial_\mu A^\dagger - iA^\dagger V_\mu\right)(\partial_\mu A + iV_\mu A) - i\bar{\chi}\bar{\sigma}^m(\partial_m + iV_m)\chi + \\ &+ i\sqrt{2}(A^\dagger\lambda\chi - \bar{\lambda}\bar{\chi}A) + A^\dagger DA\Big].\end{aligned} \tag{9.351}$$

We can complete this action by adding the kinetic term for the gauge field

$$\begin{aligned}\int d^4x \Big[&-\left(\partial_\mu A^\dagger - iA^\dagger V_\mu\right)(\partial_\mu A + iV_\mu A) - i\bar{\chi}\bar{\sigma}^m(\partial_m + iV_m)\chi + \\ &+ i\sqrt{2}(A^\dagger\lambda\chi - \bar{\lambda}\bar{\chi}A) + A^\dagger DA + \\ &-\frac{1}{2}\mathrm{Tr}(F_{mn}F^{mn}) + \mathrm{Tr}(D^2) - i\mathrm{Tr}(\lambda\sigma^n\mathcal{D}_n\bar{\lambda})\Big].\end{aligned} \tag{9.352}$$

Note that if we integrate out the field D in (9.352) by means of its equation of motion, we find a quartic contribution to the scalar potential. In fact

$$D^a + A^\dagger T^a A = 0 \quad\Rightarrow\quad D = -A^\dagger T^a A \quad\Rightarrow\quad V_D = \frac{1}{2}(A^\dagger T^a A)(A^\dagger T_a A) = \frac{1}{2}D^a D_a \tag{9.353}$$

Therefore in terms of superfields, the correctly normalized supersymmetric action for a gauge field coupled to a set of N chiral superfield Φ^I transforming in the representation R_I is

$$\begin{aligned}S = &-\int d^4x\int d^2\theta\left[\frac{\tau}{32\pi i}\mathrm{Tr}(W^\alpha W_\alpha) - \frac{\bar{\tau}}{32\pi i}\mathrm{Tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}})\right] - \int d^4x\int d^2\theta d^2\bar{\theta}\sum_s \xi^s V_{U_s(1)} - \\ &-\frac{1}{4}\int d^4x d^2\theta d^2\bar{\theta}\sum_I[\Phi^{I\dagger}e^{2V_{R_I}}\Phi^I] + \frac{1}{2}\int d^4x d^2\theta\mathcal{F}(\Phi^I) + \frac{1}{2}\int d^4x d^2\bar{\theta}\mathcal{F}^*(\Phi^{\dagger I}),\end{aligned} \tag{9.354}$$

where $W_\alpha = -\frac{1}{8}\bar{D}^2(e^{-2V}D_\alpha e^{2V})$, $V = V^a T^a_{\rm adj}$ and $V_{R_I} = V^a T^a_{R_I}$. Moreover we have allowed for a complex coupling τ, which we shall parameterize as follows

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}. \tag{9.355}$$

This will produce an additional term which is given

$$-\frac{\theta_{YM}}{64\pi^2 i}\int d^4x 4i\text{Tr}(F_{mn}\tilde{F}^{mn}) = -\frac{\theta_{YM}}{16\pi^2}\int d^4x\text{Tr}(F_{mn}\tilde{F}^{mn}), \tag{9.356}$$

namely a $\theta-$term. The sum over s runs over all the $U(1)$ factor and it takes into account the possibility of adding Fayet-Iliopulos term.

NOTE: The coefficient can be a little bit simplified if we change the normalization in the definition of the Grassmannian integral. Up to now

$$\int d^2\theta\theta^2 = -\frac{1}{2}D^2\theta^2 = -\frac{1}{2}\epsilon_{\rho\sigma}(D^\alpha(D_\alpha\theta^\rho\theta^\sigma)) = -\epsilon_{\rho\sigma}D^\rho(\theta^\sigma) = D_\sigma\theta^\sigma = 2. \tag{9.357}$$

We may choose to redefine the normalization of the integral so that $\int d^2\theta\theta^2 = 1$ and $\int d^2\bar{\theta}\bar{\theta}^2 = 1$. With this new normalization the above action takes the form

$$\begin{aligned} S = & -\int d^4x\int d^2\theta\left[\frac{\tau}{16\pi i}\text{Tr}(W^\alpha W_\alpha) - \frac{\bar{\tau}}{16\pi i}\text{Tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}})\right] - \int d^4x\int d^2\theta d^2\bar{\theta}\sum_s \xi^s V_{U_s(1)} - \\ & -\int d^4x d^2\theta d^2\bar{\theta}\sum_I [\Phi^{I\dagger}e^{2V_{R_I}}\Phi^I] + \int d^4x d^2\theta\mathcal{F}(\Phi^I) + \int d^4x d^2\bar{\theta}\mathcal{F}^*(\Phi^{\dagger I}), \end{aligned} \tag{9.358}$$

This action will naturally contain a potential for the scalar which is a simple extension of the one for the WZ model. Its general form is

$$\boxed{V = \sum_I |F^I|^2 + \frac{1}{2}\sum_a D^a D_a} \tag{9.359}$$

This means that the equation for a supersymmetric vacua are now

$$F^I = 0 \qquad D^a = 0. \tag{9.360}$$

9.1 General action for the N=1 matter-gauge system

The action that we have written in the previous section is not the most general Lagrangian for the matter-gauge system with $N = 1$ supersymmetry. A part from the superpotential all the other terms are the ones obeying to the criterium of renormalizability. We want to drop this constraint and look for a more general action.
To begin with, let us consider the matter action. We already know that the most general action is provided by

$$-\int d^4k d^2\theta d^2\bar{\theta} K(\Phi^{I\dagger}, \Phi^I). \tag{9.361}$$

We shall assume that this action possesses an global invariance of the form

$$\Phi^I \mapsto e^{-i\Lambda}\Phi^I \quad \text{and} \quad \Phi^{I\dagger} \mapsto \Phi^{I\dagger}e^{i\Lambda}. \tag{9.362}$$

We want to promote this global invariance. For a local transformation

$$\Phi^{I\dagger} \mapsto \Phi^{I\dagger} \mapsto \Phi^{I\dagger}e^{i\Lambda^\dagger(y,\theta)} \neq \Phi^{I\dagger}e^{i\Lambda(y,\theta)}, \tag{9.363}$$

instead

$$\Phi^{I\dagger}e^{2V} \mapsto \Phi^{I\dagger} \mapsto \Phi^{I\dagger}e^{i\Lambda^\dagger(y,\theta)}e^{-i\Lambda^\dagger(y,\theta)}e^{2V}e^{i\Lambda(y,\theta)} = \Phi^{I\dagger}e^{2V}e^{i\Lambda(y,\theta)}. \tag{9.364}$$

Therefore we can have local gauge invariance with the minimal substitution

$$-\int d^4x \int d^2\theta d^2\bar{\theta} K(\Phi^{I\dagger}, \Phi^I) \quad \mapsto \quad -\int d^4x \int d^2\theta d^2\bar{\theta} K(\Phi^{I\dagger}e^{2V}, \Phi^I). \tag{9.365}$$

The kinetic term for the gauge fields can be also generalized. Construct with the chiral fields W and Φ^I a scalar superfield $H(\Phi^I, W)$, which is invariant under the gauge transformations. Any function H yields an action with at most two derivatives on the gauge fields. However we shall not choose a generic H. We shall focus out attention on $H = G^{ab}(\Phi)W_a^\alpha W_{b\alpha}$, where a and b are indices in the adjoint representation. Then

$$\begin{aligned}
\frac{1}{16g^2}\int d^2\theta G^{ab}(\Phi)W_a^\alpha W_{b\alpha} &= -\frac{1}{4}D^2(G^{ab}(\Phi)W_a^\beta W_{b\beta}) = \\
&= -\frac{1}{64g^2}D^\alpha\left(G_I^{ab}(\Phi)D_\alpha\Phi^I W_a^\alpha W_{b\alpha} + 2G^{ab}(\Phi)D_\alpha W_a^\beta W_{b\beta}\right) = \\
&= -\frac{1}{64g^2}\Big(G_{JI}^{ab}(\Phi)D^\alpha\Phi^J D_\alpha\Phi^I W_a^\beta W_{b\beta} + G_I^{ab}(\Phi)D^2\Phi^I W_a^\beta W_{b\beta} - \\
&\quad -2G_I^{ab}(\Phi)D_\alpha\Phi^I D^\alpha W_a^\beta W_{b\beta} + 2G_I^{ab}(\Phi)D^\alpha\Phi^I D_\alpha W_a^\beta W_{b\beta} + 2G^{ab}(\Phi)D^2 W_a^\beta W_{b\beta} + \\
&\quad + 2G^{ab}(\Phi)D_\alpha W_a^\beta D^\alpha W_{b\beta}\Big) = \\
&= -\frac{1}{64g^2}\Big(2G^{ab}(\Phi)[D^2 W_a^\beta W_{b\beta} - D_\alpha W_{a\beta}^\beta D^\alpha W_b^\beta] - 2G_{JI}^{ab}(\Phi)\chi^I\chi^J\lambda_a\lambda_b + \\
&\quad + 4G_I^{ab}(\Phi)F^I\lambda_a\lambda_b + 4i\sqrt{2}G_I^{ab}(\Phi)\chi^I\lambda_b D_a - 4\sqrt{2}G_I^{ab}(\Phi)\chi^I\sigma^{mn}\lambda_b F_{a,mn}\Big).
\end{aligned} \tag{9.366}$$

The first term will reproduce the $N=1$ SYM action with the color indices contracted with the standard matrix G_{ab}. The other are the new couplings between gauge fields and matter: there are four fermion interactions, Pauli couplings and further coupling with the auxiliary fields.
The expansion of the matter part we shall give simply the old result for the NLS model with all the quantities covariantized with respect to the gauge group.
We can easily write the potential for the scalar once we have integrated the auxiliary fields. It is of the usual form

$$V = G_{I\bar{J}}\mathcal{F}^I\bar{\mathcal{F}}^I + \frac{1}{2}\mathrm{Re}(G_{ab})^{-1}(A^{\bar{I}}T^a K_{\bar{I}})(A^I T^b K_I) \tag{9.367}$$

