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Lectures 1, 2, 3, 4

C.P. Bachas

Laboratoire de Physique de École Normale Supérieure
24 rue Lhomond, 75231 Paris cedex, France

ABSTRACT: Preliminary draft of the first four lectures at LACES 2019.

Color code:
In red: keywords or formulae to which the reader's attention is drawn.
In blue: exercises and questions to probe your understanding.
In olive green: explanatory and other 'by the way' comments.
In brown: digressions.

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1. The free bosonic string

To set our notation and conventions we briefly review the classical equations, and the quantization of a relativistic string. We will focus in particular on allowed boundary conditions at the endpoints of open strings. This material can be found in any standard textbook [1]-[6].

1.1 Polyakov and Nambu-Gotto actions

The starting point is the Nambu-Gotto action which is proportional to the (pseudo)area wiped out by the motion of the string,

$$S_{\rm NG} = -T_{\rm F} \int d^2\sigma \, [-\det(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})]^{1/2} \ . \tag{1.1}$$

Here $T_{\rm F} := (2\pi\alpha')^{-1}$ is the tension of the string, $\mu, \nu = 0, 1 \cdots d-1$ and $\alpha, \beta = 0, 1$. The string moves in flat Minkowski spacetime and its worldsheet is parametrized by σ^α. The signature of the metric is $(-+\cdots+)$.

The above action is *classically* equivalent to a theory of 2d gravity coupled to d free scalar fields, one of which (the time X^0) has negative kinetic energy, [1]

$$S_{\rm Polyakov} = -\frac{T_{\rm F}}{2} \int d^2\sigma \, \sqrt{-g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \ . \tag{1.2}$$

Both actions are invariant under reparametrizations of the worldsheet, $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^\beta)$, with the metric in (1.2) transforming as usual so that $g_{\alpha\beta} d\sigma^\alpha d\sigma^\beta$ is left unchanged. One can choose conformal coordinates so that $g_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$. The Liouville field ϕ drops out of the classical action, leading to the field equations ($\sigma^\pm = \sigma^0 \pm \sigma^1$)

$$\partial_+ \partial_- X^\mu = 0 \, , \qquad \partial_\pm X^\mu \partial_\pm X^\nu \eta_{\mu\nu} = 0 \ . \tag{1.3}$$

The two equations in red follow from the variation of the action under the auxiliary metric,

$$\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm Polyakov}}{\delta g^{\alpha\beta}} \equiv T_{\alpha\beta} \ \propto \ \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} (\partial_\gamma X^\mu \partial^\gamma X_\mu) = 0 \ . \tag{1.4}$$

Note that this gives two equations, because the trace of the energy-momentum tensor vanishes identically. From the conservation $\partial^\alpha T_{\alpha\beta} = 0$ it follows that the constraints $T_{\alpha\beta} = 0$ need only be imposed at some initial time and will be automatically satisfied at all times. They are phase-space constraints called the Virasoro conditions.

[1] "This action is called the Polyakov action, demonstrating [in the words of Polyakov himself in *"From Quarks to Strings"*] the Arnold theorem that "things are never called after their true inventors." The trick was used by J. Douglas in the 1920's to study minimal surfaces, and later by Brink, di Vecchia and Howe, and by Wess and Zumino for the superstring.

Note that if the matter theory was not conformal the Liouville equation $T_\alpha{}^\alpha = 0$ would eliminate some matter fields. For instance a potential $V(X)$ would constrain classically the coordinates to lie on the $V = 0$ submanifold. The quantum theory is less trivial than this, and potentially more interesting.

If we start instead from (1.1) we can still choose conformal coordinates, such that the *induced* metric $\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \propto \eta_{\alpha\beta}$. The classical equations are again given by (1.3) and the Virasoro constraints are now the gauge-fixing conditions. Thus, at the classical level the two actions are equivalent.

Exercise 1: Prove this, using the formula $\delta \det M = (\det M) M^{-1} \delta M$.

Exercise 2: What happens if one adds the Einstein term $\Phi_0 \int d^2\sigma \sqrt{-g}\, R$ to the world-sheet action ? Answer: One finds $\sqrt{g} R = -\Box\phi$, so the (Euclidean) integral is a topological invariant equal to $4\pi\chi = 4\pi(2 - 2h - b)$ where $b = \#boundaries$ and $h = \#handles$. Show this for the sphere, disk and torus. Hint: the round-sphere metric is $4R^2 dz d\bar{z}/(1 + z\bar{z})^2$.

1.2 Classical motion of closed strings

The 2d wave equation $\partial_+\partial_- X = 0$ is solved by a sum of left- and right-moving waves, $X = X_R(\sigma^-) + X_L(\sigma^+)$. Assuming periodic boundary conditions, $\sigma^1 \equiv \sigma^1 + 2\pi$ at some initial time $\sigma^0 = 0$, the most general solution reads

$$\underline{\text{closed}}: \qquad X^\mu = x^\mu + \alpha' p^\mu \sigma^0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(a_n^\mu e^{-in\sigma^-} + \tilde{a}_n^\mu e^{-in\sigma^+}\right) \tag{1.5}$$

where n runs over non-zero integers. The normalizations in this mode expansion have been fixed so that $p^\mu = T_F \int_0^{2\pi} d\sigma^1 \partial_0 X^\mu$ is the center-of-mass momentum of the string, and canonical Poisson brackets imply $\{a_n^\mu, a_m^\nu\} = \{\tilde{a}_n^\mu, \tilde{a}_m^\nu\} = in\, \delta_{n+m,0}\delta^{\mu\nu}$. Reality requires $(a_n^\mu)^* = a_{-n}^\mu$ and likewise for the tilde variables.

The Virasoro constraints can be solved explicitly in light-cone gauge:

$$X^+ = \alpha' p^+ \sigma^0 \ , \qquad \partial_\pm X^- = \frac{2}{\alpha' p^+} \sum_{j=2}^{d-1} \partial_\pm X^j \partial_\pm X^j \tag{1.6}$$

with $X^\pm = X^0 \pm X^1$. We used the residual freedom under reparametrizations $\sigma^+ \to f(\sigma^+)$ and $\sigma^- \to \tilde{f}(\sigma^-)$ which preserve the conformal-gauge condition. In light-cone gauge this residual freedom is completely fixed by choosing $2X_L^+ = \alpha' p^+ \sigma^+$ and $2X_R^+ = \alpha' p^+ \sigma^-$. The phase-space of a closed string is thus parametrized by the center-of-mass positions and momenta and by the oscillation amplitudes in transverse dimensions, $\{x^\mu, p^\mu, a_n^j, \tilde{a}_n^j\}$. These are subject to the mass-shell and level-matching conditions [the integrals around the string of (1.6)]

$$\underline{\text{closed}}: \qquad M^2 = -p^\mu p_\mu = \frac{2}{\alpha'} \sum_{j=2}^{d-1} \sum_{n\neq 0} a_{-n}^j a_n^j = \frac{2}{\alpha'} \sum_{j=2}^{d-1} \sum_{n\neq 0} \tilde{a}_{-n}^j \tilde{a}_n^j \ . \tag{1.7}$$

NB: The light-cone gauge is less natural than the temporal gauge $X^0 = \sigma^0$, but it is more convenient because it linearizes the Virasoro constraints.

Exercise 3 : By going to the temporal gauge show that cusps (interior points moving at the speed of light) are generic on a cosmic string [12].

Hint: go to the center-of-mass frame and choose the gauge $X^0 = \sigma^0$, in which the Virasoro constraints read $|\partial_+ \vec{X}|^2 = |\partial_- \vec{X}|^2 = \frac{1}{4}$. Taking the sum and difference gives $\partial_0 \vec{X} \cdot \partial_1 \vec{X} = 0$ and $|\partial_0 \vec{X}|^2 + |\partial_1 \vec{X}|^2 = 1$. A cusp is a point moving at the speed of light, $|\partial_0 \vec{X}|^2 = 1$, which implies $\partial_1 \vec{X} = 0$ and hence $\partial_+ \vec{X} = \partial_- \vec{X}$. As time runs in 3+1 dimensions, the unit vectors $2\partial_\pm \vec{X} = \vec{v}_\pm$ trace trajectories on the unit 2-sphere, which will generically cross *qed*. Cusps on cosmic strings are strong emitters of gravitational waves [13].

1.3 Open strings and D-branes

For worldsheets with boundary, the variation of the Polyakov action gives

$$\begin{aligned} \delta S_{\text{Polyakov}} \propto & \int d^2\sigma \, \partial^\alpha (\delta X_\mu) \, \partial_\alpha X^\mu \\ & = \oint d\sigma^\alpha \epsilon_{\alpha\beta} \partial^\beta X^\mu \, \delta X_\mu - \int d^2\sigma \, \delta X_\mu \, \partial^\alpha \partial_\alpha X^\mu = 0 \; . \end{aligned} \tag{1.8}$$

A free endpoint, i.e. one for which δX^μ is arbitrary, must thus obey the Neumann conditions $\partial_\perp X^\mu = 0$, where $\perp$ stands for normal to the boundary. The general solution for an open string with two free endpoints reads [2]

$$\underline{\text{open}}\ (\text{NN}): \qquad X^\mu = x^\mu + 2\alpha' p^\mu \sigma^0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} a_n^\mu \left(e^{-in\sigma^-} + e^{-in\sigma^+}\right) , \tag{1.9}$$

where $n \in \mathbb{Z}$ and the boundary is at $\sigma^1 = 0, \pi$. Note that left- and right-moving amplitudes are identified, so the transverse excitations are reflected on the boundaries to form standing waves. In terms of the coordinate fields $\partial_+ X_L = \partial_- X_R$ at both endpoints.

The Neumann conditions preserve Poincaré invariance, so there is a conserved (center-of-mass) momentum and angular momentum in $\mathbb{R}^D$. The open-string phase space in light-cone gauge is parametrized by $\{x^\mu, p^\mu, a_n^j\}$ subject to the mass-shell condition

$$\underline{\text{open}}: \qquad -p^\mu p_\mu = \frac{1}{2\alpha'} \sum_{j=2}^{d-1} \sum_{n\neq 0} a_{-n}^j a_n^j \; . \tag{1.10}$$

Since $\partial_1 X^\mu = 0$ at $\sigma^1 = 0, \pi$, one infers from the Virasoro constraints that free endpoints always travel at the speed of light.

[2] The canonical choice of worldsheet parametrization ensures that the Poisson brackets are the same for open and closed strings.

Another natural boundary condition, for instance appropriate for a violin string, is the fixed endpoint or Dirichlet condition, X^j constant at $\sigma^1 = 0, \pi$ [so that $\delta X^j = 0$ on the boundaries]. The corresponding solution is

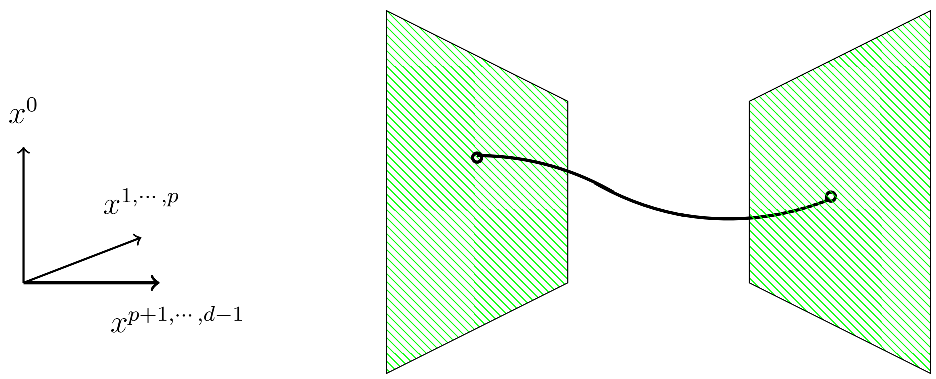
$$\underline{\text{open}}\ (\text{DD}): \qquad X^j = x^j + \frac{\sigma^1}{\pi}\Delta x^j + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}a_n^j\,(e^{-in\sigma^-} - e^{-in\sigma^+})\ . \tag{1.11}$$

The main difference with (1.9) is that there is no center-of-mass momentum p^j, since the string cannot move in the jth direction. There is however something that replaces it, stretching between x^j and $x^j + \Delta x^j$, the positions of the two fixed endpoints. In terms of coordinate fields the fixed-endpoint conditions read $\partial_+ X_L = -\partial_- X_R$.

If $X^{0,1,\cdots,p}$ obey Neumann conditions and the remaining $X^{p+1,\cdots,d-1}$ Dirichlet conditions, this describes a situation in which the string endpoints are stuck on $(p+1)$-dimensional hyperplanes in spacetime, as in the figure below. We will think of these hyperplanes as trajectories of infinite extended objects (defects of spacetime) called D(irichlet)-branes. We will soon see that they are non-perturbative, soliton-like excitations of string theory. The mass-shell condition for an open string between two parallel static Dp branes reads

$$\underline{\text{open}}: \qquad -p^\mu p_\mu = \frac{1}{2\alpha'}\sum_{j=2}^{d-1}\sum_{n\neq 0} a_{-n}^j a_n^j + |T_F\Delta\vec{x}|^2\ , \tag{1.12}$$

where μ runs over the $p+1$ Neumann directions only. The extra term on the right-hand side is indeed the mass squared of an open string stretching linearly between the D-branes. It enters in the mass formula like the momentum of some extra, hidden dimension. This is not a coincidence, it is a consequence of the deep symmetry of string theory called T-dulaity, as we will see.



