Lecture one

The fact that classical symmetries in general donot survive to quantization is now a well-established fact. Quantum theory to be powerful and consistent needs an high degree of symmetry but because Nature is asymmetric we have to account for this symmetry breaking

- Oldest and most primitive idea ⇒ approximate symmetry: Lagrangian has terms breaking a symmetry but they are small
- Spontaneous symmetry breaking → Heisenberg in condensed matter systems (1939)
  → Goldstone and Nambu in particle physics (60's)

  Dynamical equations are completely symmetric even at quantum level but the ground state is asymmetric ⇒ low-energy physics involving mesons, electroweak theory, strong interaction

- Third and more subtle mechanism ⇒ anomalous breaking of symmetries

  No energetic consideration ⇒ they appear at perturbative level because there is no regularization procedure respecting the anomalous symmetry

We will study chiral anomalies arising when chiral fermions (or axions) interacts with gauge and gravitational fields.

Anomalies seems not to be just an accident of perturbation theory ⇒ they are related to deep mathematical structures and can be sometimes determined without refering to perturbation theory

Symmetry at classical level => conserved currents

$\partial_\mu J^\mu_\alpha = 0$ => Ward identity at quantum level

anomalies $\partial_\mu J^\mu_\alpha = \underset{\substack{\uparrow \\ \text{quantum origin}}}{a_\alpha}$ => Anomalous Ward identity

Importantly : the terms destroying the classical symmetry, coming out from the regularization, are always finite => no infinite
=> there is a certain degree of ambiguity in the final form
<=> this correspond to the freedom on the local term in the renormalization procedure.

Effects and impact of anomalies on a theory

1) The anomaly affects a current J^μ_α coupled to gauge or gravitational fields => if it cannot be removed by local terms is a disaster
=> renormalizability and/or unitarity are destroyed

2) The anomaly is related to classically conserved Noether currents j^μ_α not coupled to dynamical fields => to be regulated preserving the dynamical gauge invariance but their non-conservation is not a problem => classical selection ~~rules~~ rules are changed

Let us ~~exemplify~~ discuss a simple two-dimensional example showing that anomalies must be present and are not related to perturbative expansion

D=2
$$\left.\begin{aligned} \gamma^0 &= \sigma^1 \\ \gamma^1 &= i\sigma^2 \\ \gamma_5 &= -i\sigma_3 \end{aligned}\right\} \Rightarrow \quad \varepsilon^{\mu\nu}\gamma_\nu = i\gamma^\mu\gamma_5 \qquad \varepsilon^{01} = 1 = -\varepsilon_{01}$$

$\sigma_{1,2,3}$ Pauli matrices

Then axial vector current $J^\mu_5 = \bar\psi i\gamma^\mu\gamma_5\psi$ is <u>dual</u> to vector current $J^\mu = \bar\psi\gamma^\mu\psi$ $\quad J^\mu_5 = \varepsilon^{\mu\nu}J_\nu$

⇒ dynamical non-trivial theory both current cannot be conserved (remember that in massless QED they do at classical level)

Consider vacuum correlation function

$$\langle J^\mu(x) J^\nu(y) \overset{\text{Poincaré}}{=} g^{\mu\nu}\Pi_1(x-y) - \frac{\partial^\mu\partial^\nu}{\Box}\Pi_2(x-y) + \left(\frac{\partial^\mu\varepsilon^{\nu\alpha}\partial_\alpha}{\Box} + \frac{\partial^\nu\varepsilon^{\mu\alpha}\partial_\alpha}{\Box}\right)\Pi_3(x-y)$$

$$\Rightarrow \quad \langle J^\mu_5(x) J^\nu(y)\rangle = \varepsilon^{\mu\nu}\Pi_1(x-y) - \varepsilon^{\mu\alpha}\frac{\partial_\alpha\partial^\nu}{\Box}\Pi_2(x-y) - \left(g^{\mu\nu} - 2\frac{\partial^\mu\partial^\nu}{\Box}\right)\Pi_3(x-y)$$

