Instantons in Gauge Theories

Contents

Outline

- Instantons in gauge theories: solutions of Euclidean equations of motion. Saddle points of path integrals. Instantons as self-dual connections. The $k = 1$ solution.

- ADHM construction. The moduli space of self-dual connections. Fermions in the instantons background.

- Supersymmetric theories. Supersymmetry in the instanton moduli space. $\mathcal{N} = 1$ supersymmetric gauge theories. The Affleck-Dine-Seiberg prepotential

- $\mathcal{N} = 2$ supersymmetric gauge theories. The idea of localisation. The prepotential. Multi-instanton calculus via localization.

- Seiberg-Witten curves from localisation.

1 Instantons in Gauge Theories

Correlators in Quantum Field Theories are described by path integrals over all possible field configurations

$$\left\langle \prod_i \mathcal{O}(x_i, t_i) \right\rangle = \int \mathfrak{D}\phi \prod_i \mathcal{O}(x_i, t_i)\, e^{\frac{i}{\hbar} S(\phi)} \tag{1.1}$$

For a gauge theory

$$S = -\frac{1}{4g^2} \int d^4x\, F^a_{\mu\nu} F^{a\mu\nu} + \ldots \tag{1.2}$$

In the classical limit $\hbar \to 0$, the integral is dominated by the saddle point of the action $\delta S = 0$. To compute the contribution of the saddle point to the integral is convenient to perform the analytic continuation

$$t_E = \mathrm{i}\, t \tag{1.3}$$

and write

$$i\,S=-\frac{1}{4g^2}\int dt_E\,d^3x\,F^a_{mn}F^a_{mn}+\ldots=-S_E \tag{1.4}$$

With respect to $S(\phi)$, $S_E(\phi)$ has the advantage of being positive defined. The classical limit of the path integral

$$\left\langle\prod_i \mathcal{O}(x_i,-it_{Ei})\right\rangle=\int \mathfrak{D}\phi\,\prod_i \mathcal{O}(x_i,-it_{Ei})\,e^{-\frac{1}{\hbar}S_E(\phi)} \tag{1.5}$$

is then dominated by the minima of $S_E(\phi)$. A solution of the Euclidean equations of motion is called an *instanton*.

Besides its applications in the computation of path integrals , instantons can be used also to compute tunnelling effects between different vacua of a quantum field theory. At low energies, the energy levels of a particle moving on a potential $V(x)$ can be approximated by those of the harmonic oscillator for a particle moving on a quadratic potential $V(x)\approx\frac{\omega_0}{2}(x-x_0)^2$ around a minima at $x=x_0$. In presence of a tunnelling between two vacua, each harmonic oscillator energy levels split into two with energies

$$E_n=(n+\tfrac{1}{2}\pm\tfrac{1}{2}\Delta_n)\,\omega_0 \qquad \Delta_n\sim e^{-S_E} \tag{1.6}$$

with S_E the Euclidean action for an instanton solution describing the transition between the two minima of a particle moving in the upside-down potential $V_E=-V(x)$. The factor in front of the exponential can be computed evaluating the fluctuation of the action up to quadratic order around the instanton action. We refer to Appendix 1A for details.

1.1 Gauge Instantons

In gauge theories, instantons are solutions of the Euclidean version of the Yang-Mills action

$$\begin{aligned} S_E &= \frac{\mathrm{Im}\,\tau}{8\pi}\int d^4x\,\mathrm{Tr}\,F_{mn}F_{mn}-\mathrm{i}\,\frac{\mathrm{Re}\,\tau}{8\pi}\int d^4x\,\mathrm{Tr}\,F_{mn}\tilde{F}_{mn}\\ &= \frac{1}{2g^2}\int d^4x\,\mathrm{Tr}\,F_{mn}F_{mn}+\ldots \end{aligned} \tag{1.7}$$

with

$$\begin{aligned} F_{mn} &= \partial_m A_n-\partial_n A_m+[A_m,A_n]\\ \tilde{F}_{mn} &= \tfrac{1}{2}\epsilon_{mnpq}F_{pq} \end{aligned} \tag{1.8}$$

and

$$\tau = \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g^2} \tag{1.9}$$

The dual field $\tilde{F}_{mn}$ satisfied by construction the Bianchi identity:

$$\begin{aligned} D_m \tilde{F}_{mn} &= \partial_m \tilde{F}_{mn} + [A_m, \tilde{F}_{mn}] \\ &= \tfrac{1}{2}\epsilon_{mnpq}\left(2\partial_m\partial_p A_q + \partial_m[A_p, A_q] + 2[A_m, \partial_p A_q] + [A_m, [A_p, A_q]]\right) \\ &= 0 \end{aligned} \tag{1.10}$$

The first three terms cancel between themselves using the antisymmetric property of the epsilon tensor, while the last one cancel due to the Jacobi identity satisfied by any Lie algebra.

The YM equation of motion can be written as

$$D_m F_{mn} = \partial_m F_{mn} + [A_m, F_{mn}] = 0 \tag{1.11}$$

Interestingly, this equation has precisely the same form than the Bianchi identity (1.10) with $\tilde{F}$ replaced by F. This implies that the YM equations can be solved by requiring that the field F is self or anti-self dual

$$F = \pm\tilde{F} \tag{1.12}$$

We notice that this equation can be solved only in Euclidean space since $\tilde{\tilde{F}} = -F$ in the Minkowskian space, so eigenvalues of the Poincare dual action are $\pm\mathrm{i}$. On the other hand in the Euclidean space $\tilde{\tilde{F}} = F$ and $\pm$-eigenvalues are allowed.

In the language of forms

$$F = \tfrac{1}{2}F_{mn}dx^m dx^n \qquad *F = \tfrac{1}{2}\tilde{F}_{mn}dx^m dx^n \tag{1.13}$$

with

$$F = DA = dA + A \wedge A \tag{1.14}$$

The Bianchi and field equations read

$$\begin{aligned} DF = dF + [A, F] = 2dA\,A + 2A\,dA + AA^2 - A^2A = 0 &\qquad \text{Bianchi Indentity} \\ D * F = d * F + [A, *F] = 0 &\qquad \text{FieldEquations} \end{aligned} \tag{1.15}$$

A connection satisfying (1.12) with the plus sign is called a *Yang Mills instanton* while the one with minus sing is called an *anti-instanton*. Instantons are classified by the topological integer

$$k = -\frac{1}{16\pi^2}\int d^4x \,\mathrm{tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu}) = -\frac{1}{8\pi^2}\int d^4x \,\mathrm{tr}\,(F\wedge F) \tag{1.16}$$

called, the *instanton number*. This number is the second Chern number $k = \int \mathrm{ch}_2(F)$, with the Chern character defined as

$$\mathrm{ch}(F) = \sum_n ch_n(F) = \exp\left(\frac{iF}{2\pi}\right) \tag{1.17}$$

Next we show that instantons minimize the Euclidean Yang-Mills action over the space of gauge connections with a given topological number k. To see this we start from the trivial inequality

$$\int d^4x \,\mathrm{tr}\,(F \pm \tilde{F})^2 \geq 0 \tag{1.18}$$

and use $\mathrm{tr}\, F^2 = \mathrm{tr}\, \tilde{F}^2$ to show

$$\tfrac{1}{2g^2}\int d^4x \,\mathrm{tr}\, F^2 \geq \tfrac{1}{2g^2}\left|\int d^4x \,\mathrm{tr}\, F\tilde{F}\right| = \frac{8\pi^2|k|}{g^2} \tag{1.19}$$

with the inequality saturates for instantons or anti-instantons. Plugging this value in the YM action one finds that instantons are weighted by $e^{-S_{inst}}$ with

$$-S_{\mathrm{inst}} = 2\pi\mathrm{i}\, k\,\tau = -\frac{8\pi^2 k}{g^2} + \ldots \tag{1.20}$$

with $k > 0$. On the other hands anti-instanton effects are weighted by $e^{-2\pi i\tau^*}$. The instanton corrections to correlators in gauge theories take then the form

$$\langle\mathcal{O}\rangle = \int DA\, e^{-S}\,\mathcal{O} = \sum c_k\, g^k + \sum_{k=1}^{\infty} d_k\, e^{-\frac{8\pi^2 k}{g^2}} \tag{1.21}$$

Summarizing amplitudes in gauge theories are computed by path integrals dominated by the minima of the Euclidean classical action and quantum fluctuations around them. Fluctuations around the trivial vacuum, i.e. where all fields are set to zero, give rise to loop corrections suppressed in powers of the gauge coupling. Instantons are non trivial solutions of the Euclidean classical action and lead to corrections to the correlators exponentially suppressed in the gauge coupling . These corrections are important in theories like QCD where the gauge coupling get strong at low energies. Understanding of the instanton dynamics is then crucial in addressing the study of phenomena in the strong coupling regime like confinement or chiral symmetry breaking, etc.

1.2 The $k=1$ solution

To construct self-dual connections we first observe that we can construct a self-dual two-form with values on $SU(2)$ in terms of the 2×2 matrices $\sigma_n = (i\vec{\tau}, 1)$, $\bar{\sigma} = \sigma_n^\dagger = (-i\vec{\tau}, 1)$[1]

$$\sigma_{mn} = \tfrac{1}{4}(\sigma_m \, \bar{\sigma}_n - \sigma_n \, \bar{\sigma}_m) = \tfrac{1}{2} \, \epsilon_{mnpq} \, \sigma^{pq} \tag{1.22}$$

This implies that a self-dual SU(2) connection can be written as

$$A_m = \frac{2(x-x_0)_n \sigma_{nm}}{(x-x_0)^2 + \rho^2} \tag{1.23}$$

Indeed, the field strength associated to this connection is

$$F_{mn} = \frac{4\rho^2 \sigma_{nm}}{((x-x_0)^2 + \rho^2)^2} \tag{1.24}$$

which is self-dual due to (1.22). Moreover, the instanton number is[2]

$$k = -\frac{1}{16\pi^2} \int d^4x \, \mathrm{tr}\,(F_{mn} \tilde{F}^{mn}) = 6 \frac{\rho^4 \mathrm{vol}_{S^3}}{\pi^2} \int \frac{r^3 \, dr}{(r^2+\rho^2)^4} = 1 \tag{1.25}$$

with $\mathrm{vol}_{S^3} = 2\pi^2$ the volume of the unitary sphere. The connection (1.23) represents then a $k=1$ instanton. We notice that we can find a different solution by rotating A with a matrix of $SU(2)$

$$A \to U A U^\dagger \tag{1.26}$$

so the total number of parameters describing the solution is 8: 4 positions x_0, 1 size ρ and 3 rotations U.