More generally, the two D-branes at the string endpoints can be different. If one is of lower dimensionality, or if they extend in orthogonal directions, some of the coordinates will obey Neumann conditions at one endpoint and Dirichlet conditions at the other endpoint. The modes of such 'mixed' DN coordinates have half-integer frequencies,

$$\text{open (ND)}: \qquad X^j = x^j + i\sqrt{\frac{\alpha'}{2}} \sum_{r\in\frac{1}{2}+\mathbb{Z}} \frac{1}{r}\, a_r^j \,(e^{-ir\sigma^-} + e^{-ir\sigma^+}) \;+\; c.c.\;, \tag{1.13}$$

[Verify]. Note that neither momentum nor stretching is allowed in a ND direction. Finally, let us consider the case of two D-branes that have undergone a relative rotation in the (X^1, X^2) plane by an angle ϑ. Suppose the first D-brane extends along the direction 1, so that X^1 obeys a Neumann condition and X^2 a Dirichlet condition at $\sigma^1 = 0$, while at the other endpoint, $\sigma^1 = \pi$, $(X^1 \cos\vartheta + X^2 \sin\vartheta)$ is Neumann and $(X^1 \sin\vartheta - X^2 \cos\vartheta)$ is Dirichlet (see figure 1). The general solution with these boundary conditions reads

$$X^1 + iX^2 = \sqrt{\frac{\alpha'}{2}} \left[\sum_r \frac{1}{r}(a_r\, e^{-ir\sigma^-} + a_r^* e^{ir\sigma^+}) + \sum_s \frac{1}{s}(b_s^*\, e^{is\sigma^-} + b_s e^{-is\sigma^+}) \right]\;, \tag{1.14}$$

where $r \in \mathbb{Z} + \frac{\vartheta}{\pi}$ and $s \in \mathbb{Z} - \frac{\vartheta}{\pi}$. One checks indeed that at $\sigma^1 = 0$ the complex coordinate $X^1 + iX^2$ is real, consistently with the Dirichlet condition on X^2, while $e^{-i\vartheta}(X^1 + iX^2)$ becomes real at $\sigma^1 = \pi$. The expansion (1.14) reduces to (1.13) for a rotation by 90^0, in which case X^1 become ND and X^2 becomes DN and all oscillations are half-integer modded.

Note that the open string of figure 1 prefers to sit at the intersection point. It can wonder away as one pumps in more and more energy, but it can never escape.

Exercise 4: Find the mode expansion for an open string stretched between two identical D-branes that move with relative velocity v in a transverse dimension. Repeat for two D1-branes with both relative rotation and motion in the (X^1, X^2) plane. What can you say about the motion of the intersection point? Is there a violation of causality? The answers can be found in refs. [14, 15].

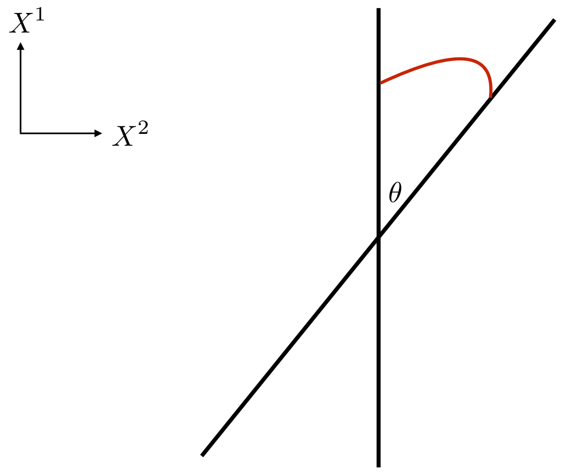


Figure 1: Two D1-branes at an angle.The stretched open string is localized (for small energy budget) near the point of intersection of the D-branes.

1.4 Light-cone quantization

The mass spectrum of a classical string is non-negative and continuous. In the quantum theory it becomes discrete, and it may include a tachyon. The first fact follows from the harmonic-oscillator algebra of the amplitudes,

$$[a_n^i, (a_m^j)^\dagger] = [\tilde{a}_n^i, (\tilde{a}_m^j)^\dagger] = n\delta_{n-m,0}\,\delta^{ij}\ . \tag{1.15}$$

The ground state is annihilated by all $n > 0$ modes, while their hermitean conjugates $a_{-n}^j = (a_n^j)^\dagger$ create excited string states. The tachyon arises from the normal ordering of the mass-squared operator. Each oscillation mode with frequency ω contributes $\frac{1}{2}\omega$ to the ground-state energy. The infinite sum over frequencies diverges, but can be regularized with a local counterterm. The result is an invariant 2d Casimir energy which can have either positive or negative sign.

Let us compute some examples. For an open string with Neumann conditions at both endpoints one finds

$$\underline{\text{open}}\,(\text{NN}): \qquad -\alpha' p^\mu p_\mu = \sum_{j=2}^{d-1}\sum_{n>0} a_{-n}^j a_n^j - \frac{d-2}{24} := \hat{N} - \frac{d-2}{24}, \tag{1.16}$$

where $\hat{N}$ is the number operator defined below, and the negative contribution is the 2d Casimir energy of the $d-2$ integer-modded coordinates with [3]

$$\sum_{n=1}^{\infty}\frac{n}{2} = \frac{1}{2}\zeta(-1) = -\frac{1}{24}\ . \tag{1.17}$$

[3]To justify the zeta-function prescription, let $\sigma^1 \in [0, \pi L]$, introduce a short-distance cutoff ϵ on the string, and make a *local* (energy-density independent of L) subtraction to render the result finite

$$\lim_{\epsilon\to 0}\Big[\sum_{n=1}^{\infty}\frac{n}{2L}e^{-n\epsilon/L} - \#\frac{L}{\epsilon^2}\Big] = -\frac{1}{24L}\ .$$

.

The result is the same if some of the coordinates are DD rather than NN. If on the other hand there are $d_\perp$ coordinates of DN type, then the mass formula becomes

$$\underline{\text{open}}\,(\text{mixed}):\quad -\alpha' p^\mu p_\mu = \sum_{j=2}^{d_\perp+1}\sum_{r>0} a_{-r}^j a_r^j \;+\; \sum_{j=d_\perp+2}^{d-1}\sum_{n>0} a_{-n}^j a_n^j - \frac{d-2}{24} + \frac{d_\perp}{16}\,. \tag{1.18}$$

Exercise 5: Verify this formula, and generalize it to the case of D-branes at angles.
Answer: For an ND coordinate $\frac{1}{2}\sum_{n=0}^{\infty}(n+\frac{1}{2}) = \frac{1}{4}\sum_{\rm odd} n = \frac{1}{4}(\sum_{\rm all} n - \sum_{\rm even} n) = \frac{1}{4}(-\frac{1}{12}+\frac{2}{12}) = \frac{1}{48} = -\frac{1}{24}+\frac{1}{16}$. More generally it is enought to consider $\vartheta\in[0,\pi/2]$ since bosonic D-branes have no orientation. Furthermore the result is an even quadratic function of ϑ. The formula $-\frac{1}{12}+\frac{1}{2}(\frac{\vartheta}{\pi})^2$ fits the answer in the two cases given above: (i) a NN plus a DD coordinate ($\vartheta=0$), and (ii) two ND coordinates ($\vartheta=\pi/2$).

The number operator $\hat N$ used in the previous formulae has commutation relations $[\hat N, a_n^\dagger] = n a_n^\dagger$ and $[\hat N, a_n] = -n a_n$. For a single unit-frequency harmonic oscillator one finds $\hat N (a^\dagger)^m|0\rangle = m(a^\dagger)^m|0\rangle$. Thus $\hat N$ is the sum of the frequencies of *all* creation operators applied to the ground state. We will call it the level of the excitation. Closed strings have independent levels in the left- and right-moving sectors, but these are identified by the level matching condition $\hat N_L = \hat N_R$.

Closed strings can also have fractional-frequency modes when they move in spaces with conical singularities called ‘orbifolds’. The simplest orbifold is the complex plane with the identification $X^1+iX^2 \equiv e^{2i\vartheta}(X^1+iX^2)$ with $\vartheta\in[0,\pi]$. Check that the left-moving closed-string modes have frequencies $\pm\frac{\vartheta}{\pi}+\mathbb{Z}$, and the same is true for the right movers. The allowed masses of twisted closed-string states are therefore the same (up to an overall factor of 4) as those of an open string stretched between two D-branes at angle ϑ.

1.5 Open and closed-string spectra

Let us now consider the spectra in the two Poincaré-invariant cases, namely closed strings in $\mathbb{R}^{1,d-1}$ and open strings with free endpoints. The ground state obeys $a_n^j|0\rangle = 0$ (NN open string) and $a_n^j|0\rangle = \tilde a_n^j|0\rangle = 0$ (closed string) for all $n>0$. Its mass squared is negative for $d>2$, it is a *tachyon* of mass $\alpha' M^2 = -1$ (open) or -4 (closed). The excited states have squared mass $-1+\hat N$ for open, or $4(-1+\hat N_L) = 4(-1+\hat N_R)$ for closed.

The states at level one have positive mass for $d<26$, and are massless at the critical dimension $d=26$. A spinning particle with only transverse polarizations must be massless, so consistency requires $d=26$. At level two the states are massive and they fill a representation of the rotation group $SO(25)$, as they should. These facts are summarized for the open string in table 1.

Exercise 7: Find the $SO(25)$ representations for the states at the third and fourth mass levels. See reference [16] for a general analysis of $SO(25)$ invariance at all mass levels.

$\alpha' m^2$	states	$SO(25)$ representation
-1	$\lvert 0\rangle$	scalar
0	$a^j_{-1}\lvert 0\rangle$	transverse vector
1	$a^j_{-1}a^k_{-1}\lvert 0\rangle, a^j_{-2}\lvert 0\rangle$	traceless symmetric tensor

Table 1: First three levels of the bosonic open NN string in $d = 26$ dimensions.

The closed string has a tachyon with $\alpha' m^2 = -4$ in the critical dimension. The massless states $a^j_{-1}\tilde{a}^k_{-1}|0\rangle$ transform as a general 2-index tensor of $SO(24)$. Its symmetric, anti-symmetric and trace parts describe fluctuations of the space-time metric $G_{\mu\nu}$, the antisymmetric *Kalb-Ramond* field $B_{\mu\nu}$, and a scalar field called the *dilaton* Φ. Higher-level states transform as tensor products of two representations of the open string at the same (left- and right-moving) level.

What if $d \neq 26$? In light-cone gauge the Lorentz symmetry has an *anomaly* as can be checked by computing the commutator of the Lorentz generators

$$J^{\mu\nu} = T \int d\sigma^1 (X^\mu \partial_0 X^\nu - X^\nu \partial_0 X^\mu) \ . \tag{1.19}$$

[The problem only arises in the commutator of the J^{-i}. Why ?]

In the *covariant* quantization, on the other hand, where one imposes the (positive-frequency) quadratic Virasoro constraints as weak conditions on physical states, there appear *negative-norm* states for all $d > 26$.

[Check this at the second level for the open string.]

For $d < 26$ there is no such problem a priori, but extra states do arise when one introduces interactions. This can be understood in the path-integral quantization, where the Liouville mode only decouples at the critical dimension, and acquires negative kinetic energy above. There do exist consistent string theories with a dynamical Liouville mode at $d < 26$, called *non-critical* string theories.[4]

[4]One may consider likewise open strings between different D-branes, but the interactions of such strings produce open strings with both endpoints on the same D-brane for which consistency requires that $d \leq 26$.

1.6 Asymptotic density and Hagedorn temperature

As a simple application, and a warm up for the coming lectures, we compute the generating function of (open NN) string states weighted with the exponential of their mass squared [This is *not* the canonical partition function with Boltzmann weights $\exp(-\beta M)$, but it is the partition function of the worldsheet theory]. Recall that $\alpha' M^2 = N - 1$, and since frequencies add in N the generating function is the product of 2d partition functions of the independent harmonic oscillators,

$$Z(q) = \sum_{\text{states}} \mathcal{N}(M) q^{\alpha' M^2} = q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-24} = q^{-1} + 24 + 324q + \cdots \tag{1.20}$$

The number of states at a given mass is the exponential of the microcanonical entropy, $\mathcal{N}(M) := e^{S(M)}$. It can be seen that this counts the number of partitions of the level N of the states into positive integers.

The function $Z(q)$ is a power of the Dedekind eta function,

$$Z(q) = \eta(q)^{-24} \qquad \text{where} \quad \eta(q) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \ . \tag{1.21}$$

This function transforms as a *modular* form of weight $1/2$ under the fractional linear transformations of τ, where $q = e^{2\pi i \tau}$. In particular [5]

$$\eta(\tau) \equiv e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2i\pi\tau n}) \ , \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau) \ . \tag{1.22}$$

Using this we can compute the asymptotic density of string states by a saddle-point approximation of the contour integral

$$\mathcal{N}(N) = \oint dq\, q^{-N} Z(q) \sim N^{-27/4} \, e^{4\pi\sqrt{N}} \ . \tag{1.23}$$

In doing the calculation we assumed that for $N \gg 1$ the integral is dominated by a saddle point at $\mathrm{Im}\tau \ll 1$, where $\eta(\tau) = (-i\tau)^{-1/2} \eta(-1/\tau) \simeq (-i\tau)^{-1/2} \exp(-i\pi/12\tau)$. Those of you with numerical skills could try to confirm the result numerically.

The result shows a linear rise of entropy with mass, characteristic of free strings. The canonical partition function is not therefore defined beyond a limiting *Hagedorn* temperature, $\beta_H = 4\pi\sqrt{\alpha'}$. The nature of the Hagedorn transition remains to this day mysterious.

Question: What is the Hagedron temperature for the closed string ?
Answer: The same because $S_{\text{closed}}(N) = 2S_{\text{open}}(N)$, where N is the level in one sector. But $\alpha' M^2_{\text{closed}} = 4N$ as opposed to N, so when expressed in terms of mass one gets the same rise of entropy.

[5] See Carl Ludwig Siegel (1954). Mathematika, 1, pp 4-4 doi:10.1112/S0025579300000462 for a simple proof of this identity.