Vector current conservation ⇒ $\Pi_1 = \Pi_2 \qquad \Pi_3 = 0$

axial vector current conservation ⇒ $\Pi_1 = 0$ ⇒ trivial theory

Plan of the lectures

Lecture 1: Introduction. Anomaly from path integral measure. Abelian anomaly and relation with index theorem. Non-abelian anomalies (introduction)

Lecture 2: Gauge anomalies from Feynman graphs. Properties of locality and finiteness. Anomaly cancellation and relations with matter representations. Anomaly cancellation in Standard model

Lecture 3 Wess-Zumino consistency condition => algebraic approach to anomalies. BRST cohomology and solutions for gauge anomalies from descent equation. Anomalies and topology: relations with index theorems in the non-abelian case.

Lecture 4 Gravitational anomalies: Lorentz anomalies and Einstein anomalies. Mixed gauge-gravitational anomalies. Examples in higher dimensional theory: IIB supergravity and its anomaly cancellation. Green-Schwarz mechanism and anomaly cancellation in low energy type I and heterotic strings.

We start by considering a gauge theory involving fermions in some given representation R of the gauge group G

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \mathcal{L}_{\text{matter}}(\bar\psi, \psi; D_\mu\psi, D_\mu\bar\psi)$$

where the generators of Lie algebra of G

$$\begin{cases} [T_a, T_b] = i f_{ab}^{\ \ c} T_c \\ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{\ bc} A^b_\mu A^c_\nu \end{cases}$$

Gauge transformations: (infinitesimal)

$$\begin{cases} \delta A^a_\mu = \partial_\mu \epsilon^a + f^a_{\ bc} A^b_\mu \epsilon^c \\ \delta A^R_\mu = \partial_\mu \epsilon^R - i[A^R_\mu, \epsilon^R] \\ \qquad = D_\mu \epsilon^R \end{cases}$$

or in matrix notation

$$\begin{cases} \epsilon^R = \epsilon^a T^R_a \\ A^R_\mu = A^a_\mu T^R_a \end{cases}$$

$$\begin{cases} \delta\psi^i = i \epsilon^a (T^R_a)^i_{\ j} \psi^j \Rightarrow \delta\psi = i\epsilon^R \psi \\ \delta\psi^\dagger = -i \psi^\dagger \epsilon^{\bar R} \end{cases}$$

($\bar R$ is the complex conjugate representation $R = \bar R$ for hermitian)

Covariant derivatives

$$\begin{cases} (D_\mu\psi)^i = \partial_\mu\psi^i - i (A^R_\mu)^i_{\ j} \psi^j \\ D_\mu\psi = \partial_\mu\psi - i A^R_\mu \psi \end{cases}$$

- $[D_\mu, D_\nu]\psi = -i F^R_{\mu\nu} \psi \qquad F^R_{\mu\nu} = F^a_{\mu\nu} T^R_a$
- Bianchi identity $D_{[\mu} F_{\nu\rho]} = 0$

Matter current ⇒ $J^{a\mu}_{matter} = \frac{\delta \mathcal{L}_{matter}}{\delta A_\mu^a}$

Field equations $(D_\mu F^{\mu\nu})^a = J^{a\nu}_{matter}$ ⟶

⇒ $D_\mu J^{\nu a}_{matter} = 0$

Let us now study the simplest example of anomaly: the so called ABELIAN ANOMALY. It is the anomaly affecting chiral (involving γ_5) transformations of massless Dirac fermions

⇒ Historically the first example of anomaly discovered: it explains the decay rate $\pi_0 \to 2\gamma$!!

Matter Lagrang.:

$$\boxed{\begin{aligned}\mathcal{L}_{matt} = -\bar\psi \not{D}\psi &= -\bar\psi(\not{\partial} - i\not{A}^R)\psi \\ &= -\bar\psi\gamma^\mu(\partial_\mu - iA_\mu^R)\psi\end{aligned}}$$

spin $\frac{1}{2}$ massless in representation R

Let us consider the path integral over the matter field:

$$\int D\bar\psi D\psi\, \underbrace{O_1(x_1)\dots O_n(x_n)}_{\text{operators}}\, e^{i\int d^4x\, \mathcal{L}_{matter}}$$