For a general group $SU(N)$, we start from the $SU(2)$ self-dual connection an embed it the 2×2 matrix inside the $N \times N$ matrix. The $k=1$ solution is then described by N^2-1 rotations minus $(N-2)^2$ rotations in the space orthogonal to the 2×2 block that leave invariant the solution, i.e. $4N-5$ parameters . Together with the size and 4 positions we get $4N$ parameters. Finally for k instantons far from each other we expect $4kN$ parameters. We conclude that the dimension of the instanton moduli space is

$$\dim \mathfrak{M}_k^{SU(N)} = 4kN \tag{1.27}$$

In the rest of this section we construct the general solution that goes under the name of ADHM construction.

---

[1] Here $\tau_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\tau_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\tau_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the Pauli matrices.

[2] Here we use the result $\int \frac{r^3\, dr}{(r^2+\rho^2)^4} = \frac{1}{12\rho^4}$ and $\mathrm{tr}\sigma_{nm}^2 = \frac{1}{2}$.

Appendix 1A Instantons in Quantum Mechanics

Besides its applications in the computation of path integrals , instantons can be used also to compute tunnelling effects between different vacua of a quantum field theory. To illustrate this points lets us consider a particle moving in a Double Well potential

$$V(x) = V_0 \left(1 - \frac{x^2}{x_0^2}\right)^2 \tag{1.28}$$

When V_0 is large, the solution of the Schrodinger equation reveals that the spectrum of energies deviates from that of the harmonic oscillator by a quantity Δ_n exponentially suppressed in the height of the barrier V_0, i.e.

$$E_n = (n + \tfrac{1}{2} \pm \tfrac{1}{2}\Delta_n)\,\omega_0 \qquad \Delta_n \sim e^{-\frac{2V_0\omega_0}{3}} \tag{1.29}$$

and $w_0^2 = \frac{8V_0}{x_0^2}$. This deviation is the result of the tunnelling effect between the two vacua. To see this, let us compute the probability that a particle moves from the vacuum at $x = -x_0$ to that one at $x = x_0$ in a time T

$$\langle x_0|e^{-HT}| - x_0\rangle = \int \mathfrak{D}x(t)\text{exp}\left(-\int_0^T dt_E \left[\frac{\dot{x}^2}{2} + V_E(x)\right]\right) \tag{1.30}$$

with

$$V_E(x) = -V(x) \tag{1.31}$$

Notice that the Hamiltonian H is nothing but the Euclidean Lagrangian with the upside-down potential V_E. The classical equation of motion are

$$-\ddot{x} - V_E'(x) = 0 \quad \Leftarrow \qquad \dot{x} = \sqrt{-2V_E(x)} = \frac{\sqrt{2V_0}}{x_0^2}(x^2 - x_0^2) \tag{1.32}$$

which is solved by the kink

$$x = x_0 \tanh t_E \sqrt{\frac{2V_0}{x_0^2}} \tag{1.33}$$

Plugging the solution into the action one finds

$$S = \int_{-\infty}^{\infty} dt_E \left[\frac{\dot{x}^2}{2} - V_E\right] = \int_{-\infty}^{\infty} dt_E\, \dot{x}^2 = \frac{2V_0}{3}\sqrt{\frac{8x_0^2}{V_0}} \tag{1.34}$$

The tunnelling amplitude

$$\mathcal{A} \sim e^{-S_E} \sim \text{exp}\left(-\frac{2V_0}{3}\sqrt{\frac{8x_0^2}{V_0}}\right) \tag{1.35}$$

gives then the correct exponential suppression factor to account for the level splitting Δ_n. The factor in front of the exponential can be computed evaluating the fluctuation of the action up to quadratic order around the instanton action.

Appendix 1B: Forms and Poincare duality

In the language of forms

$$F = \tfrac{1}{2} F_{mn} dx^m dx^n \qquad * F = \tfrac{1}{2} \tilde{F}_{mn} dx^m dx^n \tag{1.36}$$

with

$$F = dA + A \wedge A \tag{1.37}$$

and

$$*dx^m dx^n = \tfrac{1}{2}\, \epsilon^{mn}{}_{pq}\, dx^p dx^q \tag{1.38}$$

One finds

$$d^4x\, \tfrac{1}{2} F^{mn} F_{mn} = F \wedge *F \qquad d^4x\, \tfrac{1}{2} F^{mn} \tilde{F}_{mn} = F \wedge F \tag{1.39}$$

Appendix 1C: Chern Classes

Consider the first Chern class of a U(1) connection A on $\mathbb{R}^2$

$$c_1(F) = \frac{i}{2\pi} \int d^2x\, dA = \frac{i}{2\pi} \int_{S^1_\infty} A \tag{1.40}$$

At infinity A becomes a trivial gauge

$$A_m = \bar{U}\, \partial_m U \tag{1.41}$$

Take $U = e^{-ik\theta}$ with θ the coordinate of the S^1_∞ at infinity. Plugging this into (1.40) one finds

$$c_1(F) = \frac{i}{2\pi} \int_0^{2\pi} d\theta\, (-ik) = k \tag{1.42}$$

So the number k measures the winding of the connection along S^1_∞ at infinity.

Now consider the second Chern class. First we notice that $c_2(F)$ is a total derivative

$$\begin{aligned} k &= -\frac{1}{8\pi^2} \int \mathrm{tr}\, F \wedge F = -\frac{1}{8\pi^2} \int \mathrm{tr}\, d(AdA + \tfrac{2}{3} A^3) \\ &= -\frac{1}{8\pi^2} \int_{S^3_\infty} \mathrm{tr}\, (AdA + \tfrac{2}{3} A^3) = \frac{1}{24\pi^2} \int_{S^3_\infty} \mathrm{tr}\, A^3 \end{aligned} \tag{1.43}$$

where we used the fact that at infinity $F \to 0$ and therefore $dA \to -A^2$. To evaluate the integral at infinity we write A as a total gauge

$$A = \bar{U}\, dU \qquad \text{with} \qquad U = \frac{1}{r}(x_4 + i x_i \sigma^i) \tag{1.44}$$

with σ_m the pauli matrices. Consider the region where $x_4 \approx R$ and x_i are small. In this region

$$\mathrm{tr}\, A^3 \to \frac{6}{R^3}\, \mathrm{tr}\, \sigma_1 \sigma_2 \sigma_3\, dx_1 \wedge dx_2 \wedge dx_3 = \frac{12}{R^3}\, dx_1 \wedge dx_2 \wedge dx_3 = 12\, \Omega_3 \tag{1.45}$$

with Ω_3 the volume form in this region. Integrating over S^3_∞ one finds

$$k = \frac{1}{24\pi^2} \int_{S^3_\infty} \mathrm{tr}\, A^3 = \frac{1}{2\pi^2} \int_{S^3_\infty} \Omega_3 = 1 \tag{1.46}$$

In general the instanton number k measures the winding number of the map $U : S^3_\infty \to \mathfrak{g}$ from the three sphere at infinity to the Lie algebra.

2 ADHM construction

2.1 The self-dual connection

Here we review the ADHM construction of instantons in $\mathbb{R}^4$.
This construction exploits again the observation that any field strength 0f the form

$$F_{mn} \sim \sigma_{mn} \tag{2.47}$$

is self-dual in virtue of (1.22). We look for a gauge connection following the ansatz

$$A_m = \bar{U} \partial_m U \qquad\qquad \bar{U}\, U = \mathbb{1}_{[N \times N]} \tag{2.48}$$

with $U_{[(N+2k)\times N]}$. If $k = 0$ this is simply a pure gauge. We will now show that this connection is self-dual if:

- U is a normalized kernel of a matrix Δ of the form

$$\Delta = \mathbf{a} + x_n\, \mathbf{b}^n = \begin{pmatrix} w_{u,i\dot\alpha} \\ a_{i\alpha,j\dot\alpha} \end{pmatrix} + x_n \begin{pmatrix} 0 \\ \sigma^n_{\alpha,\dot\alpha} \delta_{i,j} \end{pmatrix} \tag{2.49}$$

  with $i = 1, \ldots k$, $u = 1, \ldots N$. In other words U satisfy

$$\bar{\Delta} U = \bar{U} \Delta = 0 \qquad\qquad \bar{U}\, U = \mathbb{1}_{[N \times N]} \tag{2.50}$$

  with Δ of the form (2.49). By bars we will always mean hermitian conjugates.

- If Δ satisfy the ADHM constraints

$$\bar{\Delta}^{\dot{\alpha}}_{i\,\lambda}\,\Delta_{\lambda,j\dot{\beta}} = f^{-1}_{ij}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{2.51}$$

or in components[3]

$$\bar{w}\tau^c w - i\bar{\eta}^c_{mn}[a_m, a_n] = 0 \tag{2.52}$$

with $a_{\alpha\dot{\alpha}} = a_m \sigma^m_{\alpha\dot{\alpha}}$ and $\bar{\eta}^c_{mn} = -i\,\mathrm{tr}\,(\bar{\sigma}_{mn}\tau^c)$.