Summary: We solved the classical equations of a relativistic string in light-cone gauge, both closed strings and open strings with various boundary conditions at their endpoints. The quantization introduces a mass subtraction that depends on the choice of boundary conditions. For periodic boundary conditions and for free endpoints the ground state is a tachyon. In the critical dimension $d = 26$, the first excited states include a massless spin 2 state for the closed string, and a massless spin 1 state for the open string with free endpoints.

2. The type-II superstrings

We now repeat the analysis for superstrings, limiting ourselves to the Neveu-Schwarz-Ramond (NSR) formulation. The reader should refer to [1] for a discussion of the Green-Schwarz formulation.

2.1 Worldsheet supersymmetry

Bosonic string theory can be described, as we saw, by a 2d conformal field theory coupled to 2d gravity. The graviton is an auxiliary, non-dynamical, field whose equations impose the vanishing of the energy-momentum tensor,

$$T_{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}} = 0 \; . \tag{2.1}$$

In conformal theories the trace vanishes automatically, $T^{\alpha}_{\ \alpha} = 0$. [6] This leaves the two non-trivial Virasoro conditions written, for free scalar fields, in red in (1.3).

Superstring theory can be likewise described as a 2d *superconformal* field theory coupled to 2d *supergravity*. The minimal non-chiral case is $N = (1,1)$ supersymmetry, with anticommuting parameters $\epsilon = (\epsilon_R, \epsilon_L)$ transforming as a 2d Majorana spinor. This subsection is taken from [1], chapter 4.

Let us start with *global* $N = (1,1)$ supersymmetry realized in 2d by a massless boson X and a massless Majorana fermion ψ,

$$S = \int d^2\sigma \left(-\frac{1}{2}\partial_\alpha X \partial^\alpha X + \frac{i}{2}\bar{\psi}\rho^\alpha \partial_\alpha \psi \right) \; . \tag{2.2}$$

The Dirac algebra $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ is represented by the imaginary 2×2 matrices

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; . \tag{2.3}$$

The Lorentz-boost generator $J^{01} = \frac{i}{2}\rho^0\rho^1$ is imaginary, so the two-component spinor can be chosen real. The Dirac equation

$$0 = \rho^\alpha \partial_\alpha \psi = \begin{pmatrix} 0 & -2i\partial_- \\ 2i\partial_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{2.4}$$

implies that ψ_R is only a function of σ^- and ψ_L is a function of σ^+.

The action (2.2) is invariant under the supersymmetry transformations

$$\delta X = \bar{\epsilon}\psi \; , \qquad \delta\psi = -i\rho^\alpha \partial_\alpha X \epsilon \tag{2.5}$$

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[6] For curved worldsheets the Weyl anomaly implies $T^{\alpha}_{\ \alpha} = \frac{c}{12}R$ where R is the Ricci scalar and c the central charge of the 'matter' CFT ($c = 1$ for a free scalar field and 1/2 for a free Majorana fermion). In critical string theories this anomaly cancels between 'matter' and ghost fields, $d - 26 = 0$ for the bosonic string and $\frac{3}{2}d - 15 = 0$ for superstrings.

with ϵ a constant Majorana spinor [Check]. Noether's theorem leads to the conserved (spinor) *supercurrent*

$$J_\alpha = \frac{1}{2}\rho^\beta \rho_\alpha \psi \partial_\beta X \ . \tag{2.6}$$

The supercurrent carries also a spinor index that is implicit in our notation. So it has four components in total, but two vanish automatically because the 2d Dirac-matrix identity $\rho^\alpha \rho^\beta \rho_\alpha = 0$ implies that $\rho^\alpha J_\alpha = 0$ (one spinor equation). We let the reader verify that $J_+ = \frac{1}{2}(J_0 + J_1)$ has only a lower spinor component equal to $-\psi_L \partial_+ X$, and $J_- = \frac{1}{2}(J_0 - J_1)$ has only an upper spinor component equal to $-\psi_R \partial_- X$. This is the result of superconformal invariance.

The idea is now to couple this "matter theory" to (auxiliary) supergravity, so as to force the vanishing of the supercurrents, thereby generalizing the Virasoro constraints (1.3). Coupling fermions to gravity requires an orthonormal frame for tangent space, i.e. a zweibein e_a^α where a is a "flat index". This has the property that $e_a^\alpha e_b^\beta g_{\alpha\beta} = \eta_{ab}$ or equivalently $e_\alpha^a e_\beta^b \eta_{ab} = g_{\alpha\beta}$, where e_α^a is the inverse zweibein matrix ($e_\alpha^a e_b^\alpha = \delta^a{}_b$). Spinors transform under Lorentz rotations of the local frame, but they are scalars under diffeomorphisms. Their coupling to gravity is through the spin connection $\omega_\alpha^a{}_b$. This is not an independent field if we insist that the frame be covariantly constant,

$$0 = D_\alpha e_\beta^a \implies \omega_\alpha^a{}_b = e_b^\beta (\partial_\alpha e_\beta^a - \Gamma_{\alpha\beta}^\gamma e_\gamma^a) \ . \tag{2.7}$$

The spin connection ω_α^{ab} is antisymmetric in the 'flat' indices (ab) as it should be.

Note: When the affine connection is not symmetric we cannot express ω in terms of the zweibein. Manifolds with $\Gamma_{[\nu\rho]}^\mu \neq 0$ are said to have torsion. In this case the spin connection contains an additional 'extra twisting or spinning' of the frame bundle.

The coupling of the free massless (1,1) multiplet to 2d gravity is

$$S_2 = -\frac{1}{2}\int d^2\sigma \sqrt{-g}(g^{\alpha\beta}\partial_\alpha X \partial_\beta X - i\bar{\psi}\rho^\alpha D_\alpha \psi) \ , \tag{2.8}$$

where $D_\alpha \psi \equiv (\partial_\alpha + \frac{1}{4}\omega_\alpha^{ab}\rho_{ab})\psi$, $\rho_{ab} \equiv \frac{1}{2}[\rho_a, \rho_b]$, and $\rho^a = e_\alpha^a \rho^\alpha$ are the flat-space Dirac matrices. Under local supersymmetry transformations

$$\delta S_2 = 2\int d^2\sigma \sqrt{-g}(D_\alpha \bar{\epsilon}) J^\alpha \ . \tag{2.9}$$

This can be cancelled by the variation of a gravitino term,

$$S_3 = -2\int d^2\sigma \sqrt{-g}\, \bar{\chi}_\alpha J^\alpha \ , \tag{2.10}$$

provided that $\delta\chi_\alpha = D_\alpha \epsilon$. This is, however, not the end of the story because J^α also transforms under the local supersymmetry transformations. The extra variation can be cancelled by the additional term

$$S_4 = -\frac{1}{4}\int d^2\sigma\sqrt{-g}\,(\bar{\psi}\psi)(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\chi_\beta)\ , \tag{2.11}$$

and by modifying the supersymmetry transformations as follows:

$$\delta X = \bar{\epsilon}\psi\ , \quad \delta\psi = -i\rho^\alpha\epsilon(\partial_\alpha X - \bar{\psi}\chi_\alpha) \quad \delta e^a_\alpha = -2i\bar{\epsilon}\rho^a\chi_\alpha\ , \quad \delta\chi_\alpha = D_\alpha\epsilon\ . \tag{2.12}$$

The action $S_2 + S_3 + S_4$ is invariant under diffeomorphisms, local Lorentz and local supersymmetry transformations. Note that it was not a priori guarantied that the above iterative procedure would terminate. If one starts, for instance, with a Poincaré invariant field theory and tries to gauge the subgroup of translations, an infinite number of iteration steps would be needed to obtain the complete non-linear coupling of gravity.

Exercise 1: Check the invariance of $S_2 + S_3 + S_4$ with respect to local supersymmetry transformations. Apply the Noether procedure in order to gauge the global $U(1)$ symmetry of a complex scalar field. In this case the variation of $S_3 = i\int A^\mu(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*)$ gives an extra piece that is cancelled by the quartic coupling $\int A_\mu A^\mu\phi^*\phi$ term.

I addition to local supersymmetry, the action $S_2 + S_3 + S_4$ has three extra local symmetries. It is invariant under local scale (Weyl) transformations

$$X \to X\ , \quad \psi \to v^{-1/2}\psi\ , \quad e^a_\alpha \to v e^a_\alpha\ , \quad \chi_\alpha \to v^{1/2}\chi_\alpha\ , \tag{2.13}$$

and under superWeyl transformations which only affect the worldsheet gravitino,

$$\delta\chi_\alpha = i\rho_\alpha\eta\ , \qquad \delta(\text{rest}) = 0\ . \tag{2.14}$$

[This follows again from the identity $\rho_\alpha\rho^\beta\rho^\alpha = 0$]. Using local supersymmetry one can set $\chi_\alpha = i\rho_\alpha\chi$, where χ is the partner of the Weyl factor of the metric (or Liouville field) ϕ . Superconformal invariance then ensures that ϕ and χ drop out completely from the action, and the corresponding components of the energy-momentum tensor and supercurrent, T_{+-}, J_{+R}, J_{-L}, vanish identically. [We use R and L for the upper and lower value of the spinor index.]

After all the dust has settled the entire supergravity multiplet has been gauge fixed away, leaving us with a free field theory and four phase-space (or super-Virasoro) conditions: $T_{--} = T_{++} = J_{-R} = J_{+L} = 0$.

Exercise 2: Find the residual supersymmetry transformations that leave the gravitino in the superconformal gauge. Show that they can be used to set $\psi^+ = 0$ in light-cone gauge. Assume flat metric and zero spin connection. [Beware that $\pm$ also refers to spacetime indices here, $\psi^\pm = \psi^0 \pm \psi^1$.]

Answer: $\delta\chi_\alpha = \partial_\alpha \epsilon$ with $\epsilon = (\epsilon_R, \epsilon_L)$. To stay in superconformal gauge we need $\partial_\alpha \epsilon = i\rho_\alpha \eta$ for some η (superconformal Killing equation). This means $\partial_+ \epsilon = -i\rho^- \eta = -2(0, \eta_L)$ which implies that $\partial_+ \epsilon_R = 0$ and likewise $\partial_- \epsilon_L = 0$. So residual supersymmetries have ϵ_R an arbitrary function of σ^-, and ϵ_L an arbitrary function of σ^+. Now in lightcone gauge we chose $2\partial_\pm X^+ = \alpha' P^+$ so that $\delta\psi^+ = -i\rho^+ \epsilon \partial_+ X^+ - i\rho^- \epsilon \partial_- X^+ = \alpha' p^+ (\epsilon_R, -\epsilon_L)$. This is enough residual supersymmetry to set the holomorphic and antiholomorphic components of the (on-shell) fermion ψ^+ to zero *qed*.

2.2 Neveu-Schwarz and Ramond sectors

Consider the supercoordinates (X^μ, ψ^μ) of a closed superstring. The mode expansion of the bosonic coordinates X^μ is as in eq. (1.5). The fermions obey the massless Dirac equation (2.4) and the following mode expansion:

$$\underline{\text{closed}}: \qquad (\psi_R^\mu, \psi_L^\mu) = \Big(\sum_r \psi_r^\mu \, e^{-ir\sigma^-} \, , \, \sum_{\tilde r} \tilde\psi_{\tilde r}^\mu \, e^{-i\tilde r \sigma^+} \Big) \, . \tag{2.15}$$

The canonical anti-commutation relations read $\{\psi_r^\mu, \psi_s^\nu\} = \{\tilde\psi_r^\mu, \tilde\psi_s^\nu\} = \delta_{r+s,0}\eta^{\mu\nu}$, while reality imposes $(\psi_r^\mu)^\dagger = \psi_{-r}^\mu$ and likewise for the tildes.[7] Now comes a crucial new feature compared to the bosonic string. All observables are fermion bilinears, so fermions can have either periodic or antiperiodic boundary conditions. Thus r and $\tilde r$ can take either integer or half-integer values.

We should however insist on one thing, that the supercurrents $J_{-R} = -\psi_R^\mu \partial_- X_\mu$ and $J_{+L} = -\psi_L^\mu \partial_+ X_\mu$ which generate the residual superconformal symmetries be globally well-defined. This means that when transported around the string they should come back to themselves up to a sign [since they are fermions], i.e.

$$J_{+L}(\sigma^1 + 2\pi) = \eta_L J_{+L}(\sigma^1) \qquad \text{and} \quad J_{-R}(\sigma^1 + 2\pi) = \eta_R J_{-R}(\sigma^1)$$

where $\eta_R, \eta_L = \pm$ are signs. This implies that all ψ_R^μ must be simultaneously periodic or antiperiodic, and likewise for all ψ_L^μ. The choices $\eta = +$ and $\eta = -$ are called Ramond (R), respectively Neveu-Schwarz (NS) boundary conditions. For a closed superstring we have four possibilites: NS-NS, NS-R, R-NS and R-R. All four are needed in type II superstrings.[8]

What about the open superstrings? Varying the action of a Majorana fermion on a worldsheet with boundary gives a boundary term

$$\delta S_{\text{Dirac}} = \frac{i}{2} \oint d\sigma^\alpha \epsilon_{\alpha\beta} \, \bar\psi \rho^\beta \delta\psi \, . \tag{2.16}$$

[7] Curly brackets here denote anticommutators, not Poisson brackets.

[8] In the absence of worldsheet supersymmetry, the fermion periodicities are only constrained by modular invariance. This is exploited in the fermionic formulation of the heterotic string where worldsheet supersymmetry is only realized among left movers.