10 N=2 gauge theories

An $N=2$ gauge theory reads in the $N=1$ Language as $N=1$ theory coupled to a chiral multiplet in the adjoint representation. An $N=1$ theory with this properties is given by

$$S = -\int d^4x \int d^2\theta \left[\frac{\tau}{16\pi i}\mathrm{Tr}(W^\alpha W_\alpha) - \frac{\bar{\tau}}{16\pi i}\mathrm{Tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}})\right] - \int d^4x d^2\theta d^2\bar{\theta}\Psi e^{2V}\Psi, \tag{10.368}$$

where Ψ is chiral field in the adjoint. If we define $D_m A \equiv \partial_\mu A + i[V_m, A]$, $\mathcal{D}_m\lambda \equiv \partial_\mu\lambda + i[V_m,\lambda]$ and $\mathcal{D}_m\chi \equiv \partial_\mu\chi + i[V_m,\chi]$, in components the above action reads

$$\begin{aligned}\int d^4x \mathrm{Tr}\Big[&-2(D_mA)^\dagger D^m A - 2i\bar{\chi}\bar{\sigma}^m\mathcal{D}_m\chi - 2i\bar{\lambda}\bar{\sigma}^m\mathcal{D}_m\lambda - \frac{1}{2}F_{mn}F^{mn} + D^2 + \\ &- 2i\sqrt{2}(A^\dagger\{\lambda,\chi\} - \{\bar{\lambda},\bar{\chi}\}A) + 2D[A,A^\dagger]\Big].\end{aligned} \tag{10.369}$$

where we have used the following definition $(T^a_{adj})_{bc} = -if_{abc}$ for writing everything in terms of commutators. The additional and unusual factor 2 are due to the normalization of the generators $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.
This action is invariant under $N=2$ if and only if there exists an $SU_R(2)$ invariance under which the fermions form a doublet. Let's define $(\lambda^I) = (\lambda,\chi)$, then

$$\begin{aligned}&\epsilon_{IJ}\{\lambda^I,\lambda^J\} = \epsilon_{IJ}(\lambda^I\lambda^J + \lambda^J\lambda^I) = 2\epsilon_{IJ}\lambda^{Ia}\lambda^{Jb}[T_a,T_b] = 2(\lambda^{1a}\lambda^{2b} - \lambda^{2a}\lambda^{1b})[T_a,T_b] = \\ &= 4\lambda^{1a}\lambda^{2b}[T_a,T_b] = 4\lambda^a\chi^b[T_a,T_b] = 4\{\lambda,\chi\}.\end{aligned} \tag{10.370}$$

Then we can rewrite the above action as follows

$$\begin{aligned}\int d^4x \mathrm{Tr}\Big[&-2(D_mA)^\dagger D^m A - 2i\bar{\lambda}^I\bar{\sigma}^m\mathcal{D}_m\lambda^I - \frac{1}{2}F_{mn}F^{mn} + D^2 + \\ &- \frac{i}{\sqrt{2}}(A^\dagger\epsilon_{IJ}\{\lambda^I,\lambda^J\} - \epsilon_{IJ}\{\bar{\lambda}^I,\bar{\lambda}^J\}A) + 2D[A,A^\dagger]\Big],\end{aligned} \tag{10.371}$$

which is manifestly $SU(2)$ invariant and therefore is $N=2$ supersymmetric. Notice that the an $N=2$ gauge theory possesses a scalar potential

$$V = \mathrm{Tr}([A, A^\dagger]^2), \tag{10.372}$$

and supersymmetric vacua given by $[A, A^\dagger] = 0$ (see Fawad).

We now consider the possibility of adding matter. A massless hypermultiplet contains two massless spinors of opposite helicities. Since these two spinors belong to the same supermultiplet must couple to the gauge field in the same way, namely they mast transform in the same representation. This means that $N=2$ theories are vectorlike. At the level of $N=1$ superfield content, this means that if a $N=1$ chiral superfield transforming in the representation R, an $N=1$ chiral superfield transforming in the conjugate representation $\bar{R}$ is present as well.

Therefore we can write $N=1$ gauge interactions between the vector gauge multiplet and the chiral superfield and this is the easy part. The not obvious part is to write the most general superpotential which would give origin to an $N=2$ theory. We have at our disposal, apart from the matter chiral superfields, the chiral superfield describing the $N=2$ additional gauge degrees of freedom. We shall denote the matter superfield in R representation with ϕ_a and with ϕ^a those in the $\bar{R}$ representation. The *gauge* gauge chiral superfield is written as ${\Psi^a}_b = \Psi_A {(T^A)^a}_b$, where ${(T^A)^a}_b$ are the generators in $R \otimes \bar{R}$ representation. The superpotential is we have

$$\mathcal{F} = b_a \phi^a + b^a \phi_a + {c_a}^b {\Psi^a}_b - m\phi_a\phi^a + g\phi_a {\Psi^a}_b \phi^b, \tag{10.373}$$

where we limited ourselves to renormalizable interactions. We can allow for small generalizations. If the R representation is reducible the mass and the cubic coupling might not be diagonal. Moreover if R is a real representation we can allow for cubic couplings of the type $d^{abc}\phi_a\phi_b\phi_c$, but these last couplings will be ruled out by requiring the invariance under $N=2$ supersymmetry. If we now expand, after a lengthy calculation, a manifest $SU(2)$ invariance is recovered if an only if

$$b_a = b^a = {c_a}^b = 0 \quad g = 1. \tag{10.374}$$

We remain with the following superpotential for the $N=2$ theory

$$\mathcal{F} = -m\phi_a\phi^a + \phi_a {\Psi^a}_b \phi^b. \tag{10.375}$$

What about $N=4$? In the language of $N=1$ is given by a gauge multiplet coupled to tree chiral superfields in the adjoint representations. Again we have to fix the superpotential. We directly start from the previous result and we shall call all the superfields Ψ_i since they cannot be distinguished. Then m is equal to zero since we deal with a massless multiplet. Moreover

$$(\Psi_2)^B{}_C = \Psi_A (T^A)^B{}_C = -i\Psi_A f^{AB}{}_C. \tag{10.376}$$

We are left with

$$\mathcal{F} = i f^{ABC} \Psi_{1A} \Psi_{2B} \Psi_{3C} = \mathrm{Tr}(\Psi_1 [\Psi_2, \Psi_3]). \tag{10.377}$$

One can verify that this supepotential gives origin to a theory with an $SU(4)$ invariance and thus with an $N = 4$ supersymmetry.

Comment on massive hypermultiplets transformation and the appearance of the Central charge.

10.1 N=2 superspace: the general form of N=2 supersymmetric gauge action

In the following we shall discuss some features of the $N = 2$ superspace in its simplest and naive form, neglecting all the technical details and in particular the effect of the central charges. A systematic presentation of the topic would require a set of dedicated lectures.
When we try to generalize the construction of the $N = 1$ to the $N = 2$ superpace, almost all the steps works in the same way. The main difference is that everything acquires an $SU(2)$ index which keeps track of the R-symmetry [8]:

$$(\theta, \bar\theta) \quad \mapsto \quad (\underset{\mathbf{2}}{\theta^I}, \underset{\mathbf{2}}{\bar\theta^I}) \tag{10.378a}$$

$$(Q, \bar Q) \quad \mapsto \quad (\underset{\mathbf{2}}{Q^I}, \underset{\mathbf{2}}{\bar Q^I}) \tag{10.378b}$$

$$(D, \bar D) \quad \mapsto \quad (\underset{\mathbf{2}}{D^I}, \underset{\mathbf{2}}{\bar D^I}) \tag{10.378c}$$

$$\int d^2\theta \quad \mapsto \quad \int d^2\theta^1 d^2\theta^2 \tag{10.378d}$$

All these quantities will enjoy the same properties of the $N = 1$ case.
A generic $N = 2$ scalar superfield will be a function of x, θ^I and $\bar\theta^I$ and it is a singlet of $SU(2)$. We shall denote it by $\Phi(x, \theta^I, \bar\theta^I)$. This superfield will contain a *huge* number of components. In order to reduce the number of components, as we did in $N = 1$ case, we shall impose a constraint, which preserves supersymmetry and $R-$symmetry. The natural choice is

$$\bar D^I \Psi(x, \theta^I, \bar\theta^I) = 0, \tag{10.379}$$

we shall call the solutions of (10.379) *N=2 chiral superfield* , since they are the obvious generalization of the $N = 1$ chiral superfield. The above constraint is easily solved by introducing the

[8]There are differences related to the presence of central charges. The presence of the bosonic coordinates associated to these generators leads to superfields which contains an infinite number of fields. One usually can impose consistently that the superfields do not depends on these coordinates. For this reason, we shall neglect the role of the central charges from now on. This implies that the representation of all the quantities is identical to case $N = 1$ apart from the appearance of the $SU(2)$ index.

coordinates

$$z = x + i\theta^I \sigma \bar{\theta}^I, \tag{10.380}$$

which satisfies $\bar{D}^I z = 0$. Then the above condition becomes

$$\frac{\partial}{\partial \bar{\theta}^I} \Psi(z, \theta^I, \bar{\theta}^I) = 0 \qquad \Rightarrow \qquad \Psi(x, \theta^I, \bar{\theta}^I) = \Psi(z, \theta^I). \tag{10.381}$$

The field content can be obtained by analyzing all the derivatives D^I_α of the superfield at $\theta^I = 0, \bar{\theta}^I = 0$

$$\begin{gathered} \Psi|_{\theta=\bar{\theta}^I=0} = \phi(x) \quad D^I_\alpha \Psi\big|_{\theta^I=\bar{\theta}^I=0} = \lambda^I_\alpha(x) \quad D^I_\alpha D^J_\beta \Psi\big|_{\theta^I=\bar{\theta}^I=0} = \frac{1}{2}\epsilon^{IJ} F_{mn}(x) \sigma^{mn}_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta} C^{IJ}(x) \\ \frac{1}{2}\epsilon_{IJ} D^I_\alpha D^J D^K \Psi\bigg|_{\theta^I=\bar{\theta}^I=0} = \chi^I_\alpha(x) \qquad \frac{1}{4}\epsilon_{IK}\epsilon_{JL}(D^I D^J)(D^K D^L)\Psi\bigg|_{\theta^I=\bar{\theta}^I=0} = Z(x) \end{gathered} \tag{10.382}$$

Therefore the content of the N=2 chiral superfield is

$$(\phi(x), Z(x), C^{IJ}(x), \lambda^I_\alpha(x), \chi^I_\alpha(x), F_{mn}(x)) \tag{10.383}$$

This superfield seems a natural candidate to describe gauge fields, since it naturally contains an object transforming as a field strength. However, at the moment F_{mn} is not the curl of a vector field, it is an arbitrary complex quantity. This means that we have to reduce the representation by imposing another constraint. The right condition is

$$D^I D^J \Psi = \bar{D}^I \bar{D}^J \Psi^\dagger \tag{10.384}$$

We can write down a solution of this constraint if we shall break the $SU(2)$ covariance: first of all we shall write $(\theta^I) = (\theta, \tilde{\theta})$ and subsequently we shall expand the superfield Ψ in power of $\tilde{\theta}$:

$$\Psi(z, \theta^I) = \Phi(z, \theta) + i\sqrt{2}\tilde{\theta}\mathcal{W}(z, \theta) + \tilde{\theta}^2 F(z, \theta). \tag{10.385}$$