The matrices $w_{[N\times 2k]}$, $a_{[2k\times 2k]}$ are made of pure numbers describing the instanton moduli[4]. Notice that the resulting connection is invariant under $U(k)$ rotations

$$a_m \to U a_m U^\dagger \qquad w_{\dot{\alpha}} \to U w_{\dot{\alpha}} \tag{2.53}$$

The moduli space of instantons is then defined by the $U(k)$ quotient of the hypersurface defined by (2.52) and has dimension

$$\dim_{\mathbb{R}}\mathfrak{M}_k = 4k(N+2k) - 3k^2 - k^2 = 4kN \tag{2.54}$$

Notice that equation (2.50) and (2.51) imply

$$\mathbb{1} = U\bar{U} + \Delta\, f\, \bar{\Delta} \tag{2.55}$$

To see that the gauge connection constructed in this way is self-dual let us compute F_{mn}:

$$\begin{aligned} F_{mn} &= \partial_m A_n - \partial_n A_m - [A_m A_n] \\ &= 2\partial_{[m}\bar{U}\partial_{n]}U - [\bar{U}\partial_m U, \bar{U}\partial_n U] \end{aligned} \tag{2.56}$$

Inserting the identity (2.55) into the first term in (2.56), rewriting derivatives on U's as derivatives on Δ's and using (2.50) one finds

$$\begin{aligned} F_{mn} &= 2\partial_{[m}\bar{U}\Delta\, f\, \bar{\Delta}\,\partial_{n]}U = 2\bar{U}\partial_{[m}\Delta\, f\,\partial_{n]}\bar{\Delta}\,U \\ &= 2\bar{U}\begin{pmatrix} 0 \\ \sigma_{[m}\otimes \mathbb{1}_{[k\times k]}\end{pmatrix} f \begin{pmatrix} 0 & \sigma^\dagger_{n]}\otimes\mathbb{1}_{[k\times k]}\end{pmatrix} U \\ &= 4\bar{U}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_{mn}\otimes f_{[k\times k]}\end{pmatrix} U \sim \sigma_{mn} \end{aligned} \tag{2.57}$$

---

[3]To see this, we notice that $\bar{\Delta}\,\Delta = \bar{\mathbf{a}}\,\mathbf{a} + x_n\,(\bar{\mathbf{b}}_n\,\mathbf{a} + \bar{\mathbf{a}}\,\mathbf{b}_n) + \,x_n\,x_m\,\bar{\mathbf{b}}_n\,\mathbf{b}_m$. The $\mathbf{b}$-dependent terms are all proportional to $\delta^{\dot{\beta}}_{\dot{\alpha}}$, while the first term leads to (2.52). Here $\bar{\eta}^c_{mn}$ is antisymmetric, $\bar{\eta}^c_{ab} = \epsilon_{cab}$, $\bar{\eta}^c_{4m} = \delta_{mc}$.

[4]In the mathematical literature the ADHM equations are often written as $[B_1, B_2] + IJ = 0$, $[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \xi\,\mathbf{1}_{k\times k}$. These equations follows from the identifications $B_\ell = \frac{1}{\sqrt{2}}(a_{2\ell} + i a_{2\ell-1})$ and $w = (J \;\; I^\dagger)$.

Explicit solutions

The simplest solution: $k=1$, $N=2$ at $a_{\alpha\dot\alpha}=0$:

$$
\begin{aligned}
\Delta &= \begin{pmatrix} \rho\,\mathbb{1}_{[2\times 2]} \\ x_{[2\times 2]} \end{pmatrix} \qquad \bar U = \frac{1}{(\rho^2+r^2)^{\frac{1}{2}}} \begin{pmatrix} -x_{[2\times 2]} & \rho\,\mathbb{1}_{[2\times 2]} \end{pmatrix} \\
\bar\Delta\,\Delta &= (\rho^2+r^2)\,\mathbb{1}_{[2\times 2]} \qquad \Rightarrow \quad f = \frac{1}{\rho^2+r^2} \qquad r^2 = x_m x^m \\
A_m &= \bar U \partial_m U = \frac{2 x_n \sigma_{mn}}{x^2+\rho^2} \\
F_{mn} &= 4\bar U \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma_{mn}}{(\rho^2+r^2)} \end{pmatrix} U = \frac{4\,\rho^2\,\sigma_{mn}}{(\rho^2+r^2)^2}
\end{aligned} \tag{2.58}
$$

2.2 Fermions in the instanton background

Let us consider now the gauge theory in the background of an instanton. To quadratic order in all fields beside the gauge field A_m, the Euclidean action of any supersymmetric theory can be written as

$$
\begin{aligned}
S_E \;\approx\; \int d^4x \Big\{ &\mathrm{tr}_{\mathbf{Adj}} \left(\frac{1}{2g^2} F_{mn}^2 + \frac{i\theta}{16\pi^2} F^{mn}\tilde F_{mn} \right) \\
&+2\,\mathrm{tr}_{\mathbf{R}}\, D_n \bar\Psi \bar\sigma_n \Psi + \mathrm{tr}_{\mathbf{R_a}}\, D_n \bar\phi_a D_n \phi_a + \ldots \Big\}
\end{aligned} \tag{2.59}
$$

with $\Psi = (\Lambda^A, \psi, \tilde\psi)$ the set of all fermions in the adjoint, fundamental and anti-fundamental representations and

$$
D_m \Phi = \partial_m \Phi + A_m\,\Phi \tag{2.60}
$$

the covariant derivative. To this order the equations of motion of the fields are linear and can be written as

$$
\begin{aligned}
D_m F_{mn} &\approx 0 \\
D_n(\sigma_n \bar\Psi) &\approx D_n(\bar\sigma_n \Psi) \approx 0 \\
D^2 \phi_a &\approx 0
\end{aligned} \tag{2.61}
$$

To find solutions we propose the ansatz

$$
\begin{aligned}
A_m &= \bar{U}\partial_m U \\
\Lambda^A_\alpha &= \bar{U}(\mathcal{M}^A\, f\, \bar{b}_\alpha - b_\alpha\, f\, \bar{\mathcal{M}}^A)U \\
\psi_\alpha &= \mathcal{K}\, f\, \bar{b}_\alpha\, U \\
\tilde{\psi}_\alpha &= \bar{U}\, b_\alpha\, f\, \tilde{\mathcal{K}} \\
\bar{\Lambda}_{\dot{\alpha}A} &= \bar{\psi}_\alpha = \bar{\tilde{\psi}}_\alpha = \phi_a = 0
\end{aligned}
\tag{2.62}
$$

In components one writes

$$
\begin{aligned}
(A_m)^u_v &= \bar{U}^u_\lambda \partial_m U_{\lambda v} \\
(\Lambda^A_\alpha)^u_v &= \bar{U}^u_\lambda\, \left\{\, \mathcal{M}^A_{\lambda i}\, f_{ij}\, (\bar{b}_\alpha)_{j\lambda} - (b_\alpha)_{\lambda i}\, f_{ij}\, \bar{\mathcal{M}}^A_{j\lambda'} \,\right\}\, U_{\lambda' v} \\
(\psi_\alpha)_u &= \mathcal{K}_i\, f_{ij}\, (\bar{b}_\alpha)_{j\lambda}\, U_{\lambda u} \\
(\tilde{\psi}_\alpha)^u &= \bar{U}^u_\lambda\, (\bar{b}_\alpha)_{\lambda i}\, f_{ij}\, \tilde{\mathcal{K}}_j
\end{aligned}
\tag{2.63}
$$

with $\lambda = 1,\dots N+2k$, $u = 1,\dots N$, $i = 1,\dots k$. The matrices $\mathcal{M}^A_{\lambda i}$, $\mathcal{K}_i$, $\tilde{\mathcal{K}}_i$ are made of Grassmanian numbers and[5]

$$
\bar{\partial}^{\dot{\alpha}\alpha}\Delta_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}}\, b^\alpha \qquad\qquad \bar{\partial}^{\dot{\alpha}\alpha}\bar{\Delta}^{\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}\, \bar{b}^\alpha
\tag{2.64}
$$

with $\bar{\partial}^{\dot{\alpha}\alpha} = \frac{1}{2}\bar{\sigma}^{\dot{\alpha}\alpha}_n \partial_n$. To evaluate the Dirac equations on the fermion we should evaluate the action of the covariant derivative on the fermions. To this aim we use the writing of the identity $\mathbb{1} = U\bar{U} + \Delta\, f\, \bar{\Delta}$ in order to translate derivatives on U into derivatives on Δ's. First we notice that

$$
D_n U = (U\bar{U} + \Delta\, f\, \bar{\Delta})\partial_n U + U\, A_n = -\Delta\, f\, \partial_n \bar{\Delta}\, U
\tag{2.65}
$$

and similarly

$$
D_n\, \bar{U} = -\bar{U}\, \partial_n \Delta\, f\, \bar{\Delta}
\tag{2.66}
$$

Using these equations one finds

$$
\begin{aligned}
D^{\dot{\alpha}\alpha}\Lambda^A_\alpha &= \bar{U}\, \Big\{ \mathcal{M}^A\, \bar{\partial}^{\dot{\alpha}\alpha}\, f\, \bar{b}_\alpha - \mathcal{M}^A\, f\, \bar{b}_\alpha \Delta\, f\, \bar{\partial}^{\dot{\alpha}\alpha}\bar{\Delta} - \bar{\partial}^{\dot{\alpha}\alpha}\Delta\, f\, \bar{\Delta}\, \mathcal{M}^A\, f\, \bar{b}_\alpha \\
&\qquad - b_\alpha\, \bar{\partial}^{\dot{\alpha}\alpha}\, f\, \bar{\mathcal{M}}^A + \bar{\partial}^{\dot{\alpha}\alpha}\Delta\, f\, \bar{\Delta}\, b_\alpha\, f\, \bar{\mathcal{M}}^A + b_\alpha\, f\, \bar{\mathcal{M}}^A\, \Delta\, f\, \bar{\partial}^{\dot{\alpha}\alpha}\bar{\Delta} \Big\}\; U \\
D^{\alpha\dot{\alpha}}\psi_\alpha &= \mathcal{K}\, (\bar{\partial}^{\dot{\alpha}\alpha} f\, \bar{b}_\alpha - f\, \bar{b}_\alpha\, \Delta\, f\, \bar{\partial}^{\dot{\alpha}\alpha}\bar{\Delta})\, U \\
D^{\alpha\dot{\alpha}}\tilde{\psi}_\alpha &= \bar{U}\, (b_\alpha\, \bar{\partial}^{\dot{\alpha}\alpha} f - \bar{\partial}^{\dot{\alpha}\alpha}\Delta\, f\, \bar{\Delta}\, b_\alpha\, f)\, \tilde{\mathcal{K}}
\end{aligned}
\tag{2.67}
$$

[5]Here we use $\sigma_{n,\alpha\dot{\alpha}}\, \sigma_{n,\beta\dot{\beta}} = 2\, \epsilon_{\alpha\beta}\, \epsilon_{\dot{\alpha}\dot{\beta}}$, $\bar{\sigma}^{\alpha\dot{\alpha}}_n\, \sigma^n_{\beta\dot{\beta}} = 2\delta^\alpha_\beta\, \delta^{\dot{\alpha}}_{\dot{\beta}}$.

