In these lectures we will discuss some aspects of SUSY theories in 4D (flat) space time.
We start by recalling that SUSY can be viewed as a TRANSLATION in SUPERSPACE.
What does it mean?.
Recall that a (continuous, global) symmetry is given by a UNITARY operator $U$ ($U^\dagger = U^{-1}$) acting on the Fock space $\mathcal{F}$

$$U : |\psi\rangle \longrightarrow |\psi'\rangle = U|\psi\rangle$$

for $|\psi\rangle$ = any (multiparticle) state.
For translations we write

$$U = \exp(i a^\mu P_\mu)$$

where $P_\mu^\dagger = P_\mu$, $[P_\mu, P_\nu] = 0$ also operators on $\mathcal{F}$ (Generators).

Consider now an operator $\mathcal{O}(x)$ and evaluate its expect. value between a state $|\psi\rangle$.

$$\langle \mathcal{O}(x) \rangle \equiv \langle \psi | \mathcal{O}(x) | \psi \rangle .$$

If I translate the state $|\psi\rangle$:

$$\langle \psi | \mathcal{O}(x) | \psi \rangle \to \langle \psi' | \mathcal{O}(x) | \psi' \rangle =$$

$$= \langle \psi | U^{\dagger} \mathcal{O}(x) U | \psi \rangle$$

For this to be called a translation better be:

$$\langle \psi | U^{\dagger} \mathcal{O}(x) U | \psi \rangle = \langle \psi | \mathcal{O}(x+a) | \psi \rangle$$

$$\Rightarrow \quad U^{\dagger} \mathcal{O}(x) U = \mathcal{O}(x+a)$$

$$\Rightarrow \quad [\mathcal{P}_{\mu}, \mathcal{O}(x)] = i \partial_{\mu} \mathcal{O}(x) \equiv P_{\mu} \mathcal{O}(x)$$

Notice the difference between $\mathcal{P}_{\mu}$ and $P_{\mu}$ (operator on $\mathcal{F}$ / derivative on $\mathcal{O}$)

An object like $\mathcal{O}(x)$ is called a FIELD, e.g. a free field

$$\phi(x) \sim \int d^3\mathbb{k} \left( a_{\mathbb{k}} e^{-ik\cdot x} + a^+_{\mathbb{k}} e^{ik\cdot x} \right)$$

$$k^0 = +\sqrt{\mathbb{k}^2 + m^2}$$

$$P_\mu \sim \int d^3\mathbb{k} \; k_\mu \, a^+_{\mathbb{k}} a_{\mathbb{k}}$$

Note: ① $\hat{\mathcal{O}}(x) = x^2 \mathcal{O}(x)$ is NOT a field

because $U^+ \hat{\mathcal{O}}(x) U = x^2 \mathcal{O}(x+a)$

$$\neq \hat{\mathcal{O}}(x+a) = (x+a)^2 \mathcal{O}(x+a)$$

② Rotations and bosts can be discussed in similar way...

③ On the vacuum: $P_\mu |0\rangle = 0$

$(\Leftrightarrow U|0\rangle = |0\rangle)$ we say that translation symmetry is UNBROKEN (as opposed to spontaneously broken)

This can be used to prove that, eg.

$\langle 0|\mathcal{O}(x)|0\rangle = $ const: $\partial_\mu \langle 0|\mathcal{O}(x)|0\rangle =$

$$= \langle 0|\partial_\mu \mathcal{O}(x)|0\rangle = -i\langle 0|[P_\mu, \mathcal{O}(x)]|0\rangle = 0.$$

Given a field $\Theta(x)$ AND a TRANSLATIONAL INVARIANT MEASURE ($d^4x$) I can construct a TRANSL. INVARIANT OPERATOR:

$$\Omega = \int d^4x\, \Theta(x)$$

$$U^\dagger \Omega U = \int d^4x\, U^\dagger \Theta(x) U =$$

$$= \int d^4x\, \Theta(\underbrace{x+a}_{x'}) = \int d^4x'\, \Theta(x') = \Omega .$$

Derivatives of fields are fields and (ignoring issues of regulators that CAN be sorted out in a transl. invariant way), products of fields are fields so this can be used to construct an invariant action $S = \int d^4x\, \mathcal{L}(x)$.

Time to look @ SUSY!