The superfield Ψ solves the constraint (10.384) if $\mathcal{W}_\alpha$ is the chiral field strength superfield of a vector $N = 1$ superfield V and F is given by

$$F(z, \theta) = \int d^2\bar{\theta} \Phi^\dagger(z - i\theta\sigma\bar{\theta}, \theta, \bar{\theta}) e^{2V(z - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})} \tag{10.386}$$

where $\Phi^\dagger(x, \theta, \bar{\theta})$ is understood as the one given in (7.174). Then the whole action of $N = 2$ theory is simply obtained by taking

$$\begin{aligned} S_{N=2} &= \int dz d^2\theta d^2\tilde{\theta}(\Psi^2) = \int dz d^2\theta \Phi(z, \theta) F(z, \theta) + \int dz d^2\theta W^\alpha(z, \theta) W_\alpha(z, \theta) = \\ &= \int dz \int d^2\theta d^2\bar{\theta} \Phi^\dagger(z - i\theta\sigma\bar{\theta}, \theta, \bar{\theta}) e^{2V(z - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})} \Phi(z, \theta) + \int dz d^2\theta W^\alpha(z, \theta) W_\alpha(z, \theta). \end{aligned} \tag{10.387}$$

We have also the obvious generalizations

$$
\begin{aligned}
S_{N=2} = & \int dz d^2\theta d^2\tilde{\theta} G(\Psi) = \int dz d^2\theta G_{ab}(\Phi) W^a(z,\theta) W^b(z,\theta) + \\
& + \int dz d^2\theta G_a(z,\theta) F^a(z,\theta) = \int dz d^2\theta G_{ab}(\Phi) W^a(z,\theta) W^b(z,\theta) + \\
& + \int dz \int d^2\theta d^2\bar{\theta} \left[\Phi^\dagger(z - i\theta\sigma\bar{\theta}, \theta, \bar{\theta}) e^{2V(z - i\theta\sigma\bar{\theta}, \theta, \bar{\theta})} \right]^a G_a(z,\theta),
\end{aligned}
\tag{10.388}
$$

where $G_a = \frac{\partial G}{\partial \psi^a}$ and $G_a = \frac{\partial^2 G}{\partial \psi^a \partial \psi^b}$. In N=1 langauge the most general gauge theory with a $N = 2$ supersymmetry is a gauged non-linear sigma model, where both the couplings of the vector multiplet and the Kaheler potential are determined by one function G.

A Background Field Method

When quantizing a gauge theory, the classical local symmetry $\delta A_\mu = -D_\mu \omega$ possessed by the Lagrangian[9]

$$\mathcal{L} = -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} \tag{A.389}$$

is not manifest in the intermediate steps of many perturbative calculations because of the gauge fixing procedure. Thus an *efficient* use of the constraints originating from the presence of this invariance is often difficult. A manifest (and partially *fictitious*) gauge invariance can be restored in the perturbative formalism if we introduce a classical background field $\mathbf{A}_\mu$ and we split the original field A_μ as follows

$$A_\mu = \mathbf{A}_\mu + Q_\mu, \tag{A.390}$$

where Q_μ is called the *quantum field.* The Lagrangian (A.389) written in terms of these two fields, *i.e.*

$$\mathcal{L} = -\frac{1}{2g^2} F_{\mu\nu}(Q+A) F^{\mu\nu}(Q+A), \tag{A.391}$$

exhibits two distinct local symmetries:

- **Quantum symmetry:**

$$\begin{aligned} \delta_q \mathbf{A}_\mu =& 0 \\ \delta_q Q_\mu =& - D_\mu \Omega - [\mathbf{A}_\mu, \Omega] \qquad \text{with} \quad D_\mu \cdot = \partial_\mu \cdot + [Q_\mu, \cdot] \end{aligned} \tag{A.392}$$

- **Classical symmetry:**

$$\begin{aligned} \delta_c \mathbf{A}_\mu =& - \mathbf{D}_\mu \Theta \qquad \text{with} \quad \mathbf{D}_\mu \cdot = \partial_\mu \cdot + [\mathbf{A}_\mu, \cdot] \\ \delta_c Q_\mu =& [\Theta, Q_\mu] \end{aligned} \tag{A.393}$$

When quantizing the field Q_μ, we are obviously forced to break the quantum symmetry (A.392) by introducing a gauge-fixing term, however it is not necessary to destroy the invariance (A.393). This goal is achieved by choosing a gauge-fixing term that is invariant under (A.393). We can use for example

$$S_{g.f.} = -\frac{1}{\xi} \int dx^4 (\mathbf{D}_\mu Q^\mu)^2. \tag{A.394}$$

The ghost action is consequently given by

$$S_{ghost} = 2i \int dx^4 \bar{c} \mathbf{D}_\mu (D_\mu c + [\mathbf{A}_\mu, c]) = 2i \int dx^4 \bar{c} \mathbf{D}_\mu (\mathbf{D}^\mu c + [Q^\mu, c]), \tag{A.395}$$

[9] In the following, a trace over the Lie algebra generators, which are taken anti-hermitian, is understood where necessary. Moreover we shall use the Euclidean notation. This is in fact more suitable for a systematic loop expansion.

and it is manifestly invariant under (A.393). Summarizing the total action

$$S_{tot} \equiv S_{gauge} + S_{g.f.} + S_{ghost}$$

is unchanged by the classical transformations and this symmetry is therefore unbroken at the quantum level if we use a regularization procedure that preserves it, *e.g.* dimensional regularization.

We shall now explore the effects of this background symmetry on the full effective action Γ. To begin with, we shall introduce

$$Z[J, \mathbf{A}] = \int DQ_\mu e^{-S_{tot}(Q,\mathbf{A})-J\cdot Q}. \tag{A.396}$$

This functional generator is invariant under background transformation if we assume that the current J transforms as follows

$$\delta_c J = [\Theta, J]. \tag{A.397}$$

The same invariance obviously holds for the connected generator

$$W[J, \mathbf{A}] = -\log(Z[J, \mathbf{A}]) \tag{A.398}$$

and for the effective action

$$\Gamma[\hat{Q}, \mathbf{A}] = W[J, \mathbf{A}] - J \cdot \hat{Q} \qquad \text{where} \quad \hat{Q} = \frac{\delta W[J, \mathbf{A}]}{\delta J}, \tag{A.399}$$

once we have chose the following trasformation rule for the classical field $\hat{Q}$

$$\delta\hat{Q} = [\Theta, \hat{Q}]. \tag{A.400}$$

In other words, the full effective action $\Gamma[\hat{Q}, \mathbf{A}]$ must be a gauge-invariant functional of the fields $\hat{Q}_\mu$ and $\mathbf{A}_\mu$ with respect to the background symmetry (A.393) and (A.400).

At this stage, a natural question is how to relate the background field formalism to the usual one. In order to answer this question, we perform the change of variable $Q \mapsto Q - \mathbf{A}$ in the path integral (A.396). The dependence on the background $\mathbf{A}$ disappears from S_{gauge}, which becomes the standard Yang-Mills action for the field Q. However it is still present in the gauge fixing term and consequently in the ghost action. We have then

$$Z[J, \mathbf{A}] = e^{J\cdot\mathbf{A}} \int DQ_\mu Dc D\bar{c}\ e^{-S_{gauge}(Q)+\frac{1}{\xi}\int d^4x(\mathbf{D}_\mu(Q^\mu-\mathbf{A}^\mu))^2)-2i\int d^4x\bar{c}\mathbf{D}_\mu D^\mu c-J\cdot Q} \equiv e^{J\cdot\mathbf{A}}\bar{Z}_{\mathbf{A}}[J]. \tag{A.401}$$

Here $\bar{Z}_{\mathbf{A}}[J]$ is the standard functional generator for Yang-Mills Green-functions, but with the unusual gauge-fixing $\frac{1}{\xi}(\mathbf{D}_\mu(Q^\mu - \mathbf{A}^\mu))^2$. The identity (A.401), in turn, entails

$$W[J, \mathbf{A}] = -J \cdot \mathbf{A} + W_{\mathbf{A}}[J] \tag{A.402}$$

and performing the Legendre transform

$$\Gamma[\hat{Q}, \mathbf{A}] = \Gamma_{\mathbf{A}}[\hat{Q} + \mathbf{A}]. \tag{A.403}$$

In other words the effective action in the background field formalism is the standard effective action for a gauge fixing of the form $\frac{1}{\xi}(\mathbf{D}_\mu(Q^\mu - \mathbf{A}^\mu))^2$ evaluated for $Q_{class.} = \hat{Q} + \mathbf{A}$.
At this point, there is a second question we should answer: what is the advantage of the background field formalism? If we had to compute the entire $\Gamma[\hat{Q}, \mathbf{A}]$, the advantage would be insignificant. Actually the presence of the second source, the background field $\mathbf{A}$, has increased the number of 1-PI diagrams to compute. This morally balances the simplifications coming from the recovered gauge invariance! However, this is not the end of the story.
For answering many important questions in the quantum theory, it is sufficient to know $\Gamma[0, \mathbf{A}]$: the generating function of 1-IP Green function with no external $\hat{Q}$ field. Consider, for example, the problem of determining the $\beta-$function of the theory. Since the same coupling g describes the interactions of both background and quantum fields, the complete information about $\beta-$function is already present in $\Gamma[0, \mathbf{A}]$. But this is not the only simplification. In fact, since $\Gamma[0, \mathbf{A}]$ is by construction a gauge invariant functional of the background field, a simple argument based on dimensional analysis entails that the only possible divergent term must have the form of a divergent constant times

$$F_{\mu\nu}(\mathbf{A}) F^{\mu\nu}(\mathbf{A}), \tag{A.404}$$

with

$$F_{\mu\nu}(\mathbf{A}) = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g[\mathbf{A}_\mu, \mathbf{A}_\nu]. \tag{A.405}$$

In (A.405) we have restored the coupling constant through the usual change $\mathbf{A}_\mu \mapsto \mathbf{A}_\mu g$. If we introduce the renormalized quantity

$$\mathbf{A}^R_\mu = Z_{\mathbf{A}}^{-1/2} \mathbf{A}_\mu \qquad \text{and} \qquad g_R = Z_g^{-1} g, \tag{A.406}$$

eq. (A.405) reads

$$\begin{aligned} F_{\mu\nu}(\mathbf{A}) =& Z_{\mathbf{A}}^{1/2} \partial_\mu \mathbf{A}^R_\nu - Z_{\mathbf{A}}^{1/2} \partial_\nu \mathbf{A}^R_\mu + g_R Z_{\mathbf{A}} Z_g [\mathbf{A}^R_\mu, \mathbf{A}^R_\nu] = \\ =& Z_{\mathbf{A}}^{1/2} \left(\partial_\mu \mathbf{A}^R_\nu - \partial_\nu \mathbf{A}^R_\mu + g_R Z_{\mathbf{A}}^{1/2} Z_g [\mathbf{A}^R_\mu, \mathbf{A}^R_\nu] \right). \end{aligned} \tag{A.407}$$

This expression will have a gauge covariant form in terms of the renormalized quantities, as required by the classical symmetry, if and only if the following Ward Identity holds

$$Z_g = Z_{\mathbf{A}}^{-1/2}. \tag{A.408}$$

This relation reduces the computation of the $\beta-$function of the theory to the computation of the wave-function renormalization, which is a much simpler problem. In fact,

$$g_B = g_R(\mu)\mu^\epsilon Z_g = g_R(\mu)\mu^\epsilon Z_{\mathbf{A}}^{-1/2} \qquad \Rightarrow \qquad \beta(g_R) = -\epsilon g_R(\mu) + \frac{1}{2} g_R(\mu)\mu \frac{\partial \log Z_{\mathbf{A}}}{\partial \mu}. \quad \text{(A.409)}$$

It is worth mentioning that the $\beta-$function is not the only information that one can extract from $\Gamma[0, \mathbf{A}]$: one can also build $S-$matrix of the whole theory. This topic is however beyond the scope of these notes.