The first two terms in each line cancel between each other in virtue of the identities (2.64) and

$$\bar{\partial}^{\dot{\alpha}\alpha} f = -\, f\, \bar{b}^{\alpha} \Delta^{\dot{\alpha}}\, f = -\, f\, \bar{\Delta}^{\dot{\alpha}} b^{\alpha}\, f \qquad\qquad \bar{b}^{\alpha}\, \Delta^{\dot{\alpha}} = \bar{\Delta}^{\dot{\alpha}}\, b^{\alpha} \tag{2.68}$$

that follows from (2.64) and the bosonic ADHM constraint in (2.69). The cancelation of the last terms in the first two lines requires the fermionic ADHM constraints

$$\bar{\Delta}\, \mathcal{M} + \bar{\mathcal{M}}\, \Delta = 0 \tag{2.69}$$

In components, writing

$$\Delta_{\lambda,\dot{\alpha}j} = \begin{pmatrix} w_{u,i\dot{\alpha}} \\ a_{i\alpha,j\dot{\alpha}} + x_{\alpha\dot{\alpha}}\delta_{ij} \end{pmatrix} \qquad \mathcal{M}^{A}_{\lambda,j} = \begin{pmatrix} \mu^{A}_{u,j} \\ M^{A}_{\alpha i,j} \end{pmatrix} \tag{2.70}$$

the fermionc constraints (2.69) reduce to

$$\bar{M}^{A}_{\alpha} = M^{A}_{\alpha} \qquad \bar{\mu}^{A} w_{\dot{\alpha}} - \bar{w}_{\dot{\alpha}} \mu^{A} + [M^{\alpha A}, a_{\alpha\dot{\alpha}}] = 0 \tag{2.71}$$

Summarizing, the moduli space of instantons in a supersymmetric gauge theory is characterised by the following zero modes:

$$\begin{aligned} \text{Vector}: &\quad (a_{\alpha\dot{\alpha},ij}, w_{\dot{\alpha},ui}, \bar{w}_{\dot{\alpha},iu}) \qquad D^{c} = \bar{w}\tau^{c} w - i\bar{\eta}^{c}_{mn}[a_m, a_n] = 0 \\ \dim \mathfrak{M}^{\text{gauge}}_{k,N} = &\quad 4k^2 + 4kN - 3k^2 - k^2 = 4kN \\ \text{Adj. fermions}: &\quad (\mu^{A}_{ui}, \bar{\mu}^{A}_{iu}, M^{A}_{\alpha,ij}) \qquad \lambda^{A}_{\dot{\alpha}} = \bar{\mu}^{A} w_{\dot{\alpha}} - \bar{w}_{\dot{\alpha}}\mu^{A} + [M^{\alpha A}, a_{\alpha\dot{\alpha}}] = 0 \\ \dim \mathfrak{M}^{\text{Adj.matter}}_{k,N} = &\quad (\#\text{ of Adj ferm.}) \times \big(2kN + 2k^2 - 2k^2 = 2kN\big) \\ \text{Fund matter}: &\quad \mathcal{K}_i \qquad \dim \mathfrak{M}^{\text{Fund matter}}_{k,N} = (\#\text{ of Fund. ferm.}) \times (k) \end{aligned} \tag{2.72}$$

with $i = 1, \ldots k$, $u = 1, \ldots N$. We notice that the components of matrices $a_{\alpha\dot{\alpha}}$ and M^{A}_{α} proportional to the identity,

$$a_m = x_{0,m}\, \mathbb{1}_{k\times k} + \ldots \qquad M^{A}_{\alpha} = \theta^{A}_{0,\alpha}\, \mathbb{1}_{k\times k} + \ldots \tag{2.73}$$

do not enter on the ADHM constraints. They are exact zero modes of the instanton action parametrising the position of the instanton in the superspace-time. They can be reabsorbed in a shift of the space-time supercoordinates $(x_m, \theta^{\alpha}_{\alpha})$

Finally the asymptotic behaviour or various fields at infinity can be found from the asymptotics[6]

$$\bar{\Delta} \approx \begin{pmatrix} \bar{w}^{\dot{\alpha}} & \bar{x}^{\dot{\alpha}\alpha} \end{pmatrix} \qquad U \approx \begin{pmatrix} \mathbb{1} \\ -\frac{x_{\alpha\dot{\alpha}}\, \bar{w}^{\dot{\alpha}}}{x^2} \end{pmatrix} \qquad f_{ij} \approx \frac{\delta_{ij}}{x^2} \tag{2.74}$$

[6]We use that $\bar{\sigma}^{\dot{\alpha}\alpha}_{(m}\, \sigma_{n)\alpha\dot{\beta}} = \delta_{mn}\, \delta^{\dot{\alpha}}_{\dot{\beta}}$.

leading to

$$
\begin{aligned}
A_n &\approx \frac{1}{x^2} w\,\bar{x}\,\sigma_n\,\bar{w} - \frac{2x_n}{x^2} w\,\bar{w} \qquad F_{mn} \approx \frac{1}{x^4} w\,\bar{x}\,\sigma_{mn}\,x\,\bar{w} \\
\Lambda^A_\alpha &\approx \frac{1}{x^4}\,(w_{\dot\alpha}\bar{x}^{\dot\alpha}{}_\alpha \bar\mu^A - \mu^A\,x_{\alpha\dot\alpha}\,\bar{w}^{\dot\alpha}) \\
\psi_\alpha &\approx -\frac{1}{x^4}\,x_{\alpha\dot\alpha}\,\mathcal{K}\,\bar{w}^{\dot\alpha} \qquad \tilde\psi_\alpha \approx -\frac{1}{x^4}\,\bar{x}^{\dot\alpha}{}_\alpha\;\; w_{\dot\alpha}\,\tilde{\mathcal{K}}
\end{aligned} \tag{2.75}
$$

Appendix 2A: Dirac matrices

Here we collect our conventions for Dirac matrices. We define

$$
\sigma_n = (i\vec\tau, 1) \qquad\qquad \bar\sigma = \sigma_n^\dagger = (-i\vec\tau, 1) \tag{2.76}
$$

with

$$
\tau_1 = \left(\begin{smallmatrix}0 1\\ 1 0\end{smallmatrix}\right) \qquad \tau_2 = \left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right) \qquad \tau_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \tag{2.77}
$$

We introduce the tensors

$$
\sigma_{mn} = \tfrac{1}{4}(\sigma_m\,\bar\sigma_n - \sigma_n\,\bar\sigma_m) \qquad \bar\sigma_{mn} = \tfrac{1}{4}(\bar\sigma_m\,\sigma_n - \bar\sigma_n\,\sigma_m) \tag{2.78}
$$

satisfying the (anti)self duality conditions

$$
\sigma_{mn} = \tfrac{1}{2}\,\epsilon_{mnpq}\,\sigma^{pq} \qquad \bar\sigma_{mn} = -\tfrac{1}{2}\,\epsilon_{mnpq}\,\bar\sigma^{pq} \tag{2.79}
$$

The t'Hooft symbols are defined as

$$
\begin{aligned}
\eta^c_{mn} &= -i\,\mathrm{tr}\,(\bar\sigma_{mn}\tau^c) \qquad \eta^c_{ab} = \epsilon_{abc} \qquad \eta^c_{m4} = \delta_{mc} \\
\bar\eta^c_{mn} &= -i\,\mathrm{tr}\,(\bar\sigma_{mn}\tau^c) \qquad \bar\eta^c_{ab} = \epsilon_{abc} \qquad \bar\eta^c_{4m} = \delta_{mc}
\end{aligned} \tag{2.80}
$$

We write

$$
x_{\alpha\dot\alpha} = x_n\sigma^n_{\alpha\dot\alpha} \qquad \partial_{\alpha\dot\alpha} = \frac{\partial}{\partial x_{\alpha\dot\alpha}} = \tfrac{1}{2}(\sigma_n)^{\alpha\dot\alpha}\partial_n \tag{2.81}
$$

The following identities will be used along the text

$$
\bar\sigma^{\dot\alpha\alpha}_{(m}\,\sigma_{n)\alpha\dot\beta} = \delta_{mn}\,\delta^{\dot\alpha}_{\dot\beta} \tag{2.82}
$$

$$
\bar\sigma^{\dot\alpha\alpha}_n\,\sigma^n_{\beta\dot\beta} = 2\delta^\alpha_\beta\,\delta^{\dot\alpha}_{\dot\beta} \tag{2.83}
$$

$$
\sigma_{n,\alpha\dot\alpha}\,\sigma_{n,\beta\dot\beta} = 2\,\epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta} \qquad\qquad \bar\sigma^{\dot\alpha\alpha}_n\,\bar\sigma^{\dot\beta\beta}_n = 2\,\epsilon^{\alpha\beta}\,\epsilon^{\dot\alpha\dot\beta} \tag{2.84}
$$

where $\epsilon_{12} = -\epsilon^{12} = 1$ and by by $T_{(mn)}$ we denoted the symmetric part of a tensor.