Take the boundary in the σ^0 direction, at $\sigma^1 = 0$. The variation vanishes if

$$\psi^T \rho^0 \rho^1 \delta\psi = \psi_R \delta\psi_R - \psi_L \delta\psi_L = 0 \quad \Longrightarrow \quad \psi_R = \pm \psi_L \big|_{\sigma^1=0} \,. \tag{2.17}$$

Superconformal invariance correlates the choice of sign with the choice of boundary condition for the partner bosonic coordinate. Indeed

$$J_{+L} = \eta J_{-R} \quad \text{implies} \quad \psi_L^\mu \partial_+ X_\mu = \eta\, \psi_R^\mu \partial_- X_\mu \tag{2.18}$$

for each μ separately. Thus $\psi_L = \eta\psi_R$ for a Neumann coordinate, and $\psi_L = -\eta\psi_R$ for a Dirichlet coordinate. Each fermionic coordinate goes, in other words, along for the ride with the bosonic one.

The two endpoints of an open string need not have the same η. One of the signs can be absorbed in the definition of the fermion fields, but the second is important. In the NS sector of the open string this relative sign is $-$, while in the Ramond sector it is $+$. The fermionic partners of NN or DD coordinates are half-integer modded in the NS sector, and integer-modded in the R sector (the opposite holds for partners of ND coordinates). Explicitly their expansion reads

$$\underline{\text{open}}: \qquad (\psi_R^\mu, \psi_L^\mu) = \Big(\sum_r \psi_r^\mu \, e^{-ir\sigma^-} \,,\, \sum_r \psi_r^\mu \, e^{-ir\sigma^+} \Big) \,, \tag{2.19}$$

with r integer or half-integer as explained above, and with the gluing sign at $\sigma^1 = 0$ abosrbed in a redefinition the left-moving components ψ_L^μ.

Generalize this discussion to strings stretched between D-branes at angles. Can you see a difference between $\theta = 0$ and $\theta = \pi$? Answer: At this point, no.

Digression on worldsheet symmetries. They imply by Noether's theorem the existence of conserved two-component currents $\partial_\alpha j^\alpha = \partial_+ j^+ + \partial_- j^- = 0$. When each term vanishes separately there exists a companion conserved current, $\tilde{j} = (j^+, -j^-)$, and the symmetry is doubled.[9] This is the case in particular for the energy-momentum tensor $T_{\alpha\beta}$ and for the spinor-valued supercurrent J_α, which break up as we have seen into separately conserved left-moving and right-moving currents.

Now a boundary reflects left movers to right movers, so it necessarily breaks at least half of such doubled symmetries. If the symmetries are gauged it should break no more. For example $T_{++} - T_{--} = 4T_{01}$ must vanish at the boundary, or else boundary-preserving reparametrizations of the worldsheet would not be good gauge symmetries. The same is true for the N=(1,1) superconformal currents as discussed above. There is no such requirement for global worldsheet symmetries, which are broken completely by generic D-brane configurations.

[9]This happens when the charge density is the sum of a left-moving and a right-moving one. A counterexample is that currents associated to the SO(1,9) Lorentz invariance of the target spacetime which do not factorize in this way.

2.3 GSO projections and type IIA,B supergravities

As in the case of the bosonic string, we can solve the super Virasoro constraints in the light-cone gauge. First use as before the freedom under (anti)holomorphic parametrizations to set $X^+ = \alpha' p^+ \sigma^0$. Next, the supersymmetry transformation (2.12) for ψ^+ reduces to $\delta\psi^+ = -i\alpha' p^+ \rho^0 \epsilon$ [the plus is a spacetime index]. Using this one can set $\psi^+ = 0$, while preserving the superconformal gauge $\chi_\alpha = i\rho_\alpha \chi$, as well as $X^+ = \alpha' p^+ \sigma^0$ and the fact that X^μ, ψ^μ are solutions of the massless Klein-Gordon, respectively Dirac equations [verify the statement].

The constraints are now linear in X^- and ψ^-, which can be solved to express their oscillation amplitudes in terms of those in the remaining dimensions. The mass-shell (and level-matching) conditions read :[10]

$$\underline{\mathrm{NS}}: \qquad -\frac{\alpha'}{4} p^\mu p_\mu = \sum_{j=2}^{d-1} \left(\sum_{n>0} a^j_{-n} a^j_n + \sum_{r \in \mathbb{N}+1/2} r \psi^j_{-r} \psi^j_r \right) - \frac{d-2}{16} \, ,$$

$$\underline{\mathrm{R}}: \qquad -\frac{\alpha'}{4} p^\mu p_\mu = \sum_{j=2}^{d-1} \left(\sum_{n>0} a^j_{-n} a^j_n + \sum_{r \in \mathbb{N}} r \psi^j_{-r} \psi^j_r \right) \, . \tag{2.20}$$

These conditions hold for the right-moving sector of closed superstrings. There are analogous conditions for the left movers, with a^j_n, ψ^j_r replaced by $\tilde{a}^j_n, \tilde{\psi}^j_r$. For open superstrings the only differences are that the r.h.s. must be multiplied by a factor $\frac{1}{4}$, and that the zero-point subtraction in the NS sector depends on the choice of boundary conditions at the endpoints.

Let us focus first on the closed superstring. Its states are tensor products of a left-moving with a right-moving state, subject to the level-matching condition. The lowest-mass state in the NS-NS sector is a tachyon, while at the next level we find an unconstrained 2-index tensor field,

$$\begin{array}{ll} |0\rangle_{\mathrm{NS}} \otimes |\widetilde{0}\rangle_{\mathrm{NS}} & \underline{\text{tachyon}}\ (T) \\ \psi^j_{-1/2}|0\rangle_{\mathrm{NS}} \otimes \tilde{\psi}^k_{-1/2}|\widetilde{0}\rangle_{\mathrm{NS}} & \underline{\text{graviton}} \text{ et al } \ (G_{jk}, B_{jk}, \Phi) \, . \end{array} \tag{2.21}$$

It can be decomposed into the graviton (symmetric traceless part), the dilaton (trace) and a 2-index antisymmetric tensor. All of them are massless in the critical dimension $d = 10$, as required for Lorentz invariance. Other states in the NS-NS sector have positive mass and decouple at low energies.

In the R-R sector the ground states are massless because the subtraction in (2.20) vanishes. In addition, there exist fermionic zero modes, $r = 0$, which act on

[10] In the NS sector the subtraction is $(d-2)(\sum n - \sum r) = (d-2)(-\frac{1}{24} - \frac{1}{48})$. In the R sector it cancels between bosons and fermions since they are both periodic. Recall that bosonic and fermionic oscillators have opposite zero-point energy.

the states without changing their mass. Their canonical anticommutation relations, $\{\psi_0^j, \psi_0^k\} = \{\tilde\psi_0^j, \tilde\psi_0^k\} = \delta^{jk}$, are the same as those of the Dirac matrices for SO(8). The massless R-R ground states of the superstring must represent this algebra, and hence they transform as the tensor product of two SO(8) spinors,

$$\underline{\text{RR bispinor}}\ (C_{a\tilde b}): \qquad |a\rangle_{\rm R} \otimes |\tilde b\rangle_{\rm R}\ . \tag{2.22}$$

Note that the fermionic zero modes are real, so a and $\tilde b$ label the sixteen components of a *Majorana* spinor of SO(8).[11] Bispinor states are spacetime bosons, and can be rewritten as generalized p-form gauge fields, as we will see.

How about the states in the mixed NS-R and R-NS sectors? The massless ones carry a vector and a spinor index, and they are therefore *spacetime fermions*. This means that the corresponding 'second quantized' fields must anticommute. They are the gravitini and dilatini of supergravity,[12]

$$\underline{\text{gravitini}}\ \text{et al}\ \ (\Psi_a{}^j, \widetilde\Psi_{\tilde b}{}^j): \qquad |a\rangle_{\rm R} \otimes \tilde\psi^j_{-1/2}|\tilde 0\rangle_{\rm NS}\ , \quad \psi^j_{-1/2}|0\rangle_{\rm NS} \otimes |\tilde b\rangle_{\rm R}\ . \tag{2.23}$$

Note that all states in these mixed sectors carry one spinor index and a number of vector indices; they are therefore spacetime fermions.

The theory constructed so far still includes in its spectrum a tachyon, like the bosonic string. It is possible to cure this instability by imposing the Gliozzi-Scherk-Olive (GSO) projections which only keep states of even worldsheet-fermion parity in both the left- and the right-moving sectors. The parity operators are defined by their anticommuting property,

$$\{(-)^F, \psi_r^j\} = \{(-)^{\tilde F}, \tilde\psi_r^j\} = 0\ . \tag{2.24}$$

Note that since strings can join and split, and parity is multiplicative, only the even projections are consistent. What is less obvious at first is that the NS ground state is parity odd, so that GSO projects indeed out the tachyon. The proper justification involves the construction of vertex operators for the emission of different string states, see [1]-[6]. The vertex operator for the tachyon is fermion-odd, while for the graviton it is fermion-even.

Eqs. (2.24) for the $r=0$ modes (i.e. the SO(8) Dirac matrices) imply that the parity operators act as the chirality operator on R-R states. The two Weyl-Majorana spinors of SO(8) correspond to two inequivalent representations, usually denoted $\mathbf{8}_s$ and $\mathbf{8}_c$. Which one we declare to be parity-even is a matter of convention, it depends on how we represent the algebra of fermionic zero modes in terms of gamma matrices.

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[11] In the covariant gauge $|0\rangle_{\rm R}$ is a SO(1,9) Majorana spinor that obeys the massless Dirac equation $\partial\!\!\!/\Psi = 0$. Choosing $p^\mu = (0, p^-, \vec 0)$ this implies $\Gamma^+\Psi = 0$, which can be recognized as the light-cone gauge condition $\psi_0^+|0\rangle_{\rm R} = 0$.

[12] Dilatini are analogous to the trace part in the NS-NS sector, given by the contraction $\Gamma^j_{ab}\Psi_b{}^j$.

What is physically meaningful is the *relative* chirality projection in the left- and right-moving sectors. The theory in which the two Ramond ground states (and hence also the two 10d gravitini) have opposite chirality is called type IIA , the one with the same chirality is called type IIB.

This short-cut discussion of the GSO projection is incomplete. For instance, level matching and even-parity projection in one sector only would be sufficient to remove the tachyon. Such a theory is, however, anomalous under global reparametrizations of the worldsheet. The absence of global anomalies (**'modular invariance'**) is essential for the UV finiteness of string theory, and restricts possible choices of the free-string spectrum. It excludes, for example, superstring theories with only NS-NS or only R-R sectors, as well as the theory with no fermion-parity projections, see later.
A choice without modular anomalies is to keep the NS-NS and R-R sectors only, and to perform an *overall* parity projection $(-)^F(-)^{\tilde{F}} = +$. The so obtained theories have no spacetime fermions and a tachyon, they are called type-0A and type-0B.

2.4 Ramond-Ramond gauge fields

The massless states in the NS-NS sector of type II superstrings are the excitations of the graviton around flat Minkowski spacetime, a 2-index antisymmetric tensor and the dilaton field (G^{ij}, B^{ij} and Φ). In the R-R sector, as we saw, the massless states are Weyl-Majorana bispinors,

$$\widehat{C}_{a\tilde{a}}\, |a\rangle_R \otimes |\tilde{a}\rangle_{\tilde{R}} \ . \tag{2.25}$$

Considered as a matrix (with indices suppressed) the field $\widehat{C}$ obeys two chirality conditions [the chirality operator is a real diagonal matrix so $\Gamma^T = \Gamma$, see below]

$$\Gamma\widehat{C} = \pm\widehat{C}\Gamma = \widehat{C} \qquad \text{with} \begin{cases} - & \text{for IIA} \ , \\ + & \text{for IIB} \ . \end{cases} \tag{2.26}$$

A complete basis of matrices in spinor space is given by the totally antisymmetric products $\Gamma^{i_1 i_2 .. i_n} \equiv \Gamma^{[i_1}\Gamma^{i_2}\cdots\Gamma^{i_n]}$. Thus the bispinor field can be decomposed into antisymmetric n-form fields

$$C_{i_1 i_2 .. i_n} \ \equiv \ \mathrm{tr}(\,\widehat{C}\,\Gamma_{i_1 i_2 .. i_n}) \ . \tag{2.27}$$

The chirality conditions then imply that there are only odd-n forms in the IIA theory, and only even-n forms in the IIB theory. Indeed

$$C_{i_1 i_2 .. i_n} \equiv \mathrm{tr}(\,\widehat{C}\Gamma_{i_1 i_2 .. i_n}) = \mathrm{tr}(\,\Gamma\widehat{C}\Gamma_{i_1 i_2 .. i_n}) = \pm\mathrm{tr}(\,\widehat{C}\,\Gamma\Gamma_{i_1 i_2 .. i_n}) = \pm(-)^n\mathrm{tr}(\,\widehat{C}\,\Gamma_{i_1 i_2 .. i_n}\Gamma) \ .$$

Comparing the second and fourth entries and using the cyclic property of the trace shows that even-n forms vanish in type IIA and odd-n forms vanish in type IIB.

In addition, the gamma-matrix identities

$$\Gamma\Gamma_{i_1 i_2 .. i_n} = \frac{(-)^{n(n+1)/2}}{(8-n)!}\,\epsilon_{i_1 i_2 \ldots i_8}\Gamma^{i_{n+1}\ldots i_8} \tag{2.28}$$

imply that the n-forms and the $(8-n)$ forms are related by a duality relation. Thus the type-IIA theory has independent 1-form and 3-form R-R fields, while the type-IIB theory has a 0-form, a 2-form and a self-dual 4-form. As a check note that both theories have 64 $(= 8 + 56 = 1{+}28 + 35)$ massless R-R states, as many as the $8{\times}8$ Weyl-Majorana bispinor matrix.