⇒ anomalies appear in this step! We can consider later the integration over gauge fields

Now are the relevant operators for S-matrix are <u>correlations of currents</u> that are generated by the "effective action", taking functional derivatives with respect to A_μ

$$\left\{ \begin{array}{l} e^{iW[A]} = \int D\psi D\bar\psi \; e^{iS_{matter}} \\ J^{a\mu}_{matter} = \dfrac{\delta \mathcal{L}_{matter}}{\delta A_\mu^a} \end{array} \right.$$

Let us consider the classical invariance of our matter lagrangian

$$U = \exp i\epsilon T\gamma_5 \qquad T = T^\dagger \qquad [T, T^a] = 0$$
$$[T, \gamma_5] = 0$$

$\Rightarrow$ T belongs to an <u>abelian subgroup of G</u>

$$\begin{array}{l} \psi \to U\psi \\ \bar\psi \to \bar\psi U \end{array}$$ is a symmetry $\Rightarrow$ there is an associate <u>conserved current</u>

<u>Exercise</u>

$$\boxed{\partial_\mu J^\mu_5 = 0 \qquad J^\mu_5 = i\bar\psi\gamma^\mu\gamma_5 T\psi}$$

<u>Quantum theory</u> $\Rightarrow$ The quantum translation of the conservation current is

$$\langle \partial_\mu J^\mu_5(x) \rangle_A = 0$$

The simplest way to check this relation is to promote ϵ to be a space-time dependent variable $\epsilon \to \epsilon(x)$

and then make the change of variables in the path-integral

$$\begin{cases}\psi' = U(x)\psi \\ \bar\psi' = \bar\psi\, U(x)\end{cases} \Rightarrow$$

$$\int D\psi D\bar\psi\, e^{iS_{mat}(\bar\psi,\psi,A)} = \int D\psi' D\bar\psi'\, e^{iS_{mat}[\bar\psi',\psi',A]}$$

$$= \int D\psi D\bar\psi\, J[\epsilon(x)]\, e^{iS_{mat} + i\int d^4x\, \epsilon(x)\partial_\mu J^\mu_5}$$

where $J(\epsilon(x))$ is the Jacobian of the transformation and the second term comes from the variation of the Lagrangian

Remark: The Jacobian is a feature of quantum theory: it comes from the path-integral integration. There is nothing similar at classical level

Taking $\epsilon(x)$ infinitesimal and writing $\boxed{J(\epsilon(x)) = \exp i\int d^4x\, \epsilon(x)\, a(x)}$

we get from the fact that we have done just a change of variables
(at first order in $\epsilon(x)$)

$$\int D\psi D\bar\psi\, e^{iS_{mat}(\psi,\bar\psi,A)}\, [\, i\int d^4x\, \epsilon(x)\left(a(x) + \partial_\mu J^\mu(x)\right) = 0$$

$$\Rightarrow \boxed{\partial_\mu \langle J^\mu_5(x)\rangle_A = -\langle a(x)\rangle_A}$$

If $J(\epsilon(x)) \neq 1$ the classical conservation law is violated $\Rightarrow$ the symmetry is broken at quantum level

$J(\epsilon(x))$ comes from the non-invariance of the fermionic measure!!

Let us see if $J(\epsilon(x))$ is non-trivial and, in this case, its explicit expression in $D=4$. We have to study the behavior of the fermionic measure under

$$\begin{cases}\psi \to \psi' = U(x)\psi \\ \bar\psi \to \bar\psi' = \bar\psi\,\bar U(x)\end{cases} \qquad \bar U = (i\gamma^0)U^\dagger(i\gamma^0)$$

$$\boxed{\begin{aligned} D\psi &\to D\psi' = (\det \mathcal{U})^{-1} D\psi \\ D\bar\psi &\to D\bar\psi' = (\det \bar{\mathcal{U}})^{-1} D\bar\psi \end{aligned}} \qquad \langle x|\mathcal{U}|y\rangle = U(x)\,\delta^4(x-y)$$