A.1 $\beta-$function

To illustrate the use of the background field method (BFM), we shall now compute the one-loop $\beta-$function for a gauge theory with an arbitrary matter content.
Because of the Ward Identity (B.462), we need to focus our attention on the quadratic part $\Gamma[0, \mathbf{A}]$. At one loop, this is determined by the vertices in the action which are quadratic in the quantum field Q_μ[10].

A.1.1 Gauge-contribution

In the gauge sector this term are given by

$$\mathcal{L}_{YM+g.f.} = -\frac{1}{2}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} - \left(\mathbf{D}_\mu Q_\nu \mathbf{D}^\mu Q^\nu - \left(1 - \frac{1}{\xi}\right)(\mathbf{D}_\mu Q^\mu)^2\right) - 2g\mathbf{F}^{\mu\nu}[Q_\mu, Q_\nu] + O(Q^3), \quad \text{(A.410)}$$

and they produce the following contributions:

Vertices with one $A-$line: We have just one vertex quadratic in the quantum field Q and with one $\mathbf{A}-$line. It yields

$$\begin{aligned} I_{gauge} =& \frac{1}{2}\int \frac{d^{2\omega}p d d^{2\omega}k_1 d^{2\omega}k_2}{(2\pi)^{4\omega}} \frac{d^{2\omega}q d d^{2\omega}r_1 d^{2\omega}r_2}{(2\pi)^{4\omega}} \delta^{2\omega}(p + k_1 + k_2)\delta^{2\omega}(q + r_1 + r_2) \times \\ & \times A^a_\lambda(p) A^l_\lambda(q) G^{abc}_{\lambda\mu\nu}(p, k_1, k_2) G^{lmn}_{\rho\sigma\tau}(q, r_1, r_2) \langle Q^\mu_b(k_1) Q^\nu_c(k_2) Q^\sigma_m(r_1) Q^\tau_n(r_2)\rangle_0 = \\ =& \int \frac{d^{2\omega}p d^{2\omega}q d^{2\omega}k_1 d^{2\omega}k_2}{(2\pi)^{4\omega}} \delta^{2\omega}(p + k_1 + k_2)\delta^{2\omega}(q - k_1 - k_2) \times \\ & \times A^a_\lambda(p) A^l_\lambda(q) G^{abc}_{\lambda\mu\nu}(p, k_1, k_2) G^{lmn}_{\rho\sigma\tau}(q, -k_1, -k_2) \Delta^{\mu\sigma}_{bm}(k_1)\Delta^{\mu\sigma}_{cn}(k_2) = \\ =& \int \frac{d^{2\omega}q d^{2\omega}k_1}{(2\pi)^{4\omega}} A^a_\lambda(-q) A^l_\lambda(q) G^{abc}_{\lambda\mu\nu}(-q, k_1, q - k_1) G^{lmn}_{\rho\sigma\tau}(q, -k_1, k_1 - q)\Delta^{\mu\sigma}_{bm}(k_1)\Delta^{\mu\sigma}_{cn}(q - k_1), \end{aligned} \quad \text{(A.411)}$$

[10]The terms which are linear in the quantum field can be dropped by assuming that the background field $\mathbf{A}$ satisfies the e.o.m.

where the vertex $G^{abc}_{\lambda\mu\nu}(p,k_1,k_2)$ is implicitly defined by the identity

$$
\begin{aligned}
\mathcal{V}_{\mathbf{A}QQ} =& -2g\mu^{2-\omega}\int d^{2\omega}x\left(\partial_\mu Q_\nu[\mathbf{A}^\mu,Q^\nu]-\left(1-\frac{1}{\xi}\right)(\partial_\mu Q^\mu[\mathbf{A}_\nu,Q^\nu])+2\partial^\mu\mathbf{A}^\nu[Q_\mu,Q_\nu]\right)= \\
=& -\frac{ig\mu^{2-\omega}}{2}f^{abc}\int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\delta^{2\omega}(p+k_1+k_2)\mathbf{A}^\lambda_a(p)Q^\mu_b(k_1)Q^\nu_c(k_2)\bigg((k_{1\lambda}-k_{2\lambda})\eta_{\mu\nu}+ \\
&+\left[k_{2\mu}-p_\mu+\frac{k_{1\mu}}{\xi}\right]\eta_{\lambda\nu}+\left[p_\nu-k_{1\nu}-\frac{k_{2\nu}}{\xi}\right]\eta_{\lambda\mu}\bigg)\equiv \\
\equiv& \int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\delta^{2\omega}(p+k_1+k_2)\mathbf{A}^\lambda_a(p)Q^\mu_b(k_1)Q^\nu_c(k_2)G^{\lambda\mu\nu}_{abc}(p,k_1,k_2)
\end{aligned}
\tag{A.412}
$$

We choose to work in Feynman gauge ($\xi=1$) where the vertices can be rewritten in the following simple form

$$
\begin{aligned}
\mathcal{V}_{\mathbf{A}QQ} =& -\frac{ig\mu^{2-\omega}}{2}f^{abc}\int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\delta^{2\omega}(p+k_1+k_2)\mathbf{A}^\lambda_a(p)Q^\mu_b(k_1)Q^\nu_c(k_2)\times \\
&\times\big((2k_{1\lambda}+p_\lambda)\eta_{\mu\nu}-2p_\mu\eta_{\lambda\nu}+2p_\nu\eta_{\lambda\mu}\big)
\end{aligned}
\tag{A.413}
$$

and the propagator is given by

$$
\Delta^{\mu\nu}_{ab}(p)=\frac{\delta_{ab}\delta^{\mu\nu}}{p^2}. \tag{A.414}
$$

The contribution is then given by

$$
\begin{aligned}
&\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{G^{abc}_{\lambda\mu\nu}(-q,k_1,q-k_1)G^{lbc}_{\rho\mu\nu}(q,-k_1,k_1-q)}{k_1^2(q-k_1)^2}= \\
=&\int_0^1dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{G^{abc}_{\lambda\mu\nu}(-q,k_1+tq,q(1-t)-k_1)G^{lbc}_{\rho\mu\nu}(q,-k_1-tq,k_1-q(1-t))}{[k_1^2+t(1-t)q^2]^2}
\end{aligned}
\tag{A.415}
$$

In expanding the above expression we can drop all the terms which are linear $k_{1\lambda}$ and we can perform the substitution $k_{1\lambda}k_{1\mu}\mapsto\frac{1}{2\omega}\delta_{\lambda\mu}k_1^2$

$$
\begin{aligned}
&-\frac{g^2\mu^{4-2\omega}}{4}f^{abc}f^{lbc}\int_0^1dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{\left[-2\left(\left((1-2t)^2\omega-4\right)q_\alpha q_\lambda+2(k_1\cdot k_1+2q\cdot q)\delta_{\alpha\lambda}\right)\right]}{[k_1^2+t(1-t)q^2]^2}= \\
=&-\frac{g^2\mu^{4-2\omega}}{4}f^{abc}f^{lbc}\int_0^1dt\frac{\Gamma(1-\omega)((1-t)tq\cdot q)^{\omega-2}}{2^{2\omega-1}\pi^\omega}\times \\
&\qquad\times\left((\omega-1)\left((1-2t)^2\omega-4\right)q_\alpha q_\lambda+2(((t-1)t+2)\omega-2)q\cdot q\delta_{\alpha\lambda}\right)= \\
=&-\frac{g^2\mu^{4-2\omega}}{4}\delta^{al}C_2(G)\left(\frac{2^{3-4\omega}\pi^{\frac{3}{2}-\omega}(7\omega-4)\csc(\pi\omega)(q\cdot q)^{\omega-2}\left(q\cdot q\delta_{\alpha\lambda}-q_\alpha q_\lambda\right)}{\Gamma\left(\omega+\frac{1}{2}\right)}\right)= \\
=&-g^2\mu^{4-2\omega}\delta^{al}C_2(G)\frac{5\left(q\cdot q\delta_{\alpha\lambda}-q_\alpha q_\lambda\right)}{48\pi^2(\omega-2)}+O\left((\omega-2)^0\right),
\end{aligned}
\tag{A.416}
$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation. Therefore the divergent part is

$$-\frac{5g^2\mu^{4-2\omega}C_2(G)}{48\pi^2(\omega-2)}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}A_a^{\lambda}(-q)\left(q{\cdot}q\delta_{\alpha\lambda}-q_{\alpha}q_{\lambda}\right)A_a^{\alpha}(q) \tag{A.417}$$

The contribution originating from the vertex with two $\mathbf{A}-$lines vanishes since it is proportional to the integral of $1/k_1^2$.