## NOTATION

We use 2-component Weyl spinors

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \begin{aligned} \sigma^\mu &= (\mathbb{1}_2, \sigma^i) \qquad \sigma^\mu_{\alpha\dot\alpha} \\ \bar\sigma^\mu &= (\mathbb{1}_2, -\sigma^i) \qquad \bar\sigma^{\mu\dot\alpha\alpha} \end{aligned}$$

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}_4 \qquad \eta^{\mu\nu} = \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$$

$$\frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) & 0 \\ 0 & \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) \end{pmatrix}$$

Left handed spinor: $\begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix}$

Righ handed spinor: $\begin{pmatrix} 0 \\ \bar\lambda^{\dot\alpha} \end{pmatrix}$

$$\bar\lambda^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\bar\lambda_{\dot\beta} \equiv \epsilon^{\dot\alpha\dot\beta}(\lambda_\beta)^*$$

Indices are raised / lowered with:

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \epsilon^{\alpha\beta}, \epsilon^{\dot\alpha\dot\beta} \quad (\epsilon^{12} = 1).$$

when not indicated ${}^{\alpha} \searrow {}_{\alpha} \qquad {}_{\dot\alpha} \nearrow {}^{\dot\alpha}$

eg: $\chi\xi = \chi^\alpha\xi_\alpha = \epsilon^{\alpha\beta}\chi_\beta\xi_\alpha = +\xi\chi$

for ANTI commuting spinors

SUPERSYMMETRY is a TRANSLATION in SUPERSPACE : $x^\mu, \underbrace{\theta_\alpha, \bar{\theta}^{\dot{\alpha}}}_{\text{GRASSMANN}}$

SUPER FIELD:

$$Y(x,\theta,\bar{\theta}) = \varphi(x) + \theta^\alpha \xi_\alpha(x) + \dots \quad \theta^2\bar{\theta}^2 d(x)$$

TERMINATES.

$$\begin{cases} \theta_\alpha \to \theta_\alpha + \epsilon_\alpha \\ \bar{\theta}^{\dot{\alpha}} \to \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}} \\ x^\mu \to x^\mu + i\epsilon\sigma^\mu\bar{\theta} + i\bar{\epsilon}\bar{\sigma}^\mu\theta . \end{cases}$$

mixes the components

generated by

$$\begin{cases} Q_\alpha = i\dfrac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\dfrac{\partial}{\partial x^\mu} \\ \bar{Q}_{\dot{\alpha}} = -i\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\dfrac{\partial}{\partial x^\mu} \end{cases}$$

Just like $[P_\mu, Y] = +i\partial_\mu Y = P_\mu Y$.
we have (for $Y$ commuting, otherwise $\{\ \}$):

$$[Q_\alpha, Y] = Q_\alpha Y \ , \ [\bar{Q}_{\dot{\alpha}}, Y] = \bar{Q}_{\dot{\alpha}} Y .$$

These operators obey the SUSY ALGEBRA, the most important relation being:

$$\{ Q_\alpha , \bar{Q}_{\dot\beta} \} = 2 \sigma^\mu_{\alpha\dot\beta} P_\mu .$$

The SAME RELATION holds for $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha}, \mathcal{P}_\mu$ as operators of $\mathcal{F}$

If $\mathcal{Q}_\alpha |0\rangle = \bar{\mathcal{Q}}_{\dot\alpha} |0\rangle = 0$ we say that SUSY is UNBROKEN.

We will start with studying this type of theories and come back to this issue later.

The punchline is that, if I have a TRANSLATIONALLY INVARIANT MEASURE:

$$\int d^4x \, d\theta^2 \, d\bar{\theta}^2 \qquad \left( \begin{array}{l} \int d\theta_1 \, \theta_1 = 1 \\ \int d\theta_1 \cdot 1 = 0 \end{array} \quad \text{etc...} \right)$$

Then $\int d^4x \, d\theta^2 \, d\bar{\theta}^2 \, Y(x, \theta, \bar{\theta})$ is manifestly SUSY invariant, just like $\int d^4x \, \phi(x)$ is translation invariant.

In the case of translations we were done. Here $Y$ is actually REDUCIBLE.

NOTE:

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial x^\mu}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^\mu}$$

anticommute with $Q_\alpha$, $\bar{Q}_{\dot{\alpha}}$.