$\Rightarrow$ in general $D\bar\psi' D\psi' = D\bar\psi D\psi\, J \qquad J = (\det \mathcal{U})^{-1}(\det \bar{\mathcal{U}})^{-1}$

Two main examples

1) Unitary non-chiral transformation $\qquad U(x) = e^{i\epsilon^a(x)T_a} \Rightarrow \bar U(x) = U^{-1}(x) \qquad T_a = T_a^\dagger$

$\Rightarrow$ this corresponds to usual local gauge invariance! $\qquad [T_a, \gamma_\mu] = 0$

We expect in this case $J=1$ but we have to be careful in order to compute $\det \mathcal{U}(x)$ and $\det \mathcal{U}(x)^{-1}$ $\Rightarrow$ REGULARIZATION PROCEDURE

If it preserves gauge invariance we are ok!

2) Unitary chiral transformation $\qquad U(x) = e^{i\epsilon^a(x)T_a\gamma_5}$

$T_a = T_a^\dagger$

$[T_a, \gamma_\mu] = 0$

$\bar U(x) = U(x) \qquad$ (γ_5 is part)

Now $\bar{\mathcal{U}} = \mathcal{U}$ $\qquad \bar{5}$ ''

$$\Rightarrow \quad D\bar{\psi}' D\psi' = D\bar{\psi} D\psi \left(\det \mathcal{U}\right)^{-2}$$

In general we <u>not</u> expect it is one unless $\det \mathcal{U} = 1$!!

Let us try to compute $\left(\det \mathcal{U}\right)^{-2}$

functional trace and matrix trace

$$\boxed{\left(\det \mathcal{U}\right)^{-2} = e^{-2\,\mathrm{Tr}\log \mathcal{U}} = e^{i\int d^4x\, \varepsilon^a a_a(x)}}$$

~~[illegible]~~

$$\mathrm{Tr}\log\mathcal{U} = \int d^4x \,\langle x|\,\underset{\text{matrix trace}}{\mathrm{tr}}\log\mathcal{U}\,|x\rangle = \int d^4x\, \delta^4(x-x)\,\mathrm{tr}\log\mathcal{U}(x)$$

$\langle x|f(U)|y\rangle = f(U(x))\langle x|y\rangle$

$$= \boxed{\int d^4x\, \delta^4(0)\; i\epsilon^a\, \mathrm{tr}(T_a \gamma_5)}$$

Therefore $\qquad \boxed{a_a(x) = -2\,\delta^4(0)\,\mathrm{tr}\,T_a\gamma_5}$

Ill defined result : it needs <u>regularization</u> $\qquad \delta^4(0) \to \infty$ $\qquad$ <u>Who will win!</u>

$\qquad \mathrm{Tr}[\gamma_5 T_a] \to 0$

<u>Remark</u> : it is a ultraviolet divergence $\quad \delta^4(0) = \lim\limits_{x\to y} \int d^4p\, e^{ip(x-y)}$

$\Rightarrow$ regularization

<u>Idea</u> : <u>to cut</u> the large-momenta contribution

$$\int d^4x\, \varepsilon^a q_a(x) = -2 \lim_{\Lambda\to\infty} \int d^4x\, \mathrm{tr}\left[\langle x|\, \varepsilon^a(x)\, \gamma_5 T_a\, f\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right) |x\rangle\right]$$

where f is a smooth function satisfying

$$\begin{cases} f(0)=1 \\ f(\infty)=0 \\ s f'(s) = 0 \text{ at } \begin{matrix} s=0 \\ s=\infty \end{matrix} \end{cases}$$

Going to momentum-space (exercise) to show

$$\int d^4x\, \varepsilon^a(x) a_a(x) = -2 \lim_{\Lambda\to\infty} \int d^4x\, \varepsilon^a(x) \Lambda^4 \int \frac{d^4q}{(2\pi)^4}\, \mathrm{tr}\left(\gamma_5 T_a f\left(-\left[-iq + \frac{\not{D}}{\Lambda}\right]^2\right)\right)$$

To evaluate the limit expand for large Λ

$$f\left(q^2 + 2iq^\mu D_\mu/\Lambda - \frac{\not{D}^2}{\Lambda^2}\right) \simeq f(q^2) + \dots$$