The massless fields of the two superstring theories are summarized in table 2. The type-IIA theory has the same content as 11d supergravity dimensionally-reduced to ten dimensions. [Check this using the fact that the only fields of 11d supergravity are the graviton, a 3-form A_{MNR}, and an 11d Majorana gravitino.] Weakly-coupled M-theory also has M2-branes and M5-branes in eleven dmensions. We will see that D-branes tie in nicely with the conjectured M-theory/type-IIA duality.

Sector	IIA	IIB
NS-NS	$G_{\mu\nu}, B_{\mu\nu}, \Phi$	$G_{\mu\nu}, B_{\mu\nu}, \Phi$
R-R	$C_\mu, C_{\mu\nu\rho}$	$C, C_{\mu\nu}, C^{\text{s.d.}}_{\mu\nu\rho\sigma}$
R-NS & NS-R	$\psi_{\mu a}, \tilde\psi_{\mu a}$	$\psi_{\mu a}, \tilde\psi_{\mu\dot a}$
susy Dp-branes	p even	p odd

Table 2: The massless (perturbative) string states of type-IIA and IIB theories, and corresponding supergravity fields. Greek letters denote 10d vector indices, while $a, \dot a$ are Weyl Majorana spinor indices of opposite chirality. The last entry in the table is explained in the next subsection.

Exercise 3: Show that the self-dual 4-form potential (or 5-form field strength) contains 35 physical polarization states. Is there a self-dual Maxwell field in 4 spacetime dimensions?

Answer: Write the n-index antisymmetric gauge field as a n-form $C_{(n)}$, and its field strength as the exterior derivative $F_{(n+1)} = dC_{(n)}$. The dual field strength in d dimensions is a $(d-n-1)$- form that derives locally from a $(d-n-2)$-form potential,

$$^*F_{\mu_1\cdots\mu_{d-n-1}} = \frac{1}{(n+1)!}\epsilon_{\mu_1\cdots\mu_d}F^{\mu_{d-n}\cdots\mu_d} \ .$$

When $d = 2n+2$ one can try to impose a self-duality condition. It can be checked however that for Lorentzian signature $^*(^*F) = (-)^n F$, so real self-dual form fields only exist in 2 (mod 4) dimensions. This includes the 2-form in $d = 6$ and the 4-form in d=10 , but not the ordinary Maxwell field in d=4.

To count polarization states let e.g. $p^\mu = (E, 0\cdots, 0, E)$ with $E > 0$. The 4-form has 70 transverse states ζ_{ijkl} with $i, j, k, l = 1, \cdots, 8$. Self-duality implies $\zeta_{1234} = \zeta_{5678}$ etc. reducing the number to 35. A putative self-dual Maxwell field would have a single helicity state. In QFT one learns that this is impossible because the photon is its own antiparticle and CPT flips the helicity of states.

Digression on Weyl-Majorana spinors (see e.g. appendix B of vol. 2 in [2]). Let the spacetime dimension be $d = 2k + 2$, so that the little group for massless particles is SO($2k$). Define the following $a = 1, \cdots, k$ raising and lowering combinations of the gamma matrices

$$\Gamma^{a\pm} = \frac{1}{2}(\Gamma^{2a-1} \pm i\Gamma^{2a}) \ , \quad \text{so that} \quad \{\Gamma^{a+}, \Gamma^{a-}\} = 1$$

and all other anticommutators are zero. For $k = 1$ the spinor representation is

$$\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \Gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \quad \Gamma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \Gamma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad i\Gamma^1\Gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The last matrix is the chirality operator $\Gamma = i\Gamma^1\Gamma^2$ whose eigenstates can be denoted $|\pm\rangle$. For general k we take the tensor product of k such representations, to construct a spinor of SO($2k$) of dimension 2^k. We can denote its components as $|\pm, \cdots, \pm\rangle$ so that the eigenvalue of the chirality operator, $\Gamma = i^k \prod \Gamma^j$, is given by the product of all signs. The chiral or Weyl projection reduces the dimension to 2^{k-1}.

Now since some SO(8) generators $J^{ij} = i\Gamma^i\Gamma^j$ are real, the rotations $\mathcal{R} = \exp(i\theta_{ij}J^{ij})$ are complex matrices, so the representation is complex. Complex conjugation is however implemented by a change of basis, $\mathcal{R}^* = B^\dagger \mathcal{R} B$ with [check] either

$$B = B_1 \equiv \Gamma^1\Gamma^3 \cdots \Gamma^{2k-1} \qquad \text{or} \qquad B = B_2 \equiv \Gamma B_1 \ .$$

Both Bs flip the chirality when k is odd, so the two Weyl spinors are inequivalent complex-conjugate representations in 4,8, 12 etc spacetime dimensions. When k is even the Weyl spinor ζ transforms in a self-conjugate representation, and we can attempt to impose the Majorana condition $\zeta^* = B\zeta$. But since $\zeta = B^*\zeta^* = B^*B\zeta$, we need $B^*B = 1$. A simple calculation gives $B_1^*B_1 = (-)^{k(k+1)/2}$ and $B_2^*B_2 = (-)^{k(k-1)/2}$, so with both choices of B the Weyl-Majorana condition can be imposed only for $k = 4, 8...$ *etc*, that is in 10 (mod 8) spacetime dimensions. For $k = 2, 6...$*etc* a Weyl spinor can be pseudoreal, meaning that a reality projection can be imposed if the spinor also transforms in a complex representation of a different group (such as flavor or R-symmetry) so that complex conjugation in the internal space compensates the minus sign in B^*B.

2.5 Supersymmetric Dp-branes

At the end of section 2.2 we discussed gauge and global symmetries on the worldsheet. The former include the N=(1,1) superconformal symmetry, and possibly internal gauge symmetries (e.g. if gauged WZW models are part of the 'matter' CFT). Such symmetries are redundancies of the worldsheet theory, so boundary conditions should respect them. Global worldsheet symmetries, on the other hand, give rise to gauge symmetries in target spacetime.[13] D-branes may, but need not be invariant under them. If they are not, the corresponding spacetime gauge symmetry is spontaneously broken.

Consider for example translations in the ith direction. The conserved Noether current on the string worldsheet is $(j_+, j_-) = T_{\rm F}\,(\partial_+ X^i, \partial_- X^i)$. The Neumann boundary condition conserves the charge, $j_1 = (j_+ - j_-)|_{\rm bnry} = 0$, whereas the Dirichlet condition $j_+ = -j_-|_{\rm bnry}$ does not. Indeed, D-branes break spontaneously translation symmetry in the transverse directions.

What about the spacetime supersymmetries? These are generated on the worldsheet by right- and left-moving currents, Q_α and $\tilde{Q}_\alpha$, that carry 10d (Weyl-Majorana) spinor indices. They are explicitly constructed using 2d bosonization. I present here a sketchy derivation.[14] We start by bosonizing the worldsheet fermions,

$$\psi_R^{1\pm} \equiv \psi_R^2 \pm i\psi_R^3 \equiv :\exp(\pm i\phi_R^1): \ , \quad \cdots \ , \quad \psi_R^{4\pm} \equiv \psi_R^8 \pm i\psi_R^9 \equiv :\exp(\pm i\phi_R^4): \tag{2.29}$$

with similar expressions for the left movers. Here ϕ_R^a and ϕ_L^a are the right-moving and left-moving parts of free 2d bosons ($\phi_R^a \equiv \int d\sigma^1 \partial_- \phi^a$ and $\tilde{\phi}_L^a \equiv \int d\sigma^1 \partial_+ \phi^a$), and dots stand for normal ordering. If the boson is normalized so that [in the Euclidean theory and with normal ordering henceforth implicit]

$$\langle \phi_R(z)\phi_R(0)\rangle = -\log z \ , \quad \text{then} \quad \langle e^{iq\,\phi_R(z)}\, e^{-iq\,\phi_R(0)}\rangle = z^{-q^2} \tag{2.30}$$

so the exponential operators (2.29) have indeed the 2-point function of right-moving fermions. We can next define the 2^4 right- and left-moving spin fields

$$Q = \exp(\pm\frac{i}{2}\phi_R^1 \cdots \pm \frac{i}{2}\phi_R^4) \quad \text{and} \quad \tilde{Q} = \exp(\pm\frac{i}{2}\phi_L^1 \cdots \pm \frac{i}{2}\phi_L^4) \ . \tag{2.31}$$

These are (parts of) the conserved Noether currents of spacetime supersymmetry.

The fields Q and $\tilde{Q}$ have scaling dimension $\frac{1}{2}$, while a conserved current has dimension 1. The complete currents (and associated fermion emission vertices) include a piece from the lightcone coordinates, $\exp(\pm\frac{1}{2}\phi_R^0)$ with minus sign in the light-cone gauge $\psi^+ = \psi^0 + \psi^1 = 0$, and a piece from the ghosts that restores the correct scaling dimension [17].

[13] In agreement with the general expectation that quantum gravity has no global symmetries. Note however that when the gravitational interaction is switched off (by taking the Planck scale to infinity) the isometries of the metric background become global symmetries.

[14] For an introduction to 2d CFT aimed at string theory applications the reader can consult [2][4].

Exercise 4: Show that the spin field Q defined in (2.31) transforms as an SO(8) spinor, and that the product of signs in the exponent is the spinor chirality.

Answer: The generators of rotations acting on worldsheet fermions are $J^{ij} = \frac{1}{2\pi i}\oint \psi^i \psi^j$. Go to radial coordinates $z = e^{\sigma^0 + i\sigma^1}$ so that asymptotic states are created at $z = 0$. The commutator $[J^{ij}, Q(z')]$ is given by a contour integral that picks the simple pole, if any, at $z = z'$. The operator product expansion $e^{iq\phi(z)} e^{iq'\phi(z')} = (z - z')^{qq'} e^{i(q+q')\phi} +$ subleading, shows in the complex basis $\{a\pm\}$ for the vector indices i, j each generator of the rotation group either annihilates Q or flips two of the signs in the exponent. The product of signs is left invariant and can be identified with the chirality of the spinor.

Recall now that for a Neumann coordinate $\psi^i_R = \eta\psi^i_L$ at the boundary, while for Dirichlet $\psi^i_R = -\eta\psi^i_L$. Without loss of generality we set $\eta = +$. In the D9-brane case all coordinates are Neumann, so at the boundary $\psi^{a\pm}_R = \psi^{a\pm}_L$ for all $a = 1, \cdots, 4$. Hence $\phi^a_R = \phi^a_L$ and $Q = \tilde{Q}$, which is only possible if the two spin fields have the same chirality. This is the case in the type-IIB theory which admits D9-brane boundary conditions, whereas in the type-IIA theory they are inconsistent.[15] Consider next a D8-brane transverse to the direction X^9. Now $\psi^9_R = -\psi^9_L$ and hence $\phi^4_R = -\phi^4_L$ at the boundary. This implies that $Q = \Gamma\Gamma^9\tilde{Q}$ [justify], which is only possible if the two spin fields have opposite chirality. This is a consistent boundary condition for type IIA but not for type IIB.

A straightforward extension of the reasonning shows that type-IIB theory has only Dp-branes with p odd, and type-IIA theory has only Dp-branes with p even, as anticipated in table 2. More explicitly, the type-IIB theory has D1, D3, D5 and D7-branes, as well as D(-1) instantons and space-filling D9-branes. The type-IIA theory has D0, D2, D4 and D6-branes, and 'domain wall' D8-branes.

There actually exist non-supersymmetric Dp-branes with p even in type IIB theory and odd in type IIA theory. These arise from brane-antibrane pairs in one higher dimension. They have a tachyon and can hence decay, for a discussion see [18].

As should be clear from the above discussion, an elementary planar Dp-brane preserves half of the 10d supersymmetries. The corresponding Noether currents (with an implicit Weyl-Majorana spinor index) read

$$(j^0, j^1) = (Q + \Gamma^{(p)}\tilde{Q},\ Q - \Gamma^{(p)}\tilde{Q}) \ , \qquad (2.32)$$

where $\Gamma^{(p)} = \prod_\perp(\Gamma\Gamma^\perp)$ with $\perp$ running over the 9-p tranverse directions [check that this current is indeed conserved in the bulk, $\partial_\alpha j^\alpha = 0$, and that no charge flows out of the boundary, $j^1|_{\rm bnry} = 0$]. The bulk type-II theories have 32 conserved supercharges (counting the extra degeneracy from the light-cone dimensions). A planar Dp-brane conserves half of these and breaks spontaneosuly the other half.

[15] To be precise, as will be discussed later, tadpole cancellation forbids the introduction of isolated D9-branes without anti D9-branes and/or an orientifold.

The spontaneous breaking of (super)symmetries gives rise to massless Goldstone bosons (or Goldstinos), [16] which should arise as massless states of the open string [Why ?] . Indeed, the spectrum of open strings living on a supersymmetric Dp-brane is essentially the same as the right-moving sector of the closed superstring. The GSO projection eliminates the tachyon, leaving a massless 10d vector and a massless Weyl-Majorana spinor plus massive states,

$$\underline{\text{photon}}: \ \psi^j_{-1/2}|0\rangle_{\rm NS} \ , \qquad \underline{\text{photino}}: \ |a\rangle_{\rm R} \ . \tag{2.33}$$

The corresponding massless fields, A_μ and λ_a, only depend on the coordinates of the Dp-brane worldvolume, because the center-of-mass momentum in the transverse directions is zero. The effective low-energy theory of open strings is therefore 10d supersymmetric Maxwell theory reduced to $(p+1)$ dimensions. One can identify the photino with the Goldstini of broken supersymmetry, and the components $A^{p+1,\cdots,9}$ of the gauge field (which are worldvolume scalars) with the Goldstone bosons of broken translation symmetry. The remaining components of the gauge field are actually also Goldstone bosons of a hidden (dual-translation) symmetry as will be clear in the following section. The massless spectrum of the Dp-brane open strings is therefore fixed entirely by symmetries.