So I can impose $\bar{D}_{\dot\alpha}\Phi = 0$

$$\Phi = \varphi(y) + \sqrt{2}\,\theta\psi(y) + \theta^2 f(y) =$$

$$= \varphi(x) + \sqrt{2}\,\theta\psi(x) + \theta^2 f(x) + \text{derivatives}$$

$$\left( y^\mu = x^\mu + i\,\theta\sigma^\mu\bar\theta \quad \text{s.t. } \bar{D}_{\dot\alpha} y^\mu = 0 \right).$$

$\Phi$ = <u>chiral</u> <u>Superfield</u>

$$[\epsilon Q, \Phi] = [\epsilon Q, \varphi] + \sqrt{2}\,\theta^\beta [\epsilon Q, \psi_\beta] + \cdots$$

$$\parallel$$

$$\epsilon Q\,\Phi = \sqrt{2}\,i\,\epsilon\psi + 2i\,\epsilon\theta\, f + \cdots$$

$$\Rightarrow \begin{cases} \delta_\epsilon \varphi \left(\sim [\epsilon Q, \varphi]\right) = \epsilon\psi \\ \delta_\epsilon \psi \left(\sim [\epsilon Q, \psi]\right) = \epsilon f \\ \delta_\epsilon f \left(\sim [\epsilon Q, f]\right) = 0 \end{cases}$$

$$\begin{cases} \delta_{\bar\epsilon} \varphi \left(\sim [\bar\epsilon \bar Q, \varphi]\right) = 0 \\ \delta_{\bar\epsilon} \psi \left(\sim [\bar\epsilon \bar Q, \psi]\right) = -i(\sigma^\mu\bar\epsilon)\,\partial_\mu \varphi \\ \delta_{\bar\epsilon} f \left(\sim [\bar\epsilon \bar Q, f]\right) = -i\,\bar\epsilon\bar\sigma^\mu \partial_\mu \psi . \end{cases}$$

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\; K(\phi,\bar\phi) + \left(\int d^4x\, d^2\theta\; W(\phi) + c.c.\right)$$

kähler Pot. Superpot.

Note that $\int d^4x\, d\theta^2 \{\text{chiral field}\}$ is already susy invariant since

$\delta_\epsilon f$ = total derivative.

So $\int d^4x\, d\theta^2\, W(\phi)$ is susy inv.

AS LONG AS $W$ is HOLOMORPHIC.

The renormalizable WZ model is recovered for

$$K = \bar\phi^i \phi_i$$

$$W = f^i\phi_i + \frac{1}{2} m^{ij}\phi_i\phi_j + \frac{1}{6} y^{ijk}\phi_i\phi_j\phi_k$$

$$\int d\theta^2 d\bar{\theta}^2 \,\bar{\Phi}^i \Phi_i = \partial^\mu \bar{\varphi}^i \partial_\mu \varphi_i + i \bar{\psi}^i \bar{\sigma}^\mu \partial_\mu \psi_i + \bar{f}^i f_i$$

and similarly:

$$\int d\theta^2 \, W(\Phi) = -\frac{1}{2} \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \psi_i \psi_j + \frac{\partial W}{\partial \varphi_i} f_i$$

Notice that $f_i$ (and $\bar{f}^i$) appear without derivatives and can thus be integrated out by using their (algebraic) e.o.m:

$$f_i = -\frac{\partial \bar{W}}{\partial \bar{\varphi}^i} \;, \quad \bar{f}^i = -\frac{\partial W}{\partial \varphi_i}$$

yielding a POTENTIAL $V = \dfrac{\partial \bar{W}}{\partial \bar{\varphi}^i} \dfrac{\partial W}{\partial \varphi_i}$

($V$ quartic $\Rightarrow$ $W$ cubic).

$f_i, \bar{f}^i$ are known as AUXILIARY FIELDS

The case $\int d^2\theta\, W$ is often quoted as an "exception" but it should be realized that ANY term like

$\int d^2\theta\, d\bar{\theta}^2\, Y$ can be written as

$\int d^2\theta\, \bar{D}^2 Y \equiv \int d^2\theta\, \Sigma$

$\Sigma$ manifestly chiral. (CHIRALLY EXACT).

On the contrary, $\int d^2\theta\, W$ CANNOT be written as $\int d\theta^2 d\bar{\theta}^2 \ldots$ because

$W \neq \bar{D}^2 Y$.

Objects that cannot be written as $\int d\theta^2 d\bar{\theta}^2$ are known as F-terms.

The other are D-terms.

"Chirally exact objects do not contribute to the F-terms"

Another way to reduce the superfield is to assume:

$$V = V^{+} = V^{a} T^{a}$$

(REAL SUPERFIELD)

$$V^{a} = \bar{\theta}\bar{\sigma}^{\mu}\theta A^{a}_{\mu} + \bar{\theta}^{2}\theta\lambda^{a} + \theta^{2}\bar{\theta}\bar{\lambda}^{a} + \frac{1}{2}\theta^{2}\bar{\theta}^{2}D^{a} + \dots$$

If I want to couple a chiral s.f. $\phi$ I first observe that $e^{-i\alpha(x)}\phi(x,\theta,\bar{\theta})$ is NO LONGER CHIRAL.