<u>Important</u>: we have to trace γ_5 with the expansion and take the $\lim \Lambda\to\infty$

$\Rightarrow$ we need at least 4 γ matrices for the trace but at most Λ^{-4} from the expansion $\Rightarrow$ 1 term!!

$$\int d^4x\, \varepsilon^a(x) \int \frac{d^4q}{(2\pi)^4}\, \frac{1}{2} f''(q^2)\, \mathrm{tr}\left[\gamma_5 T_a \not{D}^2\right]$$

<u>Now</u>:

$$\boxed{\int \frac{d^4q}{(2\pi)^4} \frac{1}{2} f''(q^2) \underset{\text{Wick rotation}}{=} \frac{i}{32\pi^2}}$$

independent of f!!

and

$$\mathrm{tr}\left(\gamma_5 T_a (-\not{D}^2)^2\right)$$

$$= \left(\frac{i}{4}\right)^2 \mathrm{tr}_D\left(\gamma_5 [\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\lambda]\right) \mathrm{tr}_R T_a F_{\mu\nu} F_{\rho\lambda}$$

(because $\not{D}^2 = D_\mu D^\mu - \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}$)

$$= -i\,\varepsilon^{\mu\nu\rho\lambda}\,\mathrm{tr}_R T_a F_{\mu\nu} F_{\rho\lambda}$$

Therefore we get a <u>finite</u> Jacobian where the <u>anomaly</u> $a(x)$ appears to be

$$\boxed{a_a(x) = -\frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\lambda}\,\mathrm{tr}_R\left(T_a F_{\mu\nu} F_{\rho\lambda}\right)}$$

<u>Remarks</u> – ~~[illegible]~~ Normalizing canonically the kinetic term for A_μ, it is of order g^3 and from a ~~[illegible]~~ loop counting is one-loop (it is a fermion determinant result!)
$\Rightarrow$ it comes from a 3-point function at one loop!

Specializing to our case, ϵ^a being a abelian subgroup of G defined by $T^a = T$

$$\boxed{\partial_\mu \langle J_5^\mu(x)\rangle_A = \frac{1}{16\pi^2}\,\varepsilon^{\mu\nu\rho\lambda}\langle \mathrm{tr}_R T F_{\mu\nu}(x) F_{\rho\lambda}(x)\rangle_A}$$

Remark : we obtained a gauge-invariant result in the sense that

- $\alpha(x)$ is gauge invariant
- non chiral transformations have trivial Jacobian (check!)

We have used $f(\cdot(i\frac{\not D}{\Lambda})^2)$ that transforms covariantly under gauge
=> ensures gauge invariance. The use of this regulator in all fermionic determinant of the theory forces gauge invariance

In particular it ensures the gauge invariance of $W[A]$

$$e^{iW[A]} = \int D\bar\psi D\psi \; e^{i\int d^4x\, \bar\psi \not D \psi}$$

Expand the fermionic measure in (formally at the moment)

$$\psi = \sum_n c_n \varphi_n(x) \qquad \bar\psi = \sum_n \bar c_n \varphi_n^\dagger(x) \qquad \not D \varphi_n = \lambda_n \varphi_n$$

$$e^{iW[A]} = \prod_n \int d\bar c_n\, dc_n \; e^{i\lambda_n \bar c_n c_n} \simeq \prod_n \lambda_n = \det \not D$$

The product of λ_n is of course infinite but with our regulator

$$\prod_n \lambda_n f\left(\frac{\lambda_n^2}{\Lambda^2}\right)$$ it converges and importantly =>

IT IS GAUGE INVARIANT because λ_n are gauge invariant

=> gauge invariant regularization forces A_μ dependence in the Jacobian

Let us now make more precise the invariance of effective action under chiral transformations: introduce sources for fermion fields

$$\int d^4x \, (\bar\psi \eta + \bar\eta \psi)$$

compute $W[\eta, \bar\eta, A_\mu]$ and make the Legendre transf. to go to

$$\Gamma[\psi_0, \bar\psi_0; A_\mu] \qquad (\psi_0, \bar\psi_0 \text{ "classical fields"})$$