<u>Summary</u>: In the Neveu-Schwarz-Ramond formulation, the worldsheet theory of the type-II superstring is N=(1,1) 2d supergravity coupled to matter. The Hilbert space has two sectors with spacetime bosons (NS-NS and R-R) and two with spacetime fermions (NS-R and R-NS). After GSO projection the massless states are those of the two maximal 10d supergravities, called type-IIA and type-IIB. The former is non-chiral, it has 1-form and 3-form R-R gauge fields and admits supersymmetric Dp branes with p even. The latter is a chiral theory with a 0-form, 2-form and self-dual 4-form in the R-R sector, and it admits supersymmetric Dp branes with p odd.

[16] When the symmetry is a gauge symmetry they are eaten by the Brout-Englert-Higgs mechanism.

3. D-branes as solitonic excitations

Although Dirichlet conditions had been discussed before, Polchinski's key insight [19] was to identify D-branes with the non-perturbative excitations predicted by string dualities. Considering D-branes as dynamical solitons ushered in the modern era of string theory. This section, based on chapter 13, vol. 2 of [2] and on the review [7], introduces basic aspects of their dynamics.

3.1 N=4 super Yang-Mills

A celebrated solution of non-abelian Yang-Mills theory is the 't Hooft-Polyakov monopole. Let us focus on a special case, the N=4 supersymmetric theory in four dimensions which will play a central role in what follows. [17] Its Lagrangian is most conveniently described starting with the super YM theory in ten dimensions,

$$S^{\mathrm{YM}}_{(10)} = -\frac{1}{4g^2_{(10)}} \int d^{10}x \, \mathrm{tr}\left(F_{MN}F^{MN} + 2i\bar{\lambda}\not{D}\lambda\right) \, , \tag{3.1}$$

where λ is a Weyl-Majorana fermion in the adjoint representation of the gauge group. The theory is invariant under the supersymmetry transformations $\delta A_M = -i\bar{\epsilon}\Gamma_M\lambda$ and $\delta\lambda = \frac{1}{2}\Gamma_{MN}F^{MN}\epsilon$, with ϵ also Weyl-Majorana.

Exercise 1: Show the invariance of the free action. Hint: Thanks to the Majorana condition, the terms with three gamma matrices combine to a total derivative. To prove the invariance of the non-linear action one needs 10d Fierz identities, see the original reference [21].

Upon reduction to $p+1$ dimensions, the gauge field becomes a vector and $9-p$ scalars, $A^M = (A^\mu, \Phi^{i=1,\cdots,9-p})$, whereas the 16-component Weyl-Majorana gaugino reduces to N gaugini in some representation of SO(9-p). In six dimensions ($p=5$) this is a (pseudoreal) doublet of one SU(2) factor of SO(4), while in four dimensions ($p=3$) there are N=4 Weyl gaugini in the fundamental of SU(4)$\simeq$SO(6). The scalar potential $\propto \mathrm{tr}([\Phi^i,\Phi^j][\Phi^i,\Phi^j])$ vanishes when $\langle\Phi^j\rangle$ are mutually commuting matrices. This is the Coulomb branch of the theory. The gauge symmetry SU(n) breaks to U$(1)^{n-1}$ on the Coulomb branch.

The four-dimensional theory has some special properties. It is a conformal theory with vanishing beta function of the gauge coupling g. Furthermore this latter can be complexified by adding to the action a topological term ,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \, , \qquad S_{\mathrm{top}} = -\frac{i\theta}{32\pi}\int d^4x \, \mathrm{tr}(F_{\mu\nu}\, {}^*F^{\mu\nu}) \tag{3.2}$$

where ${}^*F$ is the dual field strength. S_{top} is proportional to the instanton number, or Pontryagin index of the configuration.

[17] A nice review adapted to the contents of the present lectures is [20].

A convenient rewriting of the gauge-field action is $-\frac{1}{32\pi}\mathrm{Im}[\tau(F^{\mu\nu}+i\tilde{F}^{\mu\nu})(F_{\mu\nu}+i\tilde{F}_{\mu\nu})]$.

Last but not least, it was conjectured by Montonen and Olive that the theory is invariant under the SL(2, $\mathbb{Z}$) transformations

$$\tau \to \frac{a\tau+b}{c\tau+d}\ , \tag{3.3}$$

where a, b, c, d are integers and the determinant $ad - bc = 1$. As will become clear later, this lifts to a symmetry of the full type-IIB string theory in which τ is the vacuum expectation value (vev) of a complex scalar field.

The N=4 SYM arises as the low-energy limit of open superstrings living on a collection of D3-branes. To see why, note that the GSO-projected spectrum of an open superstring with both ends on parallel, identical Dp-branes contains a 10d vector from the NS sector, and a 10d Weyl-Majorana spinor from the R sector. [18] This is the field content of a maximally-supersymmetric vector multiplet. Since strings cannot move in the transverse directions, the corresponding momenta vanish and the theory is effectively reduced to $p+1$ dimensions. As explained at the end of the last section, for a single D-brane this field content is determined by symmetries.

For n >1 identical Dp-branes the theory becomes non-abelian, as seen from the structure of string interactions. Open strings have endpoint (Chan-Paton) charges, so their quantum states must be specified by a matrix-valued wavefunction λ_{ab}, where $a, b = 1, \cdots, n$ label the D-branes on which the endpoints are attached. The matrix λ is hermitean, $\lambda_{ba} = \lambda^*_{ab}$, because oppositely-oriented open strings attached to the same Dp-branes can annihilate into an uncharged closed string that moves away into the bulk. In addition, all classical string interactions come from the disk diagram, so the effective action is single trace, as shown in figure 2. These features, together with gauge invariance and supersymmetry, determine uniquely the effective worldvolume action to be N=4 SYM with gauge group U(n) .

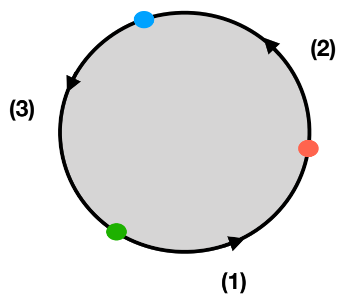


Figure 2: An interaction of open strings with endpoints on three (colored) D-branes. The corresponding interaction vertex is proportional to $\mathrm{tr}(\lambda^{(1)}\lambda^{(2)}\lambda^{(3)})$.

[18] For D9-branes, gaugini must have the same chirality as the type-IIB gravitini. This follows from closure of the vertex-operator algebra, and is necessary for anomaly cancellation in the type I superstring. For lower-dimensional Dp-branes the gaugini are non-chiral.

3.2 Effective actions

We turn now to the effective low-energy action of a single D-brane, in particular its coupling to the bulk supergravity fieds. The latter and the embedding coordinates of the D-brane correspond to massless states of, respectively, closed and open strings, so the low-energy action can be in principle extracted from the relevant string amplitudes. Symmetries constrain however the allowed terms, and reduces the problem to that of extracting the values of some free parameters of the action. This is of course the spirit of effective theories.

As warm up, consider again the N=4 SYM theory of the previous section with gauge group SU(2). The generic point on the Coulomb branch (up to 'R-symmetry' SO(6) rotations and gauge equivalence) is $\langle \Phi^1 \rangle = v\sigma_3/\sqrt{2}$ and all other $\langle \Phi^j \rangle = 0$. Out of the three vector multiplets, one remains massless and the other two obtain a mass gv. At long distance the theory is a supersymmetric Maxwell theory with massive charged particles. It also has 't Hooft-Polyakov solitons, and more generally dyonic particles with electric/magnetic charges and mass given by

$$\begin{pmatrix} Q_e \\ Q_m \end{pmatrix} = \sqrt{\frac{4\pi}{\mathrm{Im}\tau}} \begin{pmatrix} 1 & -\mathrm{Re}\tau \\ 0 & \mathrm{Im}\tau \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \;, \qquad M^2 = v^2(Q_e^2 + Q_m^2) \;, \tag{3.4}$$

where $n_e, n_m \in \mathbb{Z}$ are arbitrary integers.

Exercise 2: Show the Witten effect, i.e. that in the presence of a θ angle a magnetic pole acquires electric charge as in eq. (3.4) Comment on the Dirac quantization condition.

Answer: To see the Witten effect expand the gauge field around the monopole solution, $\vec{E} = \vec{\nabla} A^0$ and $\vec{B} = Q_m \hat{r}/4\pi r^2 + \cdots$, and linearize the Maxwell action (with $g = 1$). If the action includes a theta term, this adds a contribution proportional to

$$\frac{i\theta}{8\pi^2} \int dt d^3r \, \vec{E} \cdot \vec{B} \;\simeq\; -\frac{i\theta}{8\pi^2} \int dt d^3r \, A^0 \vec{\nabla} \cdot \vec{B} \;=\; -\frac{i\theta Q_m}{8\pi^2} \int dt d^3r \, A^0 \delta^{(3)}(\vec{r}) \;.$$

This is as if an electric charge $\sim \theta Q_m$ was sitting at the position of the monopole, $\vec{r} = 0$. For two dyons $Q_e Q_m' - Q_m Q_e' = 4\pi(n_e n_m' - n_m n_e')$, so the theory would still be consistent if electric charges were half-integer. Indeed, fundamental matter, whose charge is half that of adjoint multiplets, could be coupled consistently to the theory (though it would not preserve all supersymmetries).

At long distance scales compared to the soliton core, the monopole looks like a heavy point particle. Its effective action including linear couplins to the bulk massless fields reads [Justify]

$$S_{\mathrm{monopole}} = -\int ds \left[(M + |Q_m| D) \sqrt{-\dot{X}^\mu \dot{X}_\mu} \, + \, Q_m \tilde{A}_\mu \dot{X}^\mu \right] \;, \tag{3.5}$$

where $X^\mu(s)$ is the worldline trajectory, $\tilde{A}_\mu$ is the abelian magnetic gauge field (whose field strength is $\tilde{F}_{\mu\nu}$), and D =tr$(\sigma_3 \delta\Phi^1)/\sqrt{2}$ is a canonically-normalized massless field, the Goldstone boson of broken scale invariance. This field describes fluctuations of the symmetry breaking scale v. From the mass formula (3.4) one sees that long-wavelength fluctuations of D modify indeed the local monopole mass as in (3.5).

Interestingly, two monopoles feel an attractive scalar-force of equal strength as their Coulomb attraction or repulsion. For magentic charges of same sign the two forces cancel and the monopoles equilibrate. This continues to be the case even if their separation is comparable to the size of their core – there indeed exists an exact moduli space of multi-center monopole solutions. The deep reason is unbroken supersymmetry. The moduli space has a non-trivial metric, which geometrizes the fact that slowly-moving monopoles do have velocity-dependent interactions.

A strong piece of evidence for Montonen-Olive duality is Sen's proof that N=4 SYM has a $(n_e, n_m) = (1, 2)$ bound state. This boils down to proving the existence of a unique (anti-self-dual) normalizable form on the (Atiyah-Hitchin) two-monopole moduli space [22][20].

Now that we warmed up to the physics of solitons, let us return to D-branes. We want to think of them a solitons of type-II string theory, and write an effective action that describes their coupling to the supergravity fields, the analogs of the massless fields $\tilde{A}_\mu$ and D in the Coulomb phase of N=4 SYM. We have not actually *shown* that D-branes are solutions of closed-string field theory (a theory still under construction),[19] but we will see that this hypothesis is consistent. As it will turn out, their effective worldvolume actions are determined almost entirely by symmetries, modulo an important parameter that we need to calculate.

We begin with a dynamical relativistic p-brane and set to zero its worldvolume gauge field. The motion of the brane is described by embedding functions $Y^\mu(\zeta^\alpha)$ where $\alpha = 0, 1, \cdots, p$. The lowest-order effective action reads

$$S_{\mathrm{D}p} \;=\; -T_p \int [d\zeta]\, e^{-\Phi} \sqrt{-\det(\hat{\mathrm{G}}_{\alpha\beta})} \;+\; \rho_p \int e^{-\Phi} \hat{C}_{(p+1)} \tag{3.6}$$

where hats denote pullbacks from spacetime to the worldvolume, $\hat{G}_{\alpha\beta} = G_{\mu\nu}\partial_\alpha Y^\mu \partial_\beta Y^\nu$ and likewise for the p-form gauge field $C_{(p)}$. Apart from the coupling to the dilaton, which requires explanation, this effective action is basically fixed by general covariance and invariance under reparametrizations of the worldvolume. The tension and p-form charge (marked in red) are for the moment free parameters.