$\Rightarrow$ MUST USE a full chiral gauge parameter

$$\phi(x\theta\bar{\theta}) \to e^{-i\Lambda(x\theta\bar{\theta})}\phi(x\theta\bar{\theta})$$

But now $\bar{\Phi}\phi$ is not invariant.

$\Rightarrow$ insert $e^{V} \to \underbrace{e^{-i\bar{\Lambda}} e^{V} e^{i\Lambda}}_{\text{still real.}}$

$\bar{\Phi} e^{V} \phi$ invariant. $\equiv \tilde{\bar{\Phi}}\phi$

(or in general; $K(\tilde{\bar{\Phi}}, \phi)$)

All that remains to be done is to write the KINETIC TERMS for $V$, $(A_\mu, \lambda)$

Introduce : $\bar{\nabla}_{\dot\alpha} \equiv \bar{D}_{\dot\alpha}$

$$\nabla_\alpha = e^{-V} D_\alpha e^{V} \equiv D_\alpha + e^{-V}(D_\alpha e^{V}).$$

$$\{\nabla_\alpha, \bar{\nabla}_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu + \bar{D}_{\dot\alpha}\left(e^{-V}(D_\alpha e^{V})\right)$$

$$\equiv \nabla_{\alpha\dot\alpha}$$

$$[\bar{\nabla}^{\dot\alpha}, \nabla_{\alpha\dot\alpha}] = \bar{D}^2\left(e^{-V}(D_\alpha e^{V})\right) \equiv -4W_\alpha$$

manifestly chiral $(\bar{D}^3 = 0)$.

NB: $e^{V} \rightarrow e^{-i\bar\Lambda} e^{V} e^{+i\Lambda}$

$$\Downarrow$$

$$W_\alpha \rightarrow e^{-i\Lambda} W_\alpha e^{+i\Lambda}$$

$$W^a_\alpha = \lambda^a_\alpha + \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F^a_{\mu\nu} + \theta_\alpha D^a + \dots$$

$$\frac{1}{4}\int d^2\theta\, W^{\alpha a} W^a_\alpha + cc =$$

AUXILIARY FIELD

$$= -\frac{1}{4}F^{a2}_{\mu\nu} + i\bar\lambda^a\bar\sigma^\mu D_\mu\lambda^a + \frac{1}{2}D^{a2}$$

Even more generally, if renormaliz. is not an issue:

$$\frac{1}{4}\int d^2\theta\, f^{ab}(\phi)\, W^{a\alpha} W^b{}_\alpha + cc.$$

$\uparrow$ gauge function.

For us it will be enough:

$$f^{ab} = \delta^{ab}\left(\frac{1}{g^2} - i\frac{\Theta}{8\pi^2}\right) = \frac{\delta^{ab}}{4\pi}\cdot \tau \in \mathbb{C}.$$

$\tau$ can also be thought of as a chiral superfield. (One then rescales $g$ properly).

Note that while it is true that

$$W^\alpha W_\alpha = \frac{1}{16}\bar{D}^2 \underbrace{\left(e^{-V}(D^\alpha e^V)\bar{D}^2(e^{-V} D_\alpha e^V)\right)}$$

The object $\longrightarrow$ is NOT gauge invariant. $W^\alpha W_\alpha$ should be thought of as a "generalized" F-term.

For completeness, in components, the most general $N=1$ SUSY gauge theory can be written as

1) The pure gauge piece:
(having rescaled $g$).

$$\mathcal{L}_{gauge} = -\frac{1}{4} F^{a\,2}_{\mu\nu} + i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^{a\,2}$$

≡ PLUS ≡

2) The COVARIANTIZED WZ piece
$(\partial_\mu \to D_\mu)$
from $\bar{\Phi} e^V \Phi$.

$$\mathcal{L}_{matter} = D^\mu \bar{\varphi}^i D_\mu \varphi_i + i \bar{\psi}^i \bar{\sigma}^\mu D_\mu \psi_i + \bar{f}^i f_i +$$
$$\left( -\frac{1}{2} \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \psi_i \psi_j + \frac{\partial W}{\partial \varphi_i} f_i + c.c. \right)$$

≡ PLUS ≡

3) An extra piece from $\bar{\Phi} e^V \Phi$ aswell

$$\mathcal{L}_{extra} = -\sqrt{2} g \left( \bar{\varphi} T^a \psi \lambda^a + cc \right) + g \left( \bar{\varphi} T^a \varphi \right) D^a.$$