3 Instantons in supersymmetric gauge theories

In this section we describe the moduli space of instantons in supersymmetric gauge theories.

3.1 Supersymmetric gauge theories

Supersymmetric theories are theories symmetric under the action of fermonic generators exchanging bosonic and fermionic degrees of freedom. In a supersymmetric theory, states organise in supermultiplets with equal number of bosonic and fermionic degrees of freedom related to each other by the action of supersymmetry. In the case of a gauge theory with $\mathcal{N}=1$ supersymmetry there are two basic multiplets:

$$\begin{aligned} &\text{Vector multiplet} \qquad \mathbf{V}=(A_\mu,\Lambda_\alpha,\bar\Lambda_{\dot\alpha},D)_{\mathbf{Adj}} \\ &\text{Chiralmultiplet} \qquad \mathbf{C}=(\phi,\psi_\alpha,F)_{\mathbf{rep}} \end{aligned} \tag{3.85}$$

It is convenient to pack them in the so called Vector V and Chiral Φ superfields

$$\begin{aligned} V &= \theta\sigma^\mu\bar\theta\, A_\mu(x) - i\,\theta\bar\theta\bar\theta\,\Lambda(x) + i\theta\theta\bar\theta\bar\Lambda(x) + \tfrac{1}{2}\theta\theta\bar\theta\bar\theta D(x) \\ \Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \qquad\qquad y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta \\ &= \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\,\theta\sigma^m\bar\theta\partial_m\phi - \frac{i}{\sqrt{2}}\theta\theta\,\partial_m\psi(x)\sigma^m\bar\theta + \tfrac{1}{4}\,\theta\theta\bar\theta\bar\theta\,\Box\,\phi \end{aligned} \tag{3.86}$$

The variation of a superfield under supersymmetry is defined by

$$\delta = \epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\,\bar Q^{\dot\alpha} \tag{3.87}$$

with

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\,\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\partial_m \qquad \bar Q^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\,\theta_\alpha\,\sigma^m_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\alpha}\,\partial_m \tag{3.88}$$

The resulting supersymmetry transformations are

$$\begin{aligned} \delta A_n &= -i\bar\Lambda\bar\sigma_n\epsilon + i\bar\epsilon\bar\sigma_n\Lambda \\ \delta\Lambda &= -\sigma^{mn}\,\epsilon\,F_{mn} + i\,\epsilon\,D \\ \delta\bar\Lambda &= \bar\epsilon\bar\sigma^{mn}\,F_{mn} - i\bar\epsilon\,D \\ \delta D &= -\epsilon\sigma^m\partial_m\bar\Lambda - \partial_m\Lambda\sigma^m\bar\epsilon \end{aligned} \tag{3.89}$$

for the vector and

$$\begin{aligned}\delta\phi &= \sqrt{2}\,\epsilon\,\psi \\ \delta\psi &= i\,\sqrt{2}\,\sigma^m\,\bar{\epsilon}\,\partial_m\phi + \sqrt{2}\,\epsilon\,F \\ \delta F &= i\,\sqrt{2}\,\bar{\epsilon}\,\bar{\sigma}^m\,\partial_m\psi\end{aligned} \tag{3.90}$$

for the chiral multiplets. The general $\mathcal{N}=1$ supersymmetric gauge theory can be written as

$$L = \mathrm{tr}\int d^2\theta d^2\bar{\theta}\,K(\bar{\Phi}, e^V\,\Phi) + \mathrm{tr}\left(\int d^2\theta\left[\tfrac{\tau}{16\pi}\,W^\alpha W_\alpha + W(\Phi)\right] + \mathrm{h.c.}\right) \tag{3.91}$$

with $K(\bar{\Phi}, e^V\,\Phi)$ a real function, *the Kahler potential*, $W(\Phi)$ a holomorphic function, *the superpotential*,

$$\tau = \frac{\theta}{2\pi} + \mathrm{i}\frac{4\pi}{g_{YM}^2} \tag{3.92}$$

is the complexified gauge coupling and

$$\begin{aligned}W_\alpha &= -\tfrac{1}{4}\bar{D}\bar{D}\,D_\alpha V = \Lambda_\alpha(y) + (iD\delta_\alpha^\beta - F_{mn}\,\sigma^{mn\beta}_{\ \ \ \alpha})\,\theta_\beta + i\,\theta^2\sigma^m_{\alpha\dot{\alpha}}\partial_m\bar{\Lambda}^{\dot{\alpha}} \\ W^\alpha W_\alpha|_{\theta^2} &= \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{4}\epsilon_{\mu\nu\sigma\rho}F^{\mu\nu}F^{\sigma\rho} - D^2 + 2i\bar{\Lambda}\sigma^m\partial_m\Lambda\end{aligned} \tag{3.93}$$

When not say explicitly we will restrict to the simplest choice of Kahler potential

$$K(\bar{\Phi}, e^V\,\Phi) = \mathrm{tr}_{\mathbf{R}}\,\bar{\Phi}\,e^V\,\Phi \tag{3.94}$$

corresponding to the case where the scalar manifold is flat.

Theories with extended supersymmetry can be seen as special case of $\mathcal{N}=1$ supersymmetric theories where vector and chiral multiplets combined into bigger multiplets entering in the action in a symmetric fashion

$$\begin{aligned}&\mathcal{N}=2: \qquad \mathbf{V}_{\mathcal{N}=2} = (\mathbf{V}+\mathbf{C})_{\mathbf{Adj}} \qquad \mathbf{H} = (\mathbf{C}+\bar{\mathbf{C}})_{\mathbf{rep}} \\ &\mathcal{N}=4: \qquad \mathbf{V}_{\mathcal{N}=4} = (\mathbf{V}+3\mathbf{C})_{\mathbf{Adj}}\end{aligned} \tag{3.95}$$

The form of the Kahler potential and the superpotential is restricted by the extra supersymmetry. For example in the case of pure $\mathcal{N}=2$ supersymmetry, the action can be written in terms of a single holomorphic function $\mathcal{F}(\Phi)$ and is given by (3.91) with

$$\tau(\Phi) = \frac{\partial^2\mathcal{F}}{\partial\Phi^2} \qquad K(\bar{\Phi},\Phi) = \frac{1}{4\pi}\mathrm{Im}\,\mathrm{tr}_{\mathbf{Adj}}\,\frac{\partial\mathcal{F}}{\partial\Phi}\bar{\Phi} \qquad W(\Phi) = 0 \tag{3.96}$$

In the case of $\mathcal{N}=4$ the complete action is fixed and is given again by (3.91) with

$$K(\bar{\Phi}_a, e^V\,\Phi_a) = \sum_{a=1}^{3}\mathrm{tr}_{\mathbf{Adj}}\,\bar{\Phi}_a\,e^V\,\Phi_a \qquad W(\Phi_a) = \mathrm{tr}_{\mathbf{Adj}}\,\Phi_1[\Phi_2,\Phi_3] \tag{3.97}$$

3.2 Supersymmetry in the instanton moduli space

The space-time supersymmetry induces a supersymmetry on the instanton moduli space. In particular a symmetry under

$$Q A_n = \bar{\epsilon}_{\dot{\alpha} A} \, \bar{\sigma}_n^{\dot{\alpha}\alpha} \, \Lambda_\alpha + \ldots \tag{3.98}$$

implies that the moduli matrices Δ and $\mathcal{M}$ describing the zero modes of vector and gaugino fields

$$\delta_{\rm mod} \Delta_{\lambda,\dot{\alpha} j} = \mathcal{M}_{\lambda,j} = \bar{\epsilon}_{\dot{\alpha} A} \, \mathcal{M}^A_{\lambda,j} \tag{3.99}$$

We will write

$$\delta_{\rm mod} \Delta = \mathcal{M} \tag{3.100}$$

Acting on U one finds

$$\delta_{\rm mod} U = (U \, \bar{U} + \Delta \, f \, \bar{\Delta}) \, \delta_{\rm mod} U = U \alpha - \Delta \, f \, \bar{\mathcal{M}} \, U \tag{3.101}$$

with $\alpha = \bar{U} \, \delta_{\rm mod} U$. We notice that the α-dependent term in the right hand side is a gauge transformation $\delta_\alpha U = -U\alpha$. So if we define

$$Q = \delta_{\rm mod} + \delta_\alpha \tag{3.102}$$

with α a gauge transformation with parameter α, we find the supersymmetry transformation rule

$$Q \, U = -\Delta \, f \, \bar{\mathcal{M}} \, U \tag{3.103}$$

Similarly

$$Q \, \bar{U} = -\bar{U} \, \mathcal{M} \, f \, \bar{\Delta} \tag{3.104}$$

Acting on the gaugino one finds

$$\begin{aligned} Q \, A &= Q\bar{U} dU + \bar{U} \, d(QU) = -\bar{U} \, \mathcal{M} \, f \, \bar{\Delta} \, dU - \bar{U} \, d(\Delta \, f \, \bar{\mathcal{M}} \, U) \\ &= \bar{U} (\, \mathcal{M} \, f \, d\bar{\Delta} \, - d\Delta \, f \, \bar{\mathcal{M}}) \, U = \Lambda \end{aligned} \tag{3.105}$$

reproducing the right space-time supersymmetry transformations.