To understand the dilaton coupling, recall from section 1 that contributions to the effective action from a diagram of Euler characteristic χ are multiplied by $e^{-\chi\Phi}$. The classical D-brane action comes from worldsheets with the topology of the disk, and is therefore multiplied by $e^{-\Phi}$ as in (3.6). The bulk type-II supergravity, on the other hand, is the low-energy theory of closed strings whose classical interactions are given by spherical worldsheets. The relevant part of the bosonic action scales

[19] One *can* connect different D-branes using open string field theory, see e.g. [23][24] for reviews.

therefore with $e^{-2\Phi}$ [recall that $\chi = 2 - 2\#\text{handles} - \#\text{boundaries}$], [20]

$$S_{\text{IIA,B}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-G}\left[e^{-2\Phi}(R + 4\partial_\mu\Phi\partial^\mu\Phi) - \sum_n \frac{1}{2n!}F_{\mu_1\cdots\mu_n}F^{\mu_1\cdots\mu_n}\right] \quad (3.7)$$

with $F_{\mu_1\cdots\mu_n} = \partial_{[\mu_1}(e^{-\Phi}C_{\mu_2\cdots\mu_n]})$. Note that the vev of the dilaton defines the string coupling constant, $e^{\Phi_0} \equiv g_s$. We absorb this in κ_{10}, so that Φ in the above formulae is the dilaton field with vanishing v.e.v. [21]

The fields of the action (3.7) are in the so-called 'string frame', in which both the graviton and the R-R gauge fields mix non-trivially with the dilaton, see e.g. [25]. The kinetic terms are diagonalized in the Einstein frame, $G_{\mu\nu} = e^{\Phi/2}g^E_{\mu\nu}$ and $C_{(p)} = e^{\Phi}C^E_{(p)}$. We drop the superscript 'E' (for Einstein) in what follows. We record the Weyl-rescaling formula in $d > 2$ dimensions

$$\tilde{R} = e^{-2\omega}\left(R - \frac{4(d-1)}{(d-2)}e^{-(d-2)\omega/2}\Box e^{(d-2)\omega/2}\right)$$

where $\tilde{g}_{ij} = e^{2\omega}g_{ij}$. Using it leads to the following Einstein frame actions

$$S_{\text{IIA,B}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\left[(R - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi) - \sum_n \frac{e^{(5-n)\Phi/2}}{2n!}F_{\mu_1\cdots\mu_n}F^{\mu_1\cdots\mu_n}\right] \quad (3.8)$$

$$\text{and} \qquad S_{\text{D}p} \;=\; -T_p\int [d\zeta]\, e^{(p-3)\Phi/4}\sqrt{-\det(\hat{g}_{\alpha\beta})} \,+\, \rho_p\int \hat{C}_{(p+1)}\ . \quad (3.9)$$

We will now use these to compute the static force between two D-branes. Note in passing that D3-branes don't couple to the dilaton. This is why they play a special role in holographic dualities, see later.

Exercise 3: The bosonic action of eleven-dimensional supergravity reads

$$S_{11d} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\left[\sqrt{-G}(R - \frac{1}{48}F^2) - \frac{1}{6}A\wedge F\wedge F\right] \quad (3.10)$$

where A is a 3-form gauge potential and $F = dA$. We have included the important Chern-Simons term for future reference. Show that (3.10) reduces to the type-IIA action in ten dimensions, and find the relation between the radius of the eleventh dimension and the string coupling constant (or the dilaton).

Answer: Let r_{11} be the radius of the eleventh dimension. The Einstein-frame metric is given (up to a multiplicative constant) by $g_{\mu\nu} = r_{11}^{1/4}G_{\mu\nu}$. Computing the coefficient of the

[20] Our convention for curvature is such that the R is positive for a round sphere and negative for anti-de Sitter. Note that we omitted the action of the NS-NS 2-form $B_{\mu\nu}$ which does not couple linearly to D-branes, see later.

[21] Some authors absorb the 10d gravity coupling κ_{10} in the definition of tension and charge. We prefer to keep it explicitly in the formulae as a mnemonic for the string-loop expansion.

Maxwell term and comparing with the corresponding $n=4$ term in (3.8) gives $r_{11}^{3/4} = e^{\Phi/2}$. Restoring finally our convention $\langle\Phi\rangle = 0$ we arrive at

$$\frac{2\pi\langle r_{11}\rangle}{\kappa_{11}^2} = \frac{1}{\kappa_{10}^2} \qquad \text{and} \qquad \frac{r_{11}}{\langle r_{11}\rangle} = e^{2\Phi/3} \ . \tag{3.11}$$

Note that $\langle r_{11}\rangle = g_s^{2/3}$, so the weak-string-coupling limit is the limit of vanishingly-small radius in M theory.

3.3 Static force: Supergravity calculation

To calculate the static force, we treat the Dp-brane as heavy external sources and linearize (at leading order) their coupling to the supergravity fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $C_{(p+1)}$ and Φ. These fields are the carriers of the interaction. For a static planar Dp-brane transverse to the directions $p+1, \cdots, 9$ (denoted collectively by the symbol $\perp$) one finds

$$S_{\mathrm{D}p} \ = \ \int d^{10}x \, (T^{\mu\nu} h_{\mu\nu} + j_\Phi \Phi + j_C C_{01\cdots p}) \tag{3.12}$$

with [verify]

$$T^{\mu\nu} = \frac{1}{2} T_p \, \delta(x^\perp) \times \begin{cases} \eta^{\mu\nu} & \text{for} \quad \mu,\nu = 0,\cdots,p \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

$$j_\Phi = -\frac{p-3}{4} T_p \, \delta(x^\perp) \ , \qquad \text{and} \qquad j_C = \rho_p \, \delta(x^\perp) \ . \tag{3.14}$$

The leading interaction energy between a Dp-brane and a Dp'-brane, due to the exchange of gravitons, dilatons or R-R gauge fields, reads

$$\mathcal{E}_{\rm int} \mathrm{T} = -2\kappa_{10}^2 \int d^{10}x \int d^{10}x' \left[j_\Phi \Delta j'_\Phi - j_C \Delta j'_C + T_{\mu\nu} \Delta^{\mu\nu,\rho\sigma} T'_{\rho\sigma} \right] \ . \tag{3.15}$$

Here T is the total interaction time, and Δ, $\Delta^{\mu\nu,\rho\sigma}$ are the scalar and the graviton propagators evaluated at $x - x'$. Note the sign flip in the contribution of C due to the fact that the exchanged component is timelike [this is why Yukawa and Coulomb forces have opposite signs]. The massless propagators (in spacetime dimension $d > 2$) read, for the scalar

$$\Delta = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik(x-x')}}{k^2} \ , \tag{3.16}$$

and for the graviton in harmonic (or de Donder) gauge

$$\Delta^{\mu\nu,\rho\sigma} = \left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{d-2}\eta^{\mu\nu}\eta^{\rho\sigma} \right) \Delta \ . \tag{3.17}$$

In our case $d = 10$. Consider two identical parallel Dp-branes at positions $\vec{r}$ and $\vec{r}'$ in the transverse space. Inserting in (3.15) gives after some straightforward algebra [check]

$$\mathcal{E}_{\rm int} \;=\; 2V_p\,\kappa_{10}^2\,(\rho_p^2 - T_p^2)\,\Delta_\perp(|\vec{r}-\vec{r}\,'|)\ , \tag{3.18}$$

where V_p is the volume of the Dp-branes (which can be made finite by wrapping them on a p-hypertorus), and

$$\Delta_\perp(r) \;=\; \frac{\Gamma(d_\perp/2-1)}{4\pi^{d_\perp/2}}\ r^{2-d_\perp} \tag{3.19}$$

is the scalar propagator in the $d_\perp = 9-p$ transverse dimensions [Use the Schwinger-time representation of k^{-2} in (3.16) to verify (3.19). Recover by a similar calculation the standard Coulomb law in $d_\perp = 3$] . Notice that the graviton and dilaton exchanges combine to make the prefactor multiplying T_p^2 independent of the value of p.

Exercise 4: Calculate the force between a D-particle and a Dp-brane. Comment on the signs of the graviton and dilaton exchanges, and on the special case $p = 4$.

Answer: The graviton exchange gives $-\frac{1}{16}(7-p)T_pT_0\,\Delta_\perp(r)$ and the dilaton exchange $\frac{3}{16}(p-3)T_pT_0\,\Delta_\perp(r)$. Here $\perp$ stand for the $(9-p)$ dimensions that are transverse to the Dp-brane, and we are working in units $2\kappa_{10}^2 = 1$. Gravity is attractive all the way to $p = 7$, which corresponds to a particle moving in a conical singularity. The dilaton force changes sign at $p = 3$. Note that the D4-D0 system is force-free, this is a consequence of unbroken supersymmetry, see later.

For $p \neq 0$ the branes couple to different C fields, so there is no Maxwell force. The case $p = 6$ is however special, it corresponds to an electric charge in a magnetic-monopole field [a system know to have induced angular momentum]. The D0/D8 system is also special: it is BPS, but the no-force condition can be only understood by taking into account the anomalous effect of string creation, see later.

3.4 Static force: String theory calculation

We now repeat the calculation in string theory and compare. To this end we need to make a small digression into one-loop QFT amplitudes.

Digression: In scalar QFT in d dimensions the one-loop vacuum energy $\mathcal{E}_0$ reads

$$\mathcal{E}_0^{\rm scalar}\,\mathrm{T} = -\frac{1}{2}\log\det(-\partial^2+m^2) = -\frac{1}{2}{\rm tr}\log(-\partial^2+m^2)\ , \tag{3.20}$$

where T is the 'total time.' The trace over all particle states includes the phase-space integral $\int dkdx/h$ for each of the d spacetime dimensions. Thus (in units $\hbar = 1$ and with V the volume of space) we find

$$\mathcal{E}_0^{\rm scalar} = -\frac{V}{2}\int_0^\infty \frac{dt}{t}\int\frac{d^dk}{(2\pi)^d}\,e^{-(k^2+m^2)t} = -V\int_0^\infty\frac{dt}{2t}(4\pi t)^{-d/2}\,e^{-m^2t}\ , \tag{3.21}$$

where we reexpressed the logarithm using Schwinger's proper time.[22] For spinning particles we must multiply the integral by the number of spin states, and for fermions the overall sign should be reversed.

Treating the open string on a Dp-brane as a collection of point particles gives

$$\mathcal{E}_0^{\rm open} = -V_p \int_0^\infty \frac{dt}{2t} (4\pi^2\alpha' t)^{-(p+1)/2}\, {\rm Str}\left(e^{-m^2\pi\alpha' t}\right) , \tag{3.22}$$

where 'Str' stands for the sum over bosonic minus fermionic states of the string. The momentum integrals in the $p+1$ worldvolume dimensions have been already performed, and the dummy variable t was multiplied by $\pi\alpha'$ for later convenience. We also defined $q = e^{-\pi t}$. The supertrace is the sum over GSO-projected Neveu-Schwarz minus Ramond states,

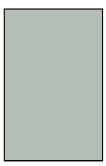
$${\rm Str}\left(q^{m^2\alpha'}\right) \;=\; \sum_{\rm NS} \frac{1}{2}\left(1+(-)^F\right) q^{m^2\alpha'} \;-\; \sum_{\rm R} \frac{1}{2}\left(1\pm(-)^F\right) q^{m^2\alpha'} \tag{3.23}$$

with $(-)^F$ the worldsheet fermion parity. Now recall that $m^2\alpha'$ is the total integer frequency (or 'level' N of the string excitation) minus a ground-state subtraction equal to $-\frac{1}{2}$ in the NS sector and 0 in the R sector. The sum over states then becomes the partition function of eight worldsheet bosons and eight worldsheet fermions with appropriately-quantized frequencies.

Since partition functions for non-interacting fields factorize, we may compute them separately. The bosons are always periodic and contribute a factor $\eta(q)^{-8}$, where we recall the definition of the Dedekind eta function from section 1.6

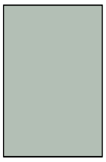
$$\eta(q) \;=\; q^{1/24} \prod_n (1-q^n) \;. \tag{3.24}$$

The fermion partition functions depend on the sector and on the presence or absence of $(-)^F$. The four terms in (3.23) read

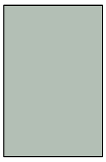
1  $= \; q^{-1/6} \prod_{n=1}^\infty (1+q^{n-1/2})^8 \;\equiv\; (\vartheta_3/\eta)^4 \;,$

NS

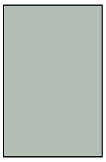
[22] This is a formal expression which is ultraviolet divergent even in the worldline formulation of quantum mechanics ($d=1$). The divergence is absent if we are only interested in the dependence of the integral on the mass m.

$(-)^F$  $= q^{-1/6}\prod_{n=1}^{\infty}(1-q^{n-1/2})^8 \equiv (\vartheta_4/\eta)^4 \; ,$

NS

1  $= 2^4\, q^{1/3}\prod_{n=1}^{\infty}(1+q^{n})^8 \equiv (\vartheta_2/\eta)^4 \; ,$

R

$(-)^F$  $= (1-1)^4\, q^{1/3}\prod_{n=1}^{\infty}(1-q^{n})^8 \equiv (\vartheta_1/\eta)^4 \; ,$

R

We have expressed these partition functions in terms of the Jacobi theta functions $\theta_a(0|q)$. The Ramond sector has equal numbers of states with opposite chirality, and hence its partition function with the insertion of $(-)^F$ vanishes.

Using the Poisson identity $\sum_n g(n) = \sum_n \hat{g}(n)$ where $\hat{g}(x) = \int_{-\infty}^{\infty} g(y) e^{2\pi i x y} dy$ is the Fourrier transform of g, one derives the following modular transformation under $\tau \to -1/\tau$ (see the appendix in Kiritsis [4], recall that $q = e^{2\pi i \tau}$):

$$\mathbf{S}: \quad \frac{\vartheta_2}{\eta} \leftrightarrow \frac{\vartheta_4}{\eta} \; , \quad \frac{\vartheta_3}{\eta} \quad \text{invariant}\, , \quad \frac{\vartheta_1}{\eta} \to e^{-i\pi/2}\frac{\vartheta_1}{\eta} \; , \quad \eta \to \sqrt{-i\tau}\,\eta \; . \tag{3.25}$$

On can also derive easily the transformations under $\tau \to \tau + 1$:

$$\mathbf{T}: \quad \frac{\vartheta_3}{\eta} \to e^{-i\pi/12}\frac{\vartheta_4}{\eta} \; , \quad \frac{\vartheta_4}{\eta} \to e^{-i\pi/12}\frac{\vartheta_3}{\eta} \; , \quad \frac{\vartheta_{2(1)}}{\eta} \to e^{i\pi/6}\frac{\vartheta_{2(1)}}{\eta} \tag{3.26}$$

The transformations $\mathbf{S}$ and $\mathbf{T}$ implement global reparametrizations of a 2d torus with modulus τ (which parametrizes equivalence classes of metrics up to diffeomorphisms and Weyl rescalings). Modular invariance is thus an important condition on closed-string loop diagrams. The annulus is half of an orthogonal torus, and the role of $\mathbf{S}$ is to exchange open- with closed-string channels, see below.