⇒ for ~~[illegible]~~ linearly realized symmetry without anomaly and constant ϵ one has that

$$\delta_\epsilon \Gamma(\psi_0, \bar\psi_0, A) = 0 \qquad (\text{ST identity for linearly realized symmetry})$$

$$\swarrow$$

This is proved by invariance of functional measure

if not we have (from our computation) for abelian chiral transformation

$$\boxed{\delta_\epsilon \Gamma(\psi_0, \bar\psi_0, A_\mu) = -\frac{\epsilon}{16\pi^2}\int d^4x \, \varepsilon^{\mu\nu\lambda\rho}\, \mathrm{Tr}_R[T F_{\mu\nu} F_{\lambda\rho}]} \rightarrow$$

We want to show now that this result is related to a very deep theorem of algebraic geometry: if you want it gives a physicist's proof of index theorem of Dirac operator.

Our result in fact smells of deep mathematics: in fact

$$\int d^4x\, \varepsilon^{\mu\nu\rho\lambda} F^a_{\mu\nu} F^a_{\rho\lambda} = \frac{64\pi^2}{g^2}\nu \qquad \nu\in\mathbb{Z}$$

$$\Rightarrow \int d^4x\, \varepsilon^{\mu\nu\rho\lambda}\, \mathrm{tr}_R F_{\mu\nu} F_{\rho\lambda} = 64\pi^2 C_R \nu$$

C_R is the casimir of R

$\mathrm{tr}_R T_a T_b = C_R \delta_{ab}$

The integer ν is called instanton number and it does not change under smooth deformation of the gauge field (the integrand is a total derivative locally and the integral just depend on the boundary condition)

Let us go in Euclidean space $\rightarrow$ $i\not{D}_E$ is the euclidean Dirac operator

γ^μ hermitean

λ_n (eigenvalue) real

$[i\not{D}_E, T]=0$

Suppose to have discrete spectrum (compactification) and take the eigenvalue problem

$$i\not{D}_E \varphi_n = \lambda_n \varphi_n$$
$$T\varphi_n = t_n \varphi_k$$

φ_n orthonormal and complete

$\mathbb{1} = \sum_n |\varphi_n\rangle\langle\varphi_n|$

$\mathrm{Tr} A = \sum_n \langle\varphi_n|A|\varphi_n\rangle$

Now we have $\gamma_5 \not{D}_E = -\gamma_5 \not{D}_E$

$$[T,\gamma_5]=0$$

$\Rightarrow$ $\gamma_5\varphi_n$ is again eigenfunction of $i\not{D}_E$ and T with eigenvalue $-\lambda_n, t_n$

$$\boxed{i\not{D}_E(\gamma_5\varphi_n) = -\gamma_5(i\not{D}_E)\varphi_n = -\lambda_n(\gamma_5\varphi_n)}$$

For $\lambda_n \neq 0 \Rightarrow$ φ_n and $\gamma_5\varphi_n$ are orthogonal

Notice that we can construct $\varphi_{n\pm} = \left(\frac{1\pm\gamma_5}{2}\right)\varphi_n$ that are eigenfunctions of γ_5 and not of $i\not{D}_E$

But they are still eigenfunctions of $(i\not{D}_E)^2$

$$\lambda_n \neq 0 \qquad \begin{cases} -\not{D}_E^2\,\varphi_{n\pm} = \lambda_n^2\,\varphi_{n\pm} \\ T\varphi_{n\pm} = t_n\,\varphi_{n\pm} \end{cases}$$

$\Rightarrow$ for $\lambda_n \neq 0$ we have eigenfunctions of $-\not{D}_E^2$ and T coming in pairs of opposite chirality

Look at the kernel: $\lambda_n = 0$

Here they are not necessary paired!