A.2 The ghost contribution

To complete the analysis in pure gauge theory we have to consider the ghost contribution. We have again two different type of vertices:

Ghost Vertex with one A−field : In momentum space it reads

$$\begin{aligned}
&2ig\mu^{2-\omega}\int d^{2\omega}x\ \left(\bar{c}[A_{\mu},\partial^{\mu}c]-(\partial_{\mu}\bar{c})[A^{\mu},c]\right)= \\
&=-g\mu^{2-\omega}\int\frac{d^{2\omega}p\ d^{2\omega}q\ d^{2\omega}k}{(2\pi)^{4\omega}}\delta(p+q+k)\bar{c}^a(p)c^b(q)A_{\mu}^c f_{abc}(q_{\mu}-p_{\mu})\equiv \\
&\equiv\int\frac{d^{2\omega}p\ d^{2\omega}q\ d^{2\omega}k}{(2\pi)^{4\omega}}\delta(p+q+k)\bar{c}^a(p)c^b(q)A_{\mu}^c S_{abc}^{\mu}(p,q,k).
\end{aligned} \tag{A.418}$$

The ghost contribution to the effective action is then given by

$$\begin{aligned}
&\frac{1}{2}\int\frac{d^{2\omega}p\ d^{2\omega}q\ d^{2\omega}k}{(2\pi)^{4\omega}}\frac{d^{2\omega}p_1\ d^{2\omega}q_1\ d^{2\omega}k_1}{(2\pi)^{4\omega}}\delta(p_1+q_1+k_1)\delta(p+q+k)\times \\
&\times\bar{c}^a(p)c^b(q)A_{\mu}^c S_{abc}^{\mu}(p,q,k)\bar{c}^l(p_1)c^m(q_1)A_{\nu}^n S_{lmn}^{\nu}(p_1,q_2,k_1)= \\
&=-\frac{1}{2}\int\frac{d^{2\omega}q\ d^{2\omega}k}{(2\pi)^{4\omega}}A_{\mu}^c(k)A_{\nu}^n(-k)\frac{S_{abc}^{\mu}(q,-k-q,k)S_{ban}^{\nu}(-q,q+k,-k)}{q^2(q+k)^2}.
\end{aligned} \tag{A.419}$$

The loop integral to be computed is

$$\begin{aligned}
&-\frac{1}{2}\int_0^1 dt\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{S_{abc}^{\mu}(q-tk,-k(1-t)-q,k)S_{ban}^{\nu}(-q+tk,q+k(1-t),-k)}{[q^2+t(1-t)k^2]^2}= \\
&=\frac{1}{2}g^2\mu^{4-2\omega}\delta_{cn}C_2(G)\int_0^1 dt\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}\frac{\left(-k_{\mu}k_{\nu}(1-2t)^2-\frac{2q\cdot q\delta_{\mu\nu}}{\omega}\right)}{[q^2+t(1-t)k^2]^2}= \\
&=\frac{1}{2}g^2\mu^{4-2\omega}\delta_{cn}C_2(G)\int_0^1 dt\frac{\Gamma(1-\omega)\left((\omega-1)k_{\mu}k_{\nu}(1-2t)^2+2(t-1)tk{\cdot}k\delta_{\mu\nu}\right)}{(4\pi)^{\omega}((1-t)tk{\cdot}k)^{2-\omega}}= \\
&=\frac{g^2\mu^{4-2\omega}}{2}\delta_{cn}C_2(G)\frac{4^{1-2\omega}\pi^{\frac{3}{2}-\omega}\csc(\pi\omega)(k.k)^{\omega-2}\left(k_{\mu}k_{\nu}-k\cdot k\delta_{\mu\nu}\right)}{\Gamma\left(\omega+\frac{1}{2}\right)}= \\
&=\frac{g^2\mu^{4-2\omega}}{2}\delta_{cn}C_2(G)\frac{k_{\alpha}k_{\lambda}-k.k\delta_{\alpha\lambda}}{48\pi^2(\omega-2)}+O\left((\omega-2)^0\right)
\end{aligned} \tag{A.420}$$

Therefore the divergent part is

$$-\frac{g^2\mu^{4-2\omega}C_2(G)}{96\pi^2(\omega-2)}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}A_a^\lambda(-q)(q.q\delta_{\alpha\lambda}-q_\alpha q_\lambda)A_a^\alpha(q)+O\left((\omega-2)^0\right) \tag{A.421}$$

The contribution originating from the ghost vertex with two $\mathbf{A}-$lines vanishes since it is proportional to the integral of $1/k^2$. Summarizing the gauge ghost contribution we have

$$-\frac{11g^2\mu^{4-2\omega}C_2(G)}{96\pi^2(\omega-2)}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}A_a^\lambda(-q)(q\cdot q\delta_{\alpha\lambda}-q_\alpha q_\lambda)A_a^\alpha(q)+O\left((\omega-2)^0\right) \tag{A.422}$$

A.3 Weyl/Majorana fermions

Next we consider the contribution of a left Weyl fermion transforming in the representation R of the gauge group. The interaction for one field is given

$$-g\mu^{2-\omega}\int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\delta(p+k_1+k_2)A_\mu^a(p)\bar{\psi}_i(k_1)\gamma^\mu T_a^{(R)}\frac{(1+\gamma_5)}{2}\psi_i(k_2), \tag{A.423}$$

and it yields the following one-loop correction to the quadratic part of the effective action

$$\begin{aligned}&\frac{g^2\mu^{4-2\omega}}{2}\int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\frac{d^{2\omega}qd^{2\omega}r_1d^{2\omega}r_2}{(2\pi)^{4\omega}}\delta(p+k_1+k_2)\delta(q+r_1+r_2)\times\\&\times A_\mu^a(p)A_\nu^b(q)\bar{\psi}_i(k_1)\gamma^\mu T_a^{(R)}\frac{(1+\gamma_5)}{2}\psi_i(k_2)\bar{\psi}(r_1)\gamma^\nu T_b^{(R)}\frac{(1+\gamma_5)}{2}\psi(r_2)=\\&=-\frac{g^2\mu^{4-2\omega}}{2}\mathrm{Tr}(T_a^{(R)}T_b^{(R)})\int\frac{d^{2\omega}qd^{2\omega}k_1}{(2\pi)^{4\omega}}A_\mu^a(-q)A_\nu^b(q)\mathrm{Tr}\left(\gamma^\mu S(q-k_1)\gamma^\nu\frac{(1+\gamma_5)}{2}S(-k_1)\right)=\\&=\frac{g^2\mu^{4-2\omega}}{2}C_2(R)\delta_{ab}\int\frac{d^{2\omega}qd^{2\omega}k_1}{(2\pi)^{4\omega}}A_\mu^a(-q)A_\nu^b(q)\mathrm{Tr}\left(\gamma^\mu S(q-k_1)\gamma^\nu\frac{(1+\gamma_5)}{2}S(-k_1)\right),\end{aligned} \tag{A.424}$$

where $S(p)$ is the fermion propagator and it is given by

$$S(p)=\frac{i}{\not{p}}=i\frac{\not{p}}{p^2} \tag{A.425}$$

and $\mathrm{Tr}(T_a^{(R)}T_b^{(R)})=-C_2(R)\delta_{ab}$. The loop integral to be performed is

$$\begin{aligned}&\frac{1}{2}\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\mathrm{Tr}\left(\frac{\gamma^\mu(\not{q}-\not{k}_1)\gamma^\nu\ \not{k}_1(1-\gamma_5)}{(q-k_1)^2k_1^2}\right)=\\&=\frac{1}{2}\int_0^1dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\mathrm{Tr}\left(\frac{\gamma^\mu(\not{q}(1-t)-\not{k}_1)\gamma^\nu(\not{k}_1+t\ \not{q})(1-\gamma_5)}{[k_1^2+t(1-t)q]^2}\right)=\\&=\frac{1}{2}\int_0^1dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\mathrm{Tr}\left(\frac{\gamma^\mu(\not{q}(1-t)-\not{k}_1)\gamma^\nu(\not{k}_1+t\ \not{q})}{[k_1^2+t(1-t)q]^2}\right)=\\&=-\frac{1}{2}\int_0^1dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{\mathrm{Tr}\left(\gamma^\mu\ \not{k}_1\gamma^\nu\ \not{k}_1\right)}{[k_1^2+t(1-t)q]^2}+\frac{1}{2}\int_0^1dt(1-t)t\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{\mathrm{Tr}\left(\gamma^\mu\ \not{q}\gamma^\nu\ \not{q}\right)}{[k_1^2+t(1-t)q]^2}=\\&=-2^{\omega-1}\int_0^1dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{(2k_1^\mu k_1^\nu-k_1\cdot k_1\delta^{\mu\nu})-t(1-t)(2q^\mu q^\nu-q\cdot q\delta^{\mu\nu})}{[k_1^2+t(1-t)q]^2}\\&=-2^{\omega-1}\int_0^1dt2^{1-2\omega}\pi^{-\omega}(t-1)t((1-t)tq.q)^{\omega-2}\Gamma(2-\omega)\left(q_\mu q_\nu-q.q\delta_{\mu\nu}\right)=\end{aligned} \tag{A.426}$$

$$
\begin{aligned}
=& -2^{-\omega}\pi^{-\omega}(q.q)^{\omega-2}\Gamma(2-\omega)\Gamma(\omega-1)\left(\frac{\Gamma(\omega+1)}{\Gamma(2\omega)}-\frac{\Gamma(\omega)}{\Gamma(2\omega-1)}\right)(q_\mu q_\nu - q.q\delta_{\mu\nu}) = \\
=& -\frac{q_\mu q_\nu - q.q\delta_{\mu\nu}}{24\pi^2(\omega-2)} + O\left((\omega-2)^0\right)
\end{aligned} \tag{A.427}
$$

The divergent contribution to the action is

$$
\frac{g^2\mu^{4-2\omega}C_2(R)}{48\pi^2(\omega-2)}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}A^a_\mu(-q)(q.q\delta^{\mu\nu}-q^\mu q^\nu)A^a_\nu(q) + O\left((\omega-2)^0\right), \tag{A.428}
$$

The contribution for right Weyl or massless majorana fermions is the same.

A.4 Scalars

The action for a scalar in the representation S is

$$
S = \int d^{2\omega x}\mathbf{D}_\mu\phi(\mathbf{D}^\mu\phi)^\dagger, \tag{A.429}
$$

with $\mathbf{D}_\mu\phi = \partial_\mu\phi + g\mathbf{A}^a_\mu T^{(S)}_a\phi$. The vertex with one **A**-line is given by

$$
\begin{aligned}
& g\int d^{2\omega}x\mathbf{A}^\mu_a(\phi T^{(S)a}\partial_\mu\phi^\dagger + \partial_\mu\phi T^{(S)a\dagger}\phi^\dagger) = \\
=& ig\int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\delta^{2\omega}(p+k_1+k_2)\mathbf{A}^\mu_a(p)\phi(k_1)T^{(S)a}\phi^\dagger(k_2)(k_{2\mu}-k_{1\mu}) \equiv \\
\equiv& \int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\delta^{2\omega}(p+k_1+k_2)\mathbf{A}^\mu_a(p)\phi_i(k_1)\phi^\dagger_j(k_2)V^{a;ij}_\mu(p,k_1,k_2)
\end{aligned} \tag{A.430}
$$

$$
\begin{aligned}
& \frac{1}{2}\int\frac{d^{2\omega}pd^{2\omega}k_1d^{2\omega}k_2}{(2\pi)^{4\omega}}\frac{d^{2\omega}qd^{2\omega}r_1d^{2\omega}r_2}{(2\pi)^{4\omega}}\delta^{2\omega}(q+r_1+r_2)\delta^{2\omega}(p+k_1+k_2)\times \\
& \times\mathbf{A}^\mu_a(p)\phi_i(k_1)\phi^\dagger_j(k_2)\mathbf{A}^\nu_b(q)\phi_k(r_1)\phi^\dagger_l(r_2)V^{a;ij}_\mu(p,k_1,k_2)V^{b;kl}_\nu(q,r_1,r_2) = \\
=& \frac{1}{2}\int\frac{d^{2\omega}qd^{2\omega}k_1}{(2\pi)^{4\omega}}\mathbf{A}^\mu_a(-q)\mathbf{A}^\nu_b(q)\Delta_{il}(k_1)\Delta_{kj}(q-k_1)V^{a;ij}_\mu(-q,k_1,q-k_1)V^{b;kl}_\nu(q,-k_1,k_1-q) = \\
=& \frac{1}{2}\int\frac{d^{2\omega}qd^{2\omega}k_1}{(2\pi)^{4\omega}}\mathbf{A}^\mu_a(-q)\mathbf{A}^\nu_b(q)\frac{V^{a;ij}_\mu(-q,k_1,q-k_1)V^{b;ji}_\nu(q,k_1-q,-k_1)}{k_1^2(q-k_1)^2}.
\end{aligned} \tag{A.431}
$$