4 $\mathcal{N} = 1$ Superpotentials

4.1 SQCD with $N_f = N - 1$ flavors

In this section we consider a $\mathcal{N} = 1$ $U(N)$ gauge theory with N_f chiral superfields Q_f and N_f superfields $\tilde{Q}$ in the fundamental and antifundamental representations respectively. In the background of the instanton the effective action is given by the moduli space integral

$$S_{\rm eff} = \sum_k q^k \Lambda^{k\beta} \int d\mathfrak{M}\, e^{-S_{{\rm inst}_k}(\mathfrak{M})} = \int d^4x_0\, d^2\theta_0\, W(Q,\tilde{Q}) \tag{4.106}$$

with

$$Q = \varphi(x_0) + \theta_0\, \psi(x_0) + \ldots \tag{4.107}$$

the quark and antiquark superfields in the moduli space,

$$W(\varphi) = \Lambda^{k\beta} \int d\widehat{\mathfrak{M}}\, e^{-S_{{\rm inst}_k}(\mathfrak{M})} = \frac{\Lambda^{k\beta}}{\varphi^{k\beta-3}} \tag{4.108}$$

the superpotential and

$$d\mathfrak{M} = d^4x_0\, d^2\theta_0\, d\widehat{\mathfrak{M}} \tag{4.109}$$

The factor $\Lambda^{k\beta}$ is included in order to make the action dimensionless. Since bosons have length dimension one and fermions half, $k\beta$ is nothing but the number of bosons n_B minus half the number of fermions n_F,

$$k\,\beta = n_B - \tfrac{1}{2}n_F = 4kN - \tfrac{1}{2}(2\,k\,N + 2\,k\,N_f) = k\,(3N - N_f) \tag{4.110}$$

We notice also that

$$\beta = 3c_{Adj} - \sum_{\mathbf{R}} c_{\mathbf{R}} \qquad c_{Adj} = N \quad c_{\rm fund} = \tfrac{1}{2} \tag{4.111}$$

is also the one-loop beta function coefficient of the gauge theory. Indeed one can write

$$q^k_{\rm eff} = q^k\,\Lambda^{k\,\beta} \qquad \text{with} \qquad \tau_{\rm eff} = \tau + \frac{\beta}{2\pi i}\ln\Lambda \tag{4.112}$$

so Λ can be interpreted as the dynamically generated scale of the theory and $\tau_{\rm eft}$ as the running gauge coupling at scale Λ.

At the instanton background with zero vevs one finds $S_{{\rm inst}_k}(\mathfrak{M}) = 0$. In order to soak the fermionic zero mode integrals we have to turn on a vev for φ in Q and brings down Yukawa term interactions from the action. Since Yukawa terms combine an Adjoint and a fundamental zero mode, a non-trivial result is found only if the number of adjoint and fundamental fermionic zero modes match besides the two extra θ's, i.e.

$$n_{\rm Adj} - n_{\rm fund} = 2\,k\,(N - N_f) = 2 \quad \Rightarrow \quad N_f = N - 1 \quad , \quad k = 1 \tag{4.113}$$

The instanton measure becomes

$$d\widehat{\mathfrak{M}} = \delta(D^c)\delta(\lambda_{\dot\alpha})\, d^{4N}w\, d^{2N}\mu\, d^{N-1}\mathcal{K}\, d^{N-1}\tilde{\mathcal{K}} \tag{4.114}$$

Finally let us evaluate $S_{\text{inst}_k}(\mathfrak{M})$. We first notice that ,once evaluated at the solutions of the equations of motion, the YM action is a total derivative. Schematically

$$\begin{aligned} S_{\text{inst}_k}(\mathfrak{M}) &= \int d^4x\, Q\, e^V\, \bar{Q}\big|_{\theta^2\bar\theta^2} + \ldots \qquad = \int d^4x\, (|D\varphi|^2 + \psi\,\Lambda\,\bar\varphi + \ldots) \\ &= \int d^4x\, \left[\partial^m(\bar\varphi\, D_m\varphi) + (-D^2\varphi + \psi\,\Lambda)\bar\varphi + \ldots\right] \end{aligned} \tag{4.115}$$

We can compute then $S_{\text{inst}_k}(\mathfrak{M})$ by using the asymptotic expansion of the instanton solution. From (2.75) one finds

$$\bar\varphi A_m A_m \varphi \sim \frac{1}{x^4}\,\bar\varphi\, w_{\dot\alpha}\, \bar{w}^{\dot\alpha}\,\varphi \qquad\qquad \psi^\alpha\,\Lambda_\alpha\,\bar\varphi \sim \frac{1}{x^4}\,\mathcal{K}\,\bar\mu\,\bar\varphi \tag{4.116}$$

with similar expressions for tilted fields. Notice that both contributions are quadratic in the instanton moduli, so the integrals become Gaussian. To simplify further the computation we take the vev matrix φ, $\tilde\varphi$ in the diagonal form:

$$\varphi_u^s = \varphi_s\,\delta_u^s \qquad\qquad \tilde\varphi_s^u = \tilde\varphi_s\,\delta_s^u \tag{4.117}$$

with $u = 1,\ldots N$, $s = 1,\ldots N-1$. For this choice, the zero modes $\mu_N, \bar\mu_N$ appear only on $\delta(\lambda_{\dot\alpha})$ and can be solved in favour of the others. The integral over $\mathcal{K}_s, \tilde{\mathcal{K}}_s, \mu_s, w_{s\dot\alpha}$, lead to the determinant

$$W(\varphi) \sim \prod_{s=1}^{N-1} \frac{\bar\varphi_s\,\bar{\tilde\varphi}_s}{|\varphi_s|^2|\bar{\tilde\varphi}_s|^2} = \frac{1}{\det m} \tag{4.118}$$

with

$$m_{ss'} = \varphi_s^u\varphi_{us'} \tag{4.119}$$

the Meson matrix. The remaining integrals are φ-independent, so they give a numerical factor. Collecting all pieces one finds

$$S_{\text{eff}} = c\int d^4x d^2\theta\, \frac{\Lambda^{2N+1}}{\det M(x,\theta)} \tag{4.120}$$

with c a constant and M the super field version of the meson matrix m.

5 The $\mathcal{N} = 2$ prepotential

In this section we will compute the instanton corrections to the prepotential of $\mathcal{N} = 2$ theories. The instanton corrections to the prepotential are given by the moduli space integral

$$S_{\text{eff}} \;=\; \sum q^k \int d\mathfrak{M}_k \, e^{S_{\text{mod}}(\Phi)} = \int d^4x d^4\,\theta \mathcal{F}_{\text{non-pert}}(\Phi) \tag{5.121}$$

with

$$\mathcal{F}_{\text{non-pert}}(a_u) = \sum q^k \int d\widehat{\mathfrak{M}}_k e^{S_{\text{mod}}(a_u)} \tag{5.122}$$

Here we denote by $q = \Lambda^\beta \, e^{2\pi i \tau}$, with Λ^β compensating for the length dimension of instanton moduli space measure. The matrix of gauge couplings is defined as

$$\tau_{uv} = \frac{\partial^2 \mathcal{F}}{\partial a_u \partial a_v} \tag{5.123}$$

5.1 The idea of localization

Theorem of Localization:

- Let $\mathfrak{g} = U(1)$ a group action on a manifold M of complex dimension ℓ specified by the vector field $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$,

  $$\delta_\xi x^i = \xi^i(x) \tag{5.124}$$

  with isolated fixed points x_0^s, i.e. points where $\xi^i(x_0^s) = 0 \; \forall i$.

- Let Q_ξ an equivariant derivative,

  $$Q_\xi \equiv d + i_\xi \tag{5.125}$$

  with d the exterior derivative and

  $$i_\xi dx^i \equiv \delta_\xi x^i \qquad i_\xi (dx^i \wedge dx^j) = \delta_\xi x^i \, dx^j - dx^i \, \delta_\xi x^j \tag{5.126}$$

  a contraction with ξ.

- Let α an equivariantly closed form, i.e. a form satisfying

  $$Q_\xi \, \alpha = 0 \tag{5.127}$$

Then, the integral of alpha is given by the localisation formula

$$\int_M \alpha = (-2\pi)^\ell \sum_s \frac{\alpha_0(x_0^s)}{\det^{\frac{1}{2}} Q_\xi^2(x_0^s)} \tag{5.128}$$

with α_0 the zero-form part of α. We notice that

$$Q_\xi^2 = di_\xi + i_\xi d = \delta_\xi \tag{5.129}$$

and therefore $Q^2{}_i^j = \partial_i \xi^j$ can be viewed as the map $Q^2 : T_{x_0}M \to T_{x_0}M$ induced by the action of the vector field ξ.

Example: Gaussian integral via $U(1)$ localization on $\mathbb{R}^2$. Let us consider the integral

$$I = \int_{\mathbb{R}^2} e^{-a(x^2+y^2)}\, dxdy \tag{5.130}$$

We first notice that $\mathbb{R}^2$ admit the action of the rotation group induced by the vector field

$$\xi = \epsilon \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \qquad \Rightarrow \qquad Q^2 = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \tag{5.131}$$

This action has only the origin $x = y = 0$ as a fixed point. An equivariantly closed form is given by

$$\alpha = e^{-a(x^2+y^2)}\, dxdy - \frac{\epsilon}{2a} e^{-a(x^2+y^2)} \tag{5.132}$$

Using the Localization formula one finds

$$I = \int_{\mathbb{R}^2} \alpha = 2\pi \frac{\epsilon\, e^{-a(x_0^2+y_0^2)}}{2a\epsilon} = \frac{\pi}{a} \tag{5.133}$$

with $x_0 = y_0 = 0$ the critical point. Notice that the right hand side does not depend on ϵ as expected.

5.2 The equivariant charge

For $\mathcal{N} = 2$ gauge theories if we identify the R-symmetry index A with $\dot{\alpha}$, the supersymmetry parameter $\xi_{\dot{\alpha}A}$ become a scalar $\epsilon_{\dot{\alpha}A} = \xi \epsilon_{\dot{\alpha}A}$. The supersymmetry variation relate then fields with the same quantum numbers. This identification is known as a *topological twist* since the resulting supersymmetry variation satisfies $\delta^2 = 0$, defining a BRS charge. By construction the moduli space action is a δ-variation of something, so the resulting theory in the moduli space is topological. The action of the BRS charge can be written in general as

$$Q\Phi = \Psi \qquad Q\Psi = Q^2\Phi = 0 \tag{5.134}$$

The list of multiplets (Φ, Ψ), transformation properties, and the eigenvalues λ_Φ for gauge and matter moduli are displayed in table 5.2. The second and third columns display the multiplets and the spin statistics of their lowest component. The fourth and fifth columns display the transformation properties under the symmetry groups.