We record for future reference the Jacobi θ functions with non-vanishing second argument [a different, often-used convention replaces $z \to \pi z$ on the r.h.s.]

$$\theta_1(z|q) = 2q^{1/8}\sin z \prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2iz})(1-q^n e^{-2iz}) \; ,$$

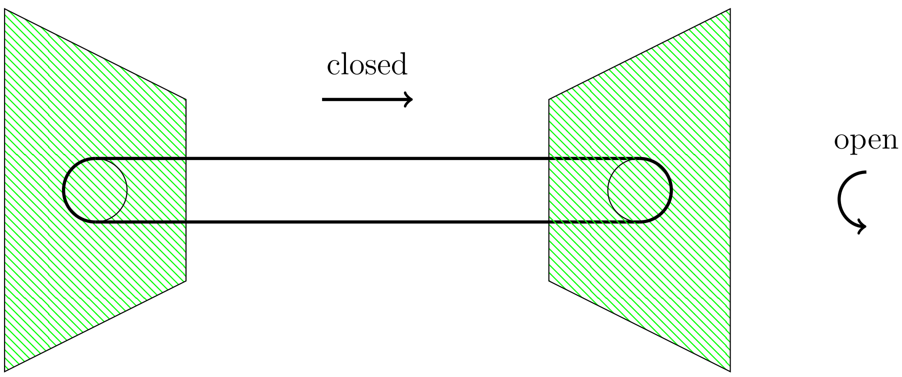
$$\theta_2(z|q) = 2q^{1/8}\cos z \prod_{n=1}^{\infty}(1-q^n)(1+q^n e^{2iz})(1+q^n e^{-2iz}) ,$$
$$\theta_3(z|q) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n+1/2} e^{2iz})(1+q^{n+1/2} e^{-2iz}) ,$$
$$\theta_4(z|q) = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n+1/2} e^{2iz})(1-q^{n+1/2} e^{-2iz}) . \tag{3.27}$$

Collecting pieces we arrive at the following one-loop amplitude of an open string whose endpoints are attached to identical parallel Dp-branes, located at the positions $\vec{r}$ and $\vec{r}'$ in the transverse space:

$$\mathcal{E}_0^{\rm open} = -2\, V_p \times \int_0^\infty \frac{dt}{2t}(4\pi^2\alpha' t)^{-(p+1)/2}\, e^{-|\vec{r}-\vec{r}'|^2 t/4\pi\alpha'}\, Z_{\rm open}(q=e^{-\pi t})$$
$$\text{with} \qquad Z_{\rm open} = \frac{1}{2\eta^8}\left[(\frac{\theta_3}{\eta})^4 - (\frac{\theta_4}{\eta})^4 - (\frac{\theta_2}{\eta})^4 \pm (\frac{\theta_1}{\eta})^4\right] . \tag{3.28}$$

Marked in red is the tensile contribution to the mass squared due to the stretching of the open string, $m^2\alpha' = |\vec{r}-\vec{r}'|^2/4\pi^2\alpha'$, and an important factor of 2 accounting for the fact that strings are oriented objects.

NB: Compare with the calculation of the force between two plates in QED. If the plates are charged they are repelled by the Coulomb force. If they are conducting, they alter the fluctuations of the photon field and thus feel a Casimir attraction. The Coulomb force is a classical force, while the Casimir force is quantum. For D-branes these two forces coexist, and they are given by the same one-loop diagram.
This is because the UV regime of fundamental open strings (which include 'photons') is the IR regime of fundamental closed strings (including the graviton) as illustrated in the figure below.



For distant D-branes ($|\vec{r}-\vec{r}'| \gg \sqrt{\alpha'}$) the stretched open string has a large number of extremely soft excitations. The cumulative Casimir force of these soft open-string modes gives the classical supergravity force between D-branes.

3.5 Ramond-Ramond charges

The first thing to notice about the expression (3.28) is that it vanishes by Jacobi's abstruse identity $\theta_3^4 - \theta_4^4 - \theta_2^4 = 0$. [Verify the first few levels]. We conclude that the force between identical D-branes vanishes to <u>all orders</u> in the α' expansion, i.e. even when the D-branes approach to substringy distance (this is analogous to the existence of an exact multimonopole moduli space in N=4 SYM, even for separations $\ll m_{\rm W}$). The existence of a moduli space is a consequence of unbroken supersymmetry. The abstruse identity shows that at each excitation level N there is an equal number of bosonic and fermionic string states whose Casimir energies cancel.

Comparing with (3.18) we find that $\rho_p = T_p$, i.e. the 'charge' of the Dp-brane with respect to $C_{(p+1)}$ is equal to its tension. So tension is charge, but how to compute its value? What makes this possible is the channel duality of fundamental-string amplitudes. We can think of the open-string loop as the classical exchange of a closed string, and try to separate the NS-NS from the R-R contributions. To do this, we must determine the periodicity of worldsheet fermions around the circle in each of the four terms in eq. (3.28).

In ordinary QFT one computes the thermal partition function either as a sum over states with Boltzmann weights, or as a path integral with periodic boundary conditions in Euclidean time. Fermions are a priori antiperiodic, [this implements Fermi statistics and is the reason why finite-temperature breaks supersymmetry]. But they become periodic if we attach $(-)^F$ to the Boltzmann weights. The closed-string R-R exchange is therefore given by the second and fourth terms of (3.28).

On sees this in ordinary QM with a complex anticommuting coordinate $\psi(t)$. Canonical quantization implies $\{\psi, \psi^\dagger\} = 1$, and algebra realized on two states, $\psi^\dagger|0\rangle = |1\rangle$, $\psi|1\rangle = |0\rangle$ and $\psi^\dagger|1\rangle = \psi|0\rangle = 0$. Define the 'coherent' bra and ket states

$$|\theta\rangle \equiv e^{\psi^\dagger\theta}|0\rangle = |0\rangle - \theta|1\rangle \ , \qquad \langle\bar\theta| \equiv \langle 0|e^{\bar\theta\psi} = \langle 0| - \langle 1|\bar\theta \ ,$$

which are eigenstates of the fermion operators with anticommuting eigenvalues,

$$\psi|\theta\rangle = \theta|\theta\rangle \ , \qquad \langle\bar\theta|\psi^\dagger = \langle\bar\theta|\bar\theta \ .$$

They are the analogs of position eigenstates. From the rules of Grassman integration one finds $\int d\bar\theta d\theta\, e^{-\bar\theta\theta}\, |\theta\rangle\langle\bar\theta| \ = \ \mathbf{1}$, and for any 2×2-matrix operator A in the space $\{|0\rangle, |1\rangle\}$

$$\mathrm{tr}(A) = \int d\bar\theta d\theta\, e^{-\bar\theta\theta}\, \langle -\bar\theta|A|\theta\rangle \ ,$$

$$\mathrm{tr}\left((-)^F A\right) = \int d\bar\theta d\theta\, e^{-\bar\theta\theta}\, \langle\bar\theta|A|\theta\rangle \ .$$

Using these identities in the derivation of the Feynman-Kac formula shows that fermions must be antiperiodic for $\mathrm{tr}(e^{-\beta H})$, and periodic for $\mathrm{tr}((-)^F e^{-\beta H})$.

We are now ready to extract the $(p+1)$-form exchange from the $|\vec{r}-\vec{r}'| \to \infty$ limit of the second term in (3.28) (the fourth as we saw vanishes). The integral is dominated by the $t \to 0$ integration region where [see (3.25) with $\tau = it/2$]

$$\frac{{\theta_4}^4}{\eta^{12}}\bigg|_{q=e^{-\pi t}} = \left(\frac{t}{2}\right)^6 \frac{{\theta_2}^4}{\eta^{12}}\bigg|_{q=e^{-4\pi/t}} = \left(\frac{t}{2}\right)^6 2^4 \ + O(e^{-\pi t/2}) \ .$$

Inserting in (3.28) gives the RR potential energy

$$\begin{aligned}\mathcal{E}_0^{\rm RR} &= V_p \times \int_0^\infty \frac{dt}{2t} (4\pi^2\alpha' t)^{-(p+1)/2}\, e^{-|\vec{r}-\vec{r}'|^2 t/4\pi\alpha'} \times \frac{t^6}{4} \\ &= V_p \times 2\pi (4\pi^2\alpha')^{3-p}\, \Delta_\perp(|\vec{r}-\vec{r}\,'|) \ . \end{aligned} \qquad (3.29)$$

Comparing with the supergravity result (3.18) we arrive at

$$\boxed{T_p^2 = \rho_p^2 = \frac{\pi}{\kappa_{10}^2}\,(4\pi^2\alpha')^{3-p} \ .} \qquad (3.30)$$

The importance of this formula appears when verifying Dirac's quantization condition. We have seen that $C_{(p+1)}$ and $C_{(7-p)}$ are not independent gauge fields, since their field strengths are related by Hodge duality in 10d. Thus Dp-branes and D$(6-p)$-branes are like electric and magnetic charges for the field $C_{(p+1)}$, i.e. they source its Maxwell equation and Bianchi identity [equivalently the Bianchi identity and Maxwell equation for the dual field $C_{(p+1)}$]. Consider for concreteness a D6-brane which sources the Bianchi identity of the 1-form field C_μ. The field created by the D6-brane, ${}^*F_{(8)} = F_{(2)}$, is the area form of the surrounding 2-sphere. In the standard polar coordinates of $\mathbb{R}^3$ it reads [23]

$$F_{(2)} = 2\kappa_{10}^2\, \frac{\rho_6}{4\pi}\, d(1-\cos\vartheta) d\varphi \ . \qquad (3.31)$$

The Dirac (Nepomechie-Teitelboim) condition is the condition that the string singularity of the potential at the south pole, $\vartheta = \pi$, cannot be observed by the phase of an electric charge transported around it,

$$2\kappa_{10}^2\, \rho_6 \rho_0 \ \in \ 2\pi\,\mathbb{Z} \ . \qquad (3.32)$$

Inserting (3.30) we see that, not only is this condition indeed obeyed, but furthermore D-branes obey it minimally. Contrary to N=4 SYM, which could be coupled consistently to ('charge-$\frac{1}{2}$') matter in the fundamental representation of U(n), string theory does not admit more elementary R-R charges than the D-branes.

[23] The magnetic-monopole field expressed as a 1-form gauge potential with canonical Maxwell action is $C = (Q_m/4\pi)\, d(1-\cos\theta) d\varphi$, where Q_m is the monopole charge. In our case this must be multiplied by $2\kappa_{10}^2$ because of the normalization (3.8) of the Maxwell action.

Question: Interpret this condition for $p = -1$. Answer: It quantizes the discontinuity of the RR scalar $C_{(0)}$ when transported around a D7-brane. This ensures that the (Euclidean) action of the D-instanton, $S_{\rm inst} = T_{(-1)} e^{-\Phi} + i\rho_{(-1)} C_{(0)}$, is well defined in the presence of the D7-brane.

One final remark on (3.30): The tension $T_p = \rho_p$ scales with the string coupling as $1/g_s$. In perturbation theory D-branes are therefore much lighter than the solitonic excitations of supergravity whose tension scales like $1/g_s^2$. They are, however, heavy compared either to the mass of fundamental strings or to interaction energies at D-brane separation larger than string scale. Both of these are $O(1)$ in g_s.

Exercise 5: Compute the annulus diagram for the D$(p+4)$-Dp system. Interpret the result both in the open- and in the closed-string channel.

Answer: In lightcone gauge there are four NN or DD (super)coordinates (X^j, ψ^j) and four DN coordinates (X^A, ψ^A). In the NS sector the ψ^j have 1/2-integer modes and the ψ^A integer modes, while in the R sector their roles are exchanged. Thus $|0\rangle_{\rm NS}$ is a massless spacetime scalar in the $(\frac{1}{2}, 0)$ representation of SU(2)×SU(2)≃SO(4), and $|0\rangle_{\rm R}$ is a SO(1,5) spinor reduced to $p+1$ dimensions. For $p = 3$ this is the content of a N=2 hypermultiplet. The open-string partition function replacing (3.28) reads

$$Z_{\rm open} = Z_{\rm F} \times Z_{\rm B} = \frac{1}{2}\left[(\frac{\theta_3}{\eta})^2 (\frac{\theta_2}{\eta})^2 - (\frac{\theta_2}{\eta})^2 (\frac{\theta_3}{\eta})^2 \right] \times \frac{1}{\eta^4} (\frac{\eta}{\theta_4})^2 \; = \; 0 \; . \tag{3.33}$$

We wrote it as a product of fermionic and bosonic partition functions which the reader is invited to verify. The contributions with insertion of $(-)^F$ vanish individually, so there is no RR exchange in the closed-string channel, as expected. The calculation confirms the dilaton coupling of eq. (3.9).

Exercise 6: Compute the force between a Dp-brane and anti-Dp-brane.

Answer: The anti-D-brane has the same tension but opposite RR charge, so the two terms in the supergravity result (3.18) have the same sign and the force is attractive.[24] In string theory this corresponds to flipping the sign of the GSO projection in the open channel. At $|\vec{r} - \vec{r}'| = \pi\sqrt{2\alpha'}$ a tachyon sets in and the pair annihilates. .

[24] In QFT charge-conjugate probes always attract as a consequence of reflection positivity [26]. For an extension to classical GR see [27].

Summary: Dp-branes can be consistently interpreted as solitonic excitations of string theory, endowed with proper dynamics. They are characterized by a tension (T_p) and a charge (ρ_p) sourcing the RR (p+1)-form field. By computing the cylinder diagram [Casimir energy for open strings] we showed that $T_p = \rho_p$, and that the charges obey the minimal Dirac quantization condition.

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