Then the loop integral to be computed is

$$
\begin{aligned}
&-\frac{g^2}{2}\mathrm{Tr}(T^{(S)a}T^{(S)b})\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{(q_\mu-2k_{1\mu})(q_\nu-2k_{1\nu})}{k_1^2(q-k_1)^2}=\\
=&\frac{g^2}{2}C_2(S)\delta^{ab}\int_0^1 dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{(q_\mu(1-2t)-2k_{1\mu})(q_\nu(1-2t)-2k_{1\nu})}{[k_1^2+t(1-t)q^2]^2}=\\
=&\frac{g^2}{2}C_2(S)\delta^{ab}\int_0^1 dt\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{q_\mu q_\nu(1-2t)^2+4k_{1\mu}k_{1\nu}}{[k_1^2+t(1-t)q^2]^2}=\\
=&\frac{g^2}{2}C_2(S)\delta^{ab}\int_0^1 dt\left[(4\pi)^{-\omega}(1-2t)^2\Gamma(2-\omega)q_\alpha q_\lambda((1-t)tq\cdot q)^{\omega-2}+\right.\\
&\left.+\,2^{1-2\omega}\pi^{-\omega}\Gamma(1-\omega)\delta_{\alpha\lambda}((1-t)tq\cdot q)^{\omega-1}\right]=\\
=&\frac{g^2}{2}C_2(S)\delta^{ab}\frac{4^{1-2\omega}\pi^{\frac{3}{2}-\omega}\csc(\pi\omega)(q\cdot q)^{\omega-2}\left(q\cdot q\delta_{\alpha\lambda}-q_\alpha q_\lambda\right)}{\Gamma\left(\omega+\frac{1}{2}\right)}=\\
=&\frac{g^2C_2(S)\delta^{ab}}{96\pi^2(\omega-2)}(q.q\delta_{\alpha\lambda}-q_\alpha q_\lambda)+O\left((\omega-2)^0\right)
\end{aligned}
\tag{A.432}
$$

Since the vertex with two scalar and two $\mathbf{A}-$limes yields a vanishing result, the divergent contribution to the action is

$$
\frac{g^2\mu^{4-2\omega}C_2(S)}{96\pi^2(\omega-2)}\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}A^a_\mu(-q)(q.q\delta^{\mu\nu}-q^\mu q^\nu)A^a_\nu(q)+O\left((\omega-2)^0\right). \tag{A.433}
$$

A.5 Summary

In a gauge theory with n_f Weyl fermions in the representation R and n_s (complex) scalar field in the representation S the divergent contributions is

$$
\begin{aligned}
&-\frac{g^2\mu^{4-2\omega}}{32\pi^2(\omega-2)}\left(\frac{11}{3}C_2(G)-\frac{2n_f}{3}C_2(R)-\frac{n_s}{3}C_2(S)\right)\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}A^a_\mu(-q)(q.q\delta^{\mu\nu}-q^\mu q^\nu)A^a_\nu(q)=\\
&=\frac{g^2\mu^{4-2\omega}}{(4\pi)^2(\omega-2)}\left(\frac{11}{3}C_2(G)-\frac{2n_f}{3}C_2(R)-\frac{n_s}{3}C_2(S)\right)\frac{1}{2}\int d^{2\omega}xF_{\mu\nu}(x)F^{\mu\nu}(x)
\end{aligned}
\tag{A.434}
$$

where we have used

$$
\begin{aligned}
&\int\frac{d^{2\omega}q}{(2\pi)^{2\omega}}A^a_\mu(-q)(q.q\delta^{\mu\nu}-q^\mu q^\nu)A^a_\nu(q)=\\
=&\int d^{2\omega}x\int\frac{d^{2\omega}qd^{2\omega}p}{(2\pi)^{4\omega}}e^{i(p+q)x}A^a_\mu(p)(q.q\delta^{\mu\nu}-q^\mu q^\nu)A^a_\nu(q)=\\
=&\int d^{2\omega}xA^a_\mu(x)(-\Box\delta^{\mu\nu}+\partial^\mu\partial^\nu)A^a_\nu(x)=\\
=&\int d^{2\omega}x(\partial_\nu A^a_\mu(x)\partial^\nu A^{a\mu}(x)-(\partial^\nu A^a_\nu(x))^2)=\\
=&\frac{1}{2}\int d^{2\omega}xF^a_{\mu\nu}(x)F^{\mu\nu}_a(x)=-\int d^{2\omega}xF_{\mu\nu}(x)F^{\mu\nu}(x)
\end{aligned}
\tag{A.435}
$$

The wave-function renormalization $Z_{\mathbf{A}}$ is then

$$Z_{\mathbf{A}} = 1 - \frac{g^2\mu^{4-2\omega}}{(4\pi)^2(\omega-2)}\left(\frac{11}{3}C_2(G) - \frac{2n_f}{3}C_2(R) - \frac{n_s}{3}C_2(S)\right) \tag{A.436}$$

Recalling that

$$\beta(g_R) = \frac{1}{2}g_R(\mu)\mu\frac{\partial \log Z_{\mathbf{A}}}{\partial\mu}\bigg|_{\omega=2} = -\frac{g_R^3(\mu)}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{2n_f}{3}C_2(R) - \frac{n_s}{3}C_2(S)\right). \tag{A.437}$$

Below, we give a table summarizing the different contributions to the $\beta-$function

Gauge fields	Weyl fermions	complex scalars
$\frac{11}{3}C_2(G)$	$-\frac{2}{3}C_2(R)$	$-\frac{1}{3}C_2(S)$

B Anomalies

The Noether theorem translates the invariance of a classical field theory under a continuous global symmetry into the existence of a divergenceless current

$$\partial_\mu J^\mu = 0, \tag{B.438}$$

which, in turn, implies the presence of a conserved charge

$$Q = \int_{t=cost.} d^3x J^0(x). \tag{B.439}$$

At the quantum level, the invariance of the theory under a continuous global symmetry is instead naturally expressed in terms of some Ward identities obeyed by the the correlation functions. We must have

$$\langle 0|\mathrm{T}\partial_\mu J^\mu(x)\prod_{i=1}^{N}\Phi_i(x_i)|0\rangle = -\sum_{i=1}^{N}\delta(x-x_i)\langle 0|\mathrm{T}\delta\Phi(x_i)\prod_{j\neq i}\Phi_j(x_j)|0\rangle. \tag{B.440}$$

Here $\delta\Phi(x)$ denote the variation of the operator $\Phi(x)$ under the symmetry. Obviously, the simplest Ward identity to be satisfied is

$$\langle 0|\partial_\mu J^\mu(x)|0\rangle = 0. \tag{B.441}$$

Consider, now, a theory of a massless Dirac fermion coupled to a $U(1)$ gauge field A_μ. Its action is given by

$$S = i\int d^4x\bar{\psi}\gamma^\mu\left(\partial_\mu + ieA_\mu\right)\psi. \tag{B.442}$$

At the classical level, this action, a part from the obvious $U(1)$ vector symmetry, is also invariant under axial transformations, namely

$$\psi \mapsto e^{\alpha\gamma_5}\psi \qquad \bar{\psi} \mapsto \bar{\psi}e^{\alpha\gamma_5}. \tag{B.443}$$

Classically, we have the conserved current

$$J_A^\mu(x) = \bar{\psi}(x)\gamma_5\gamma^\mu\psi(x). \tag{B.444}$$

Is this current still conserved at the quantum level? To answer this question, let us check perturbatively the Ward identity

$$\langle 0|\partial_\mu J_A^\mu(x)|0\rangle \stackrel{?}{=} 0. \tag{B.445}$$

In four dimension, because of the presence of γ_5, the first non trivial contribution is

$$\begin{aligned}
\mathrm{WI} =& \frac{(ie)^2}{2}\int d^4y_1\int d^4y_2 A_\alpha(y_1)A_\beta(y_2)\partial_\mu\langle \mathrm{T} J_A^\mu(x)\bar{\psi}(y_1)\gamma^\alpha\psi(y_1)\bar{\psi}(y_2)\gamma^\beta\psi(y_2)\rangle = \\
=& \frac{(ie)^2}{2}\int d^4y_1\int d^4y_2 A_\alpha(y_1)A_\beta(y_2)\partial_\mu\langle \mathrm{T}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\bar{\psi}(y_1)\gamma^\alpha\psi(y_1)\bar{\psi}(y_2)\gamma^\beta\psi(y_2)\rangle = \\
=& -(ie)^2\int d^4y_1\int d^4y_2 A_\alpha(y_1)A_\beta(y_2)\partial_\mu \mathrm{Tr}\left(\gamma^\mu\gamma^5 S(x-y_1)\gamma^\alpha S(y_1-y_2)\gamma^\beta S(y_2-x)\right).
\end{aligned} \tag{B.446}$$

The evaluation of this amplitude corresponds to the *celebrated* **AVV**-diagram drawn in the figure below.