	(Φ, Ψ)	$(-)^{F_\phi}$	$\mathbf{R}_G$	$\mathrm{SU}(2)^2$	λ_Φ
Gauge	$(a_{\alpha\dot{\alpha}}; M_{\alpha\dot{\alpha}})$	$+$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{2})$	$\chi_{ij}+\epsilon_1, \chi_{ij}+\epsilon_2$
	$(\lambda_c; D_c)$	$-$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{3})$	$\chi_{ij}+\epsilon \ , \ \chi_{ij}^{\frac{1}{2}}$
	$(\bar{\chi}, \lambda)$	$+$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{1})$	$\chi_{ij}^{\frac{1}{2}}$
	$(w_{\dot{\alpha}}; \mu_{\dot{\alpha}})$	$+$	$\mathbf{k}\,\overline{\mathbf{N}}$	$(\mathbf{1},\mathbf{2})$	$\chi_i - a_u + \frac{1}{2}\epsilon$
	$(\bar{w}_{\dot{\alpha}}; \bar{\mu}_{\dot{\alpha}})$	$+$	$\bar{\mathbf{k}}\,\mathbf{N}$	$(\mathbf{1},\mathbf{2})$	$a_u - \chi_i + \frac{1}{2}\epsilon$
Adjoint	$(M^a_\alpha; h^a_\alpha)$	$-$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{2},\mathbf{1})$	$\chi_{ij}+\epsilon_1-m, \chi_{ij}+\epsilon_2-m$
	$(\chi_{\dot{\alpha}a}; \lambda_{\dot{\alpha}a})$	$+$	$\mathbf{k}\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{2})$	$\chi_{ij}-m \ , \ \chi_{ij}+\epsilon-m$
	$(\mu_a; h_a)$	$-$	$\mathbf{k}\,\bar{\mathbf{N}}_{\mathbf{c}}$	$(\mathbf{1},\mathbf{1})$	$\chi_i - a_u + \frac{1}{2}\epsilon - m$
	$(\bar{\mu}_a; \bar{h}_a)$	$-$	$\bar{\mathbf{k}}\,\mathbf{N}_{\mathbf{c}}$	$(\mathbf{1},\mathbf{1})$	$a_u - \chi_i + \frac{1}{2}\epsilon - m$
Fund	$(\mathcal{K}; h)$	$-$	$\mathbf{k}\,\overline{\mathbf{N}}_{\mathbf{f}}$	$(\mathbf{1},\mathbf{1})$	$\chi_i - m_f$
Anti-F	$(\tilde{\mathcal{K}}; \tilde{h})$	$-$	$\mathbf{N}_{\tilde{\mathbf{f}}}\,\bar{\mathbf{k}}$	$(\mathbf{1},\mathbf{1})$	$\tilde{m}_f - \chi_i$

Table 1: Instanton moduli space for $\mathcal{N}=2$ gauge theories.

We introduce the auxiliary fields $\bar{\chi}$, h, $\tilde{h}$, h^a_α, h^a to account for the extra degrees of freedom in $\lambda_{\dot{\alpha}\dot{\beta}}$, $\mathcal{K}$, $\tilde{\mathcal{K}}$, M^a_α and $\bar{\mu}^a$ respectively. Here $a = 3, 4$. The sign indicates the spin statistics of the given field. In particular, multiplets with negative signs are associated to constraints subtracting degrees of freedom. We notice that for $N_f = 2N$ or $\mathcal{N} = 4$ we have the same number of bosonic and fermionic supermultiplets as expected for a conformal theory.

To evaluate the integral over the instanton moduli space with the help of the localisation formula we should first find a BRS equivariant charge $Q_\xi = \delta + i_\xi$. For $\mathcal{N}=2$ gauge theories the instanton moduli space is invariant under the symmetry

$$G = U(k) \times U(N) \times SO(4) \times U(N_f) \times U(N_{\tilde{f}}) \tag{5.135}$$

We parametrize the Cartan of this group by the parameters

$$(\chi_i, a_u, \epsilon_\ell, m_f, \tilde{m}_f) \qquad i=1,\dots k\ ,\ u=1,\dots N\ ,\ ,\ \ell=1,2\ ,\ f=1,\dots N_f \tag{5.136}$$

and the masses $(m_f, \tilde{m}_f)$ and M for the case of (anti)fundamental and adjoint matter respectively. The action of the equivariant BRS charge on the moduli space can then be written generically as

$$Q_\xi \Phi = \Psi \qquad Q_\xi^2 \Phi = \delta_\xi \Phi = \lambda_\Phi\, \Phi \tag{5.137}$$

with Φ a complex field that can be either a boson in the case of a multiplet containing a physical field or a fermion for multiplets involving auxiliary fields. The list of multiplets (Φ, Ψ) and their eigenvalues Λ_Φ are displayed in table 5.2. We use the shorthand notation $\epsilon = \epsilon_1 + \epsilon_2$.

5.3 The instanton partition function and the prepotential

We will regularize the volume factor by introducing some $\epsilon_{1,2}$-deformations of the four-dimensional geometry and recover the flat space result from the limit $\epsilon_{1,2} \to 0$. More precisely we will find

$$\mathcal{F}_{\rm non-pert}(a_u, q) = -\lim_{\epsilon_\ell \to 0} \epsilon_1 \epsilon_2\, \ln Z(\epsilon_\ell, a_u, q) \tag{5.138}$$

with

$$Z = \sum q^k\, Z_k \qquad Z_k = \int d\mathfrak{M}_k e^{S_{\rm mod}(a_u, \epsilon_\ell)} \tag{5.139}$$

The factor $\epsilon_1\epsilon_2$ in (5.138) takes care of the volume factor $\int d^4x d^4\theta \sim \frac{1}{\epsilon_1 \epsilon_2}$.

The k-instanton partition function Z_k is given by the moduli space integral

$$Z_k = \int d\mathfrak{M}_k\ e^{-S_{\rm mod}} = \int \frac{d\chi}{\text{vol } U(k)} \prod_\Phi \lambda_\Phi^{-(-1)^F} = \int \frac{1}{k!} \prod_{i=1}^k \frac{d\chi_i}{2\pi \mathrm{i}}\ z_k^{\rm gauge} z_k^{\rm matter} \tag{5.140}$$

with

$$\frac{d\chi}{\text{vol} U(k)} = \frac{1}{k!} \int \prod_{i=1}^k \frac{d\chi_i}{2\pi\mathrm{i}} \prod_{i\neq j} \chi_{ij} \tag{5.141}$$

Using the eigenvalues in Tab. 5.2, one finds

$$\begin{aligned}
z_k^{\text{gauge}} &= (-1)^k \prod_{i,j} \frac{\chi_{ij}^{1-\delta_{ij}}(\chi_{ij}+\epsilon)}{(\chi_{ij}+\epsilon_1)(\chi_{ij}+\epsilon_2)} \prod_{i,u} \frac{1}{(\chi_i - a_u + \frac{\epsilon}{2})(-\chi_i + a_u + \frac{\epsilon}{2})} \\
z_k^{\text{Adjoint}} &= \prod_{i,j} \frac{(\chi_{ij}+\epsilon_1-m)(\chi_{ij}+\epsilon_2-m)}{(\chi_{ij}-m)(\chi_{ij}+\epsilon-m)} \prod_{i,u} (\chi_i - a_u + \tfrac{\epsilon}{2} - m)(-\chi_i + a_u + \tfrac{\epsilon}{2} - m) \\
z_k^{\text{fund}} &= \prod_{i,f} (\chi_i - m_f) \\
z_k^{\text{anti-fund}} &= \prod_{i,f} (-\chi_i + \tilde{m}_f)
\end{aligned} \tag{5.142}$$

with $\epsilon = \epsilon_1 + \epsilon_2$. The integral over χ_i has to be thought of as a multiple contour integral around M-independent poles with

$$\operatorname{Im} \epsilon_1 >> \operatorname{Im} \epsilon_2 >> \operatorname{Im} a_u > 0 \tag{5.143}$$

Poles are in one-to-one correspondence of N-sets of two-dimensional Young Tableaux $Y = \{Y_u\}$ with total number of k boxes. Given Y, the eigenvalues χ_i can be written as

$$\chi_i^Y = \chi_{I_u,J_u}^{Y_u} = a_u + (I_u - \tfrac{1}{2})\epsilon_1 + (J_u - \tfrac{1}{2})\epsilon_2 \tag{5.144}$$

with I_u, J_u running over the rows and columns of the u^{th} tableaux Y_u. The partition function can then be written as

$$Z_k = \sum_{Y; k=|Y|} Z_Y \tag{5.145}$$

with

$$Z_Y = \operatorname{Res}_{\chi=\chi^Y} z_k^{\text{gauge}} z_k^{\text{matter}} = \prod_{\Phi} \lambda_{\Phi,Y}^{(-1)^{F+1}} \tag{5.146}$$

5.4 Examples

5.4.1 $U(1)$ plus adjoint matter: k=1,2

From (5.140) and (5.142) one finds

$$\begin{aligned}
Z_{\square} &= \frac{(m+\epsilon_1)(m+\epsilon_2)}{\epsilon_1\epsilon_2}\ , \\
Z_{\square\square} &= \frac{(m+\epsilon_1)(m+\epsilon_2)(m+\epsilon_2-\epsilon_1)(m+2\epsilon_1)}{2\epsilon_1^2\epsilon_2(\epsilon_2-\epsilon_1)}\ ,
\end{aligned} \tag{5.147}$$

which lead to the non-perturbative prepotential

$$F_{\text{n.p.}} = -\lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2\Big[q\, Z_{\square} + q^2\Big(Z_{\square\square} + Z_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} - \frac{1}{2} Z_{\square}^2\Big)\Big] = -q\, m^2 - \frac{3}{2}\, q^2\, m^2 + \cdots \ . \tag{5.148}$$