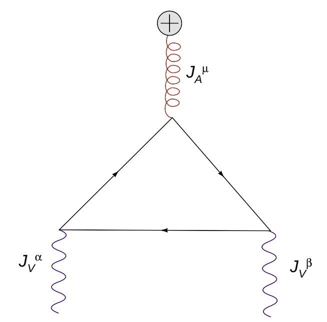

Figure 2: The **AVV**−diagram

A rigorous evaluation of this diagram can be very delicate because of the necessary regularization procedure. Here we shall just sketch the procedure of the computation. The first step is to use the momentum space, where

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i\,\not{p}}{p^2+i\epsilon}e^{-ip(x-y)}. \tag{B.447}$$

Then

$$\begin{aligned}
\mathrm{WI} =& -(ie)^2 i\int \frac{d^4k_1 d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}(k_{1\mu}+k_{2\mu})\times \\
&\times \int \frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma^5\frac{i\,\not{p}_1}{p_1^2+i\epsilon}\gamma^\alpha\frac{i(\not{k}_1+\not{p}_1)}{(k_1+p_1)^2+i\epsilon}\gamma^\beta\frac{i(\not{k}_1+\not{k}_2+\not{p}_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right).
\end{aligned} \tag{B.448}$$

Let's check the Ward identity:

$$
\begin{aligned}
=&-(ie)^2\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\times\\
&\times\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left((\not{k}_1+\not{k}_2+\not{p}_1-\not{p}_1)\gamma^5\frac{\not{p}_1}{p_1^2+i\epsilon}\gamma^\alpha\frac{(\not{k}_1+\not{p}_1)}{(k_1+p_1)^2+i\epsilon}\gamma^\beta\frac{(\not{k}_1+\not{k}_2+\not{p}_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)=\\
=&-(ie)^2\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\times\\
&\times\int\frac{d^4p_1}{(2\pi)^4}\left[\mathrm{Tr}\left(\frac{\gamma^5\gamma^\alpha(\not{k}_1+\not{p}_1)}{(k_1+p_1)^2+i\epsilon}\frac{\gamma^\beta(\not{k}_1+\not{k}_2+\not{p}_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)-\mathrm{Tr}\left(\frac{\gamma^5\gamma^\beta\ \not{p}_1}{p_1^2+i\epsilon}\frac{\gamma^\alpha(\not{k}_1+\not{p}_1)}{(k_1+p_1)^2+i\epsilon}\right)\right]=\\
=&e^2\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\times\\
&\times\int\frac{d^4p_1}{(2\pi)^4}\left[\mathrm{Tr}\left(\frac{\gamma^5\gamma^\alpha(\not{k}_1+\not{p}_1)}{(k_1+p_1)^2+i\epsilon}\frac{\gamma^\beta(\not{k}_1+\not{k}_2+\not{p}_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)-\mathrm{Tr}\left(\frac{\gamma^5\gamma^\alpha\ \not{p}_1}{p_1^2+i\epsilon}\frac{\gamma^\beta(\not{k}_1+\not{p}_1)}{(k_1+p_1)^2+i\epsilon}\right)\right]=\\
=&e^2\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\times\\
&\times\int\frac{d^4p_1}{(2\pi)^4}\left[\mathrm{Tr}\left(\frac{\gamma^5\gamma^\alpha(\not{k}_1+\not{p}_1)}{(k_1+p_1)^2+i\epsilon}\frac{\gamma^\beta(\not{k}_1+\not{k}_2+\not{p}_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)-\mathrm{Tr}\left(\frac{\gamma^5\gamma^\alpha\ \not{p}_1}{p_1^2+i\epsilon}\frac{\gamma^\beta(\not{k}_2+\not{p}_1)}{(k_2+p_1)^2+i\epsilon}\right)\right].
\end{aligned}
\tag{B.449}
$$

If we perform the change of variable $p_1\mapsto p_1+k_1$ in the second term of the integrand, the two terms cancel yielding a vanishing result. However this conclusion is naive. The original integral (B.462) and the integral (B.449) are linearly divergent and a shift in the integration variable is not a legal operation. Consider the following example

$$
I=\int_{-\infty}^{\infty}dx(f(x+a)-f(x)),
\tag{B.450}
$$

where $f(x)$ is a regular function such that $\lim_{x\to\pm\infty}f(x)=\text{finite}$. Then

$$
I=\int_{-\infty}^{\infty}dx\sum_{n=1}^{\infty}\frac{a^n}{n!}f^{(n)}(x)=\sum_{n=1}^{\infty}\frac{a^n}{n!}[f^{(n-1)}(\infty)-f^{(n-1)}(-\infty)]=a(f(\infty)-f(-\infty)).
\tag{B.451}
$$

Recall that for a regular function $\lim_{x\to\pm\infty}f(x)=\text{finite}\Rightarrow f^{(n)}(\pm\infty)=0\quad\forall n\geq 1$. Notice that for a logarithmic divergent integral $f(\pm\infty)=0$ and therefore the translation in the integration variable is legal. This result can be easily generalize to a $n-$dimensional integral and we find

$$
\int d^nx(f(x+a)-f(x))=\lim_{|x|\to\infty}\frac{a\cdot x}{|x|}f(x)S_{n-1}(|x|),
\tag{B.452}
$$

where $S_n(|x|)$ is the volume of $n-$dimensional sphere of radius $|x|$. An additional factor i is present in (B.452) if the integral is performed over the Minkowski space due to the Wick rotation. Therefore an apparently vanishing integral can produce a finite result because of a linear divergence. Let us apply this result to our integral (B.449). Performing the trace over the Dirac matrice, the above expression takes the form

$$\begin{aligned}\mathrm{WI} =&4ie^2\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\epsilon^{\alpha\mu\beta\nu}k_{2\nu}\times\\&\times\int\frac{d^4p_1}{(2\pi)^4}\left[\frac{(k_{1\mu}+p_{1\mu})}{[(k_1+p_1)^2+i\epsilon][(k_1+k_2+p_1)^2+i\epsilon]}-\frac{p_{1\mu}}{[p_1^2+i\epsilon][(k_2+p_1)^2+i\epsilon]}\right]\end{aligned}\tag{B.453}$$

and we can directly apply (B.452)

$$\begin{aligned}\mathrm{WI} =&4ie^2\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\epsilon^{\alpha\mu\beta\nu}k_{2\nu}\lim_{|p|\to\infty}\frac{1}{(2\pi)^4}(2\pi^2)ip^3\frac{k_1\cdot p}{p}\frac{p_\mu}{p^4}=\\=&-4e^2\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\epsilon^{\alpha\mu\beta\nu}k_{2\nu}\frac{1}{8\pi^2}k_1^\sigma\lim_{|p|\to\infty}\frac{p_\mu p_\sigma}{p^2}=\\=&-\frac{e^2}{8\pi^2}\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}\epsilon^{\alpha\mu\beta\nu}k_{2\nu}k_{1\mu}=\\=&-\frac{e^2}{8\pi^2}\int\frac{d^4k_1d^4k_2}{(2\pi)^8}e^{i(k_1+k_2)x}\epsilon^{\mu\alpha\nu\beta}k_{1\mu}A_\alpha(k_1)k_{2\nu}A_\beta(k_2)=\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x),\end{aligned}\tag{B.454}$$

where we have set $\lim_{|p|\to\infty}\frac{p_\mu p_\sigma}{p^2}=\frac{1}{4}\eta_{\mu\sigma}$. This result is partially ambiguous: since we have lost the invariance under momentum translation in the loop, the final result will depend on the choice of the momentum in the initial loop integral. For example if we had begun with the loop integral

$$\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma^5\frac{i(\not p_1+\not a)}{(p_1+a)^2+i\epsilon}\gamma^\alpha\frac{i(\not k_1+\not p_1+\not a)}{(k_1+p_1+a)^2+i\epsilon}\gamma^\beta\frac{i(\not k_1+\not k_2+\not p_1+\not a)}{(k_1+k_2+p_1+a)^2+i\epsilon}\right),\tag{B.455}$$

we would have found an additional contribution of the form

$$\begin{aligned}&\frac{(i)^3}{(2\pi)^4}2\pi^2i\lim_{|p|\to\infty}\mathrm{Tr}(\gamma^\mu\gamma_5\gamma^{\nu_1}\gamma^\alpha\gamma^{\nu_2}\gamma^\beta\gamma^{\nu_3})p^3\frac{p\cdot a}{p}\frac{p_{\nu_1}p_{\nu_2}p_{\nu_3}}{p^6}=\\=&\frac{(i)^3}{(2\pi)^4}2\pi^2i4i\epsilon^{\mu\nu_1\alpha\beta}\lim_{|p|\to\infty}p^5\frac{p\cdot a}{p}\frac{p_{\nu_1}}{p^6}=\frac{i}{8\pi^2}\epsilon^{\mu\alpha\beta\nu}a_\nu.\end{aligned}\tag{B.456}$$

This produces in the WI an additional term given by

$$\mathrm{WI}_{add.}=-(ie)^2i\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}(k_{1\mu}+k_{2\mu})\frac{i}{8\pi^2}\epsilon^{\mu\alpha\beta\nu}a_\nu\tag{B.457}$$

If we write $a=c(k_1+k_2)-bk_2$, the above expression becomes

$$\mathrm{WI}_{add.}=\frac{e^2b}{8\pi^2}\int\frac{d^4k_1d^4k_2}{(2\pi)^8}A_\alpha(k_1)A_\beta(k_2)e^{i(k_1+k_2)x}k_{1\mu}k_{2\nu}\epsilon^{\mu\alpha\beta\nu}=\frac{e^2b}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x).\tag{B.458}$$

There we obtain the following generalized WI

$$\langle 0|\partial_\mu j_A^\mu(x)|0\rangle=\frac{e^2(1+b)}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x).\tag{B.459}$$

Here the parameter b seems arbitrary. In order to fix b, consider again the triangle graph and multiply by $k_{1\alpha}$. The vanishing of this contraction is equivalent to require that the $U(1)$ vector current is conserved. We find

$$
\begin{aligned}
&-\frac{ik_{1\alpha}}{2}\left[\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{\not p_1}{p_1^2+i\epsilon}\gamma^\alpha\frac{(\not k_1+\not p_1)}{(k_1+p_1)^2+i\epsilon}\gamma^\beta\frac{(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)+\right.\\
&\left.+\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{\not p_1}{p_1^2+i\epsilon}\gamma^\beta\frac{(\not k_2+\not p_1)}{(k_2+p_1)^2+i\epsilon}\gamma^\alpha\frac{(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)\right]=\\
&=-\frac{i}{2}\left[\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\frac{\gamma^\mu\gamma_5\not p_1}{p_1^2+i\epsilon}\frac{\gamma^\beta(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)-\mathrm{Tr}\left(\frac{\gamma^\mu\gamma_5(\not k_1+\not p_1)}{(k_1+p_1)^2+i\epsilon}\frac{\gamma^\beta(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)+\right.\\
&\left.+\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\frac{\gamma^\mu\gamma_5\not p_1}{p_1^2+i\epsilon}\frac{\gamma^\beta(\not k_2+\not p_1)}{(k_2+p_1)^2+i\epsilon}\right)-\mathrm{Tr}\left(\frac{\gamma^\mu\gamma_5\not p_1}{p_1^2+i\epsilon}\frac{\gamma^\beta(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)\right]=\\
&=-\frac{i}{2}\left[\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\frac{\gamma^\mu\gamma_5\not p_1}{p_1^2+i\epsilon}\frac{\gamma^\beta(\not k_2+\not p_1)}{(k_2+p_1)^2+i\epsilon}\right)-\mathrm{Tr}\left(\frac{\gamma^\mu\gamma_5(\not k_1+\not p_1)}{(k_1+p_1)^2+i\epsilon}\frac{\gamma^\beta(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)\right]=\\
&=\frac{i}{16\pi^2}\epsilon^{\mu\beta\rho\sigma}k_{1\rho}k_{2\sigma}
\end{aligned}
\tag{B.460}
$$

If we consider a different choice in the integration variables, we have the following ambiguity:

$$
\begin{aligned}
&\frac{1}{2}k_{1\alpha}\left(\frac{i}{8\pi^2}\epsilon^{\mu\alpha\beta\nu}(a(k_{1\nu}+k_{2\nu})-bk_{2\nu})+\frac{i}{8\pi^2}\epsilon^{\mu\beta\alpha\nu}(a(k_{1\nu}+k_{2\nu})-bk_{1\nu})\right)=\\
=&\frac{i}{16\pi^2}\epsilon^{\mu\alpha\beta\nu}k_{1\alpha}\left((a(k_{1\nu}+k_{2\nu})-bk_{2\nu})-a(k_{1\nu}+k_{2\nu})+bk_{1\nu})\right)=-\frac{ib}{16\pi^2}\epsilon^{\mu\beta\alpha\nu}k_{1\alpha}k_{2\nu}.
\end{aligned}
\tag{B.461}
$$

Summarizing, we obtain

$$
\begin{aligned}
&-\frac{ik_{1\alpha}}{2}\left[\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{\not p_1}{p_1^2+i\epsilon}\gamma^\alpha\frac{(\not k_1+\not p_1)}{(k_1+p_1)^2+i\epsilon}\gamma^\beta\frac{(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)+\right.\\
&\left.+\int\frac{d^4p_1}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{\not p_1}{p_1^2+i\epsilon}\gamma^\beta\frac{(\not k_2+\not p_1)}{(k_2+p_1)^2+i\epsilon}\gamma^\alpha\frac{(\not k_1+\not k_2+\not p_1)}{(k_1+k_2+p_1)^2+i\epsilon}\right)\right]=\\
&=-\frac{i}{16\pi^2}(1-b)\epsilon^{\mu\beta\rho\sigma}k_{1\rho}k_{2\sigma}.
\end{aligned}
\tag{B.462}
$$

Therefore the vector current is conserved if and only $b=1$, but with this choice the axial current is anomalous and we have

$$\langle 0|\partial_\mu j^\mu_A(x)|0\rangle=\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x). \tag{B.463}$$

In other words if we require that the vector current is conserved, the axial current is not conserved and it satisfies the relation (B.463).

The next step is to generalize this result to the non-abelian case, namely a theory of a massless Dirac fermion transforming in a representation R of gauge group G gauge and coupled to the gauge field A_μ of the same group G. Its action is given by

$$S=i\int d^4x\bar\psi\gamma^\mu\left(\partial_\mu+eA^a_\mu T^{(R)}_a\right)\psi. \tag{B.464}$$

Classically, we have the conserved axial current

$$J^{\mu}_{aA}(x) = \bar{\psi}(x)\gamma_5\gamma^{\mu}T_a^{(R)}\psi(x). \tag{B.465}$$

We can repeat the analysis and what changes is the group theoretical factor in front of the anomalous term. In fact

$$\begin{aligned}
\mathrm{WI} &= \frac{(e)^2}{2}\int d^4y_1 d^4y_2 A^a_\alpha(y_1)A^b_\beta(y_2)\partial_\mu\langle \mathrm{T}J^\mu_{cA}(x)\bar{\psi}(y_1)\gamma^\alpha T_a^{(R)}\psi(y_1)\bar{\psi}(y_2)\gamma^\beta T_b^{(R)}\psi(y_2)\rangle = \\
&= \frac{(ie)^2}{2}\int d^4y_1\, d^4y_2 A^a_\alpha(y_1)A^b_\beta(y_2)\partial_\mu\langle \mathrm{T}\bar{\psi}(x)\gamma^5\gamma^\mu T_c^{(R)}\psi(x)\bar{\psi}(y_1)\gamma^\alpha T_a^{(R)}\psi(y_1)\bar{\psi}(y_2)\gamma^\beta T_b^{(R)}\psi(y_2)\rangle = \\
&= -\frac{(e)^2}{2}\int d^4y_1 d^4y_2 A^a_\alpha(y_1)A^b_\beta(y_2)\times \\
&\qquad\times\Big[\mathrm{Tr}(T_c^{(R)}T_a^{(R)}T_b^{(R)})\partial_\mu \mathrm{Tr}\left(\gamma^\mu\gamma^5 S(x-y_1)\gamma^\alpha S(y_1-y_2)\gamma^\beta S(y_2-x)\right) + \\
&\qquad\quad + \mathrm{Tr}(T_c^{(R)}T_b^{(R)}T_a^{(R)})\partial_\mu \mathrm{Tr}\left(\gamma^\mu\gamma^5 S(x-y_2)\gamma^\beta S(y_2-y_1)\gamma^\alpha S(y_1-x)\right)\Big] = \\
&= -e^2\mathrm{Tr}(T_c^{(R)}T_a^{(R)}T_b^{(R)})\int d^4y_1 d^4y_2 A^a_\alpha(y_1)A^b_\beta(y_2)\partial_\mu \mathrm{Tr}\left(\gamma^\mu\gamma^5 S(x-y_1)\gamma^\alpha S(y_1-y_2)\gamma^\beta S(y_2-x)\right) = \\
&= -\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(T_c^{(R)}F_{\mu\nu}(x)F_{\alpha\beta}(x)).
\end{aligned} \tag{B.466}$$

Therefore the coefficient of the anomaly is proportional to the group theoretical factor[11]

$$d_{abc} = \frac{1}{2}\mathrm{Tr}\left(T_c^{(R)}\{T_a^{(R)}, T_b^{(R)}\}\right) \tag{B.467}$$

and vanishes when this coefficient is zero. This coefficient vanishes when the representation R satisfies the condition

$$\mathrm{Tr}\left(T_c^{(R)}T_a^{(R)}T_b^{(R)}\right) = -\mathrm{Tr}\left(T_c^{(R)}T_b^{(R)}T_a^{(R)}\right). \tag{B.468}$$

There is a very simple way to satisfies this condition. Notice that

$$\mathrm{Tr}\left(T_c^{(R)}T_a^{(R)}T_b^{(R)}\right) = \mathrm{Tr}\left(T_c^{(R)}T_a^{(R)}T_b^{(R)}\right)^T = \mathrm{Tr}\left(T_b^{(R)T}T_a^{(R)T}T_c^{(R)T}\right). \tag{B.469}$$

If

$$T_c^{(R)T} = -S^{-1}T_c^{(R)}S, \tag{B.470}$$

then we find

$$\mathrm{Tr}\left(T_c^{(R)}T_a^{(R)}T_b^{(R)}\right) = -\mathrm{Tr}\left(T_b^{(R)}T_a^{(R)}T_c^{(R)}\right) = -\mathrm{Tr}\left(T_c^{(R)}T_b^{(R)}T_a^{(R)}\right). \tag{B.471}$$

A representation[12] which satisfies the condition (B.470) is said *real* if it is equivalent to a real representation (*e.g* the adjoint representation of SU(2)) and *pesudo-real* if it is not (*e.g* the

[11]The origin of the anticommutator is due to the fact that the anomaly expression is symmetric in the exchange of the $F^{\mu\nu}$.

[12]Recall that we are using antihermitian generators.

fundamental representation of SU(2)). In general a representation for which $d_{abc} = 0$ is called *safe* representation.
This analysis can be generalized to the case of chiral theory and local symmetries, namely a theory where the left and the right part of the Dirac fermion transform in different representation of the gauge group. In this case, by means of the above results, it is not difficult to check that the absence of anomalous term is equivalent to require[13]

$$d^L_{abc} - d^R_{abc} = 0. \tag{B.472}$$

Here $d^{L,R}_{abc}$ is the quantity defined in (B.467) for the left and right representation respectively.

C Conventions

Given a compact Lie group G, we shall denote its Lie algebra with $\mathfrak{G}$ and the generators T_a are chosen to be antihermitian $T_a^\dagger = -T^a$. They satisfies

$$[T_a, T_b] = f_{ab}{}^c T_c \tag{C.473}$$

and they are normalized so that

$$\mathrm{Tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}. \tag{C.474}$$

The Jacobi identity implies the following relation between the structure costant

$$\begin{aligned} 0 =&[T_a,[T_b,T_c]] + [T_b,[T_c,T_a]] + [T_c,[T_a,T_b]] = f_{bc}{}^n[T_a,T_n] + f_{ca}{}^n[T_b,T_n] + f_{ab}{}^n[T_c,T_n] = \\ =& f_{bc}{}^n f_{anr}T^r + f_{ca}{}^n f_{bnr}T^r + f_{ab}{}^n f_{cnr}T^r \qquad \Rightarrow \\ & f_{bc}{}^n f_{anr} + f_{ca}{}^n f_{bnr} + f_{ab}{}^n f_{cnr} = 0 \end{aligned} \tag{C.475}$$

Useful Traces

$$\mathrm{Tr}(T_a[T_b,T_c]) = f_{bc}{}^m \mathrm{Tr}(T_a T_m) = -\frac{1}{2} f_{bc}{}^m \delta_{ma} = -\frac{1}{2} f_{bca} = -\frac{1}{2} f_{abc}. \tag{C.476}$$

$$\mathrm{Tr}([T_a,T_b][T_c,T_d]) = f_{ab}{}^m f_{cd}{}^n \mathrm{Tr}(T_m T_n) = -\frac{1}{2} f_{ab}{}^m f_{cd}{}^n \delta_{mn} = -\frac{1}{2} f_{abn} f_{cd}{}^n. \tag{C.477}$$

[13]The case where we have just left or right fermions is obtained by assuming that the other sector is decoupled, that its d_{abc} vanishes

D Some useful result on antisymmetric matrices

Lemma 1 *Every real antisymmetric matrix with determinant different from zero can be brought into the following form*

$$UXU^T = \begin{pmatrix} ix_1\sigma_2 & & \\ & \ddots & \\ & & ix_n\sigma_2 \end{pmatrix}. \tag{D.1}$$

where U is an orthogonal real matrix and $\{x_1, \ldots, x_n\}$ are real numbers.

Lemma 2 *Given two commuting real antisymmetric matrices X and Y with determinant different from zero there exists an orthogonal matrix such that UXU^T and UYU^T are of the form considered in the previous lemma.*

Lemma 3 *Given an antisymmetric unitary matrix Q we can find a unitary matrix U such that*

$$UXU^T = \begin{pmatrix} i\sigma_2 & & \\ & \ddots & \\ & & i\sigma_2 \end{pmatrix}. \tag{D.2}$$

Lemma 4 *Given a complex antisymmetric matrix Z, we can always find an unitary transformation U such that UZU^T takes the following form*

$$UZU^T = \begin{pmatrix} iz_1\sigma_2 & & \\ & \ddots & \\ & & iz_n\sigma_2 \end{pmatrix},$$

if the dimension of the matrix is even or the form

$$UZU^T = \left(\begin{array}{c|cccc} 0 & \ldots & 0 & \ldots & \\ \hline \vdots & z_1 i\sigma_2 & & & \\ 0 & & \ddots & & \\ \vdots & & & z_n i\sigma_2 & \end{array}\right),$$

if the dimension is odd. In both cases $x_1, x_2, \ldots, x_n$ are real numbers