5.4.2 Example: Pure $SU(2)$ gauge theory: k=1,2

For pure $SU(2)$ gauge theory we take $T_{a_1} = T_{a_2}^{-1} = T_a$. For the first few tableaux one finds

$$\begin{aligned}
\mathbf{T}(\square,\bullet) &= T_1 + T_2 + T_{-2a} + T_1 T_2 T_{2a} \\
\mathbf{T}(\square\square,\bullet) &= T_1 + T_1^2 + T_2 + \frac{T_2}{T_1} + T_{-2a} + T_1 T_2 T_{2a} + T_1^2 T_2 T_{2a} + \frac{T_{-2a}}{T_1} \\
\mathbf{T}(\square,\square) &= 2T_1 + 2T_2 + T_1 T_{2a} + T_2 T_{2a} + T_1 T_{-2a} + T_2 T_{-2a}
\end{aligned} \tag{5.149}$$

with similar expressions for $\mathbf{T}(\bullet,\square)$,$\mathbf{T}(\bullet,\square\square)$ obtained from the first two lines replacing $a \to -a$, for $\mathbf{T}(\begin{smallmatrix}\square\\ \square\end{smallmatrix},\bullet)$ obtained from the second line exchanging $\epsilon_1 \leftrightarrow \epsilon_2$ and $\mathbf{T}\big(\bullet,\begin{smallmatrix}\square\\ \square\end{smallmatrix}\big)$ obtained from the second line after replacing $a \to -a$ and $\epsilon_1 \leftrightarrow \epsilon_2$ simultaneously. For the partition functions one finds

$$\begin{aligned}
Z_{(\square,\bullet)} &= -\frac{1}{2a\,\epsilon_1\epsilon_2(2a+\epsilon_1+\epsilon_2)} \\
Z_{(\square\square,\bullet)} &= \frac{1}{4a\epsilon_1^2\epsilon_2(\epsilon_2-\epsilon_1)(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)(2a+\epsilon_1)} \\
Z_{(\square,\square)} &= \frac{1}{\epsilon_1^2\epsilon_2^2(4a^2-\epsilon_1^2)(4a^2-\epsilon_2^2)}
\end{aligned} \tag{5.150}$$

leading to the prepotential

$$\begin{aligned}
\mathcal{F}_{np} &= -\lim_{\epsilon_\ell\to 0} \epsilon_1\epsilon_2 \left[q\, Z_{(\square)} + q^2 (Z_{(\square\square)} + Z_{(\begin{smallmatrix}\square\\ \square\end{smallmatrix})} - \tfrac{1}{2} Z^2_{(\square)}) \right] \\
&= \frac{q}{2a^2} + \frac{5}{64}\frac{q^2}{a^6} + \ldots
\end{aligned} \tag{5.151}$$

6 Seiberg-Witten curves

For simplicity we take pure $\mathcal{N}=2$ SYM with gauge group $SU(N)$. We write

$$Z(q) = \sum_k q^k Z_k = \sum_k \frac{q^k}{k!} \int \prod_{i=1}^k \frac{d\chi_i}{2\pi \mathrm{i}}\ e^{\ln z_k^{\text{gauge}}} = \sum_k \int \prod_{i=1}^k \frac{d\chi_i}{2\pi \mathrm{i}}\ e^{\frac{1}{\epsilon_1\epsilon_2}\mathcal{H}_k(\chi_i)} \tag{6.152}$$

with

$$z_k^{\rm gauge} = \prod_{i,j} \frac{\chi_{ij}^{1-\delta_{ij}}(\chi_{ij}+\epsilon)}{(\chi_{ij}+\epsilon_1)(\chi_{ij}+\epsilon_2)} \prod_i \frac{1}{P(\chi_i+\frac{\epsilon}{2})P(\chi_i-\frac{\epsilon}{2})} \tag{6.153}$$

and

$$P(x) = \prod_{u=1}^{N} P(x-a_u+\tfrac{\epsilon}{2}) \tag{6.154}$$

In the limit $\epsilon_\ell \to 0$ one finds

$$\begin{aligned} \mathcal{H}_k(\chi_i) &= \epsilon_1\epsilon_2 \left[\sum_{ij} \ln\left(\frac{\chi_{ij}(\chi_{ij}+\epsilon)}{(\chi_{ij}+\epsilon_1)(\chi_{ij}+\epsilon_2)} \right) - \sum_i \ln P(\chi_i+\tfrac{\epsilon}{2})P(\chi_i-\tfrac{\epsilon}{2}) + k\ln q \right] \\ &\approx \left[-\epsilon_1^2\epsilon_2^2 \sum_{ij} \frac{1}{\chi_{ij}^2} - 2\epsilon_1\epsilon_2 \sum_i \ln P(\chi_i) + k\epsilon_1\epsilon_2 \ln q \right] \end{aligned} \tag{6.155}$$

Introducing the density function

$$\rho(x) = \epsilon_1\epsilon_2 \sum_i \delta(x-\chi_i) \tag{6.156}$$

one can rewrite (6.155) as

$$\mathcal{H}_k(\rho) = -\int dx dz \frac{\rho(x)\rho(z)}{(x-z)^2} - 2\int dz \rho(z) \ln P(z) + \ln q \int dz \rho(z) \tag{6.157}$$

The Saddle point equation becomes

$$\frac{d\mathcal{H}_k(\rho)}{d\rho(x)} = -2\int_{\mathbb{R}} dz \frac{\rho(z)}{(x-z)^2} - 2\ln P(x) + \ln q = 0 \qquad x \in \bigcup_{u=1}^{N} \Sigma_u \tag{6.158}$$

Defining

$$y(x) = \exp\left[-\int_{\mathbb{R}} dz \frac{\rho(z)}{(x-z+i0^-)^2} - \ln P(x+i0^-) \right] \tag{6.159}$$

The saddle point equation can be written as

$$|y(x)|^2 = q \qquad x \in \bigcup_{u=1}^{N} \Sigma_u \tag{6.160}$$

with boundary condition $\lim_{x\to\infty} y(x) = x^{-N}$. A solution can be written as

$$y(x) = \tfrac{1}{2}\left(P_N(x) - \sqrt{P_N(x)^2 - 4\,q} \right) \tag{6.161}$$

that satisfy the saddle point equation in the intervals

$$\Sigma_u = [\alpha_u^-, \alpha_u^+] \qquad\qquad P_N^2 - 4q = \prod_{u=1}^{N}(x-\alpha_u^-)(x-\alpha_u^+) \tag{6.162}$$

Here $P_N(x)$ is a polynomial of order N that we will be write as

$$P_N(x) = \prod_{u=1}^{N}(x-e_u) \tag{6.163}$$

We notice that $y(x)$ is solution of the quadratic equation

$$y(x)^2 - P_N(x)y(x) + q = 0 \tag{6.164}$$

that is called *Seiberg-Witten* equaltion. We notice that using the definition (6.159), one can identify a_u with the period integrals

$$a_u = \frac{1}{2\pi i}\int_{\Sigma_u} x\,\lambda(x) \tag{6.165}$$

with

$$\lambda(x) = -dx\,\partial_x \ln y(x) = \frac{P_N'(x)}{\sqrt{P_N(x)^2-4q}}\,dx = \frac{P_N'(x)}{P_N(x)} + \frac{2qP_N'(x)}{P(x)^3} + \dots \tag{6.166}$$

leading to

$$a_u = e_u + 2q\,\mathrm{Res}_{x=e_u}\frac{xP_N'(x)}{P_N(x)^3} + \dots \tag{6.167}$$

These relations can be inverted to find e_u in terms of a_u. To leading order

$$e_u = a_u - 2q\,\mathrm{Res}_{x=a_u}\frac{xP'(x)}{P(x)^3} + \dots \tag{6.168}$$

Similarly one can compute chiral correlators

$$\langle \mathrm{tr}\,\Phi^J\rangle = \sum_u \int_{\Sigma_u} x^J\,\lambda(x) = \sum_{u=1}^{N}\left(e_u^J - 2q\,\mathrm{Res}_{x=e_u}\frac{x^J P_N'(x)}{P_N(x)^3} + \dots\right) \tag{6.169}$$

In particular the prepotential $\mathcal{F}$ can be found from the relation

$$\langle \mathrm{tr}\,\Phi^2\rangle = 2q\frac{\partial \mathcal{F}}{\partial q} \tag{6.170}$$

Example: pure $SU(2)$ gauge theory

We take

$$P_N(x) = x^2 - e^2 \tag{6.171}$$

leading to

$$a = \mathrm{Res}_{x=e} x\, P_N' \left(\frac{1}{P_N} + \frac{2\,q}{P_N^3} + \frac{6q^2}{P_N^7} + \ldots \right) = e - \frac{q}{4e^3} - \frac{15\,q^2}{64\,e^7} + \ldots \tag{6.172}$$

Inverting this relation one finds

$$e = a + \frac{q}{4a^3} + \frac{3\,q^2}{64\,a^7} + \ldots \tag{6.173}$$

One the other hand for the chiral correlator one finds

$$\langle \mathrm{tr}\,\Phi^2 \rangle = 2e^2 = 2a^2 + \frac{q}{a^2} + \frac{5\,q^2}{16\,a^6} + \ldots = 2a^2 + 2\,\mathcal{F}_1\,q + 4\,\mathcal{F}_2\,q^2 + \ldots \tag{6.174}$$

leading to

$$\mathcal{F}_{\mathrm{inst}} = \frac{q}{2a^2} + \frac{5q^2}{64a^6} \tag{6.175}$$

in agreement with (5.151).