$$i\not{D}_E\varphi_u = 0 = i\not{D}_E\varphi_d$$

$$\gamma_5\varphi_u = \varphi_u \qquad u = 1\cdots n_+$$

$$\gamma_5\varphi_d = -\varphi_d \qquad d = 1\cdots n_-$$

Let us compute the regularized trace of $\gamma_5 T$ as before

$$\mathrm{Tr}\,\gamma_5 T = \lim_{\Lambda\to\infty}\mathrm{Tr}\left\{\gamma_5 T f\left(-\frac{\not{D}_E^2}{\Lambda^2}\right)\right\} = \sum_n \langle\varphi_n|\gamma_5 T f\left(-\frac{\not{D}^2}{\Lambda^2}\right)|\varphi_n\rangle$$

$$= \sum_n f\left(\frac{\lambda_n^2}{\Lambda^2}\right) t_n \langle\varphi_n|\gamma_5|\varphi_n\rangle = \sum_{u=1}^{n_+} f(0)\,t_u - \sum_{d=0}^{n_-} f(0)\,t_d$$

$$= \sum_{u=1}^{n_+} t_u - \sum_{d=1}^{n_-} t_d$$

Take now $T=1$ $\quad \overset{\#}{=} n_+ - n_-$ difference between the number of positive and negative chirality zero-modes of $i\not{D}_E$

$$- \quad \text{index}(i\not{D}_E) = n_+ - n_- = \lim_{\Lambda\to\infty} \mathrm{Tr}\,\gamma_5\, f\left(-\frac{\not{D}_E^2}{\Lambda^2}\right)$$

$$\Rightarrow \quad \int d^4x\, a(x) = -2\,\text{index}(i\not{D}_E)$$

Now we expect the anomaly through our previous computation
Going to euclidean space we have $-i$ from γ-trace and i from $p \to p_E$ so that

$$\boxed{\text{index}(i\not{D}_E) = \frac{1}{32\pi^2}\int dx_E\, \varepsilon^E_{\mu\nu\rho\lambda}\, \mathrm{tr}_R(F^{\mu\nu}F^{\rho\lambda})}$$

ATIYAH-SINGER index theorem

Let us discuss the generalization in general $D=2d$ dimension: steps in Euclidean signature $\Rightarrow$ the index is still given by the same formula but in $D=2d$
$\gamma_E = i^d \gamma_E^1 \cdots \gamma_E^{2d}$. Take $f = e^{-s}$

$$\text{index}\left[i\not{D}_E^{(2d)}\right] = \lim_{\Lambda\to\infty} \mathrm{Tr}\,\gamma_E\, f\left(-\frac{\not{D}_E^2}{\Lambda^2}\right) = \lim_{\Lambda\to\infty} \int d^{2d}x_E\, \Lambda^{2d} \int \frac{d^{2d}q_E}{(2\pi)^{2d}}\, \mathrm{tr}\left(\gamma_E\, e^{-q_E^2 - 2iq_E^\mu \frac{D_\mu^E}{\Lambda} + \frac{\not{D}_E^2}{\Lambda^2}}\right)$$

$$= \int d^{2d}x \int \frac{d^{2d}q}{(2\pi)^d}\, e^{-q^2}\, \mathrm{Tr}\left[\gamma_E \frac{(\not{D}_E^2)^d}{d!}\right]$$

Now
$$\not{D}_E^2 = D_\mu^E D^\mu_E - \frac{i}{2}\gamma^\mu_E \gamma^\nu F_{\mu\nu}$$

and
$$\mathrm{Tr}_D(\gamma_E \gamma^{\mu_1} \cdots \gamma^{\mu_{2d}}) = (-i)^d\, 2^d\, \varepsilon^{\mu_1 \cdots \mu_d}$$

Therefore
$$\mathrm{tr}\left(\gamma_E (\not{D}_E^2)^d\right) = \overset{\nearrow\ (-\frac{i}{2})^d\, \mathrm{Tr}_D(\gamma_E \gamma^{\mu_1} \cdots \gamma^{\mu_{2d}})\, \mathrm{Tr}_R(F_{\mu_1\mu_2} \cdots F_{\mu_{2d-1},\mu_{2d}})}{(-1)^d\, \varepsilon^{\mu_1 \cdots \mu_{2d}}\, \mathrm{tr}_R(F_{\mu_1\mu_2} \cdots F_{\mu_{2d-1}\mu_{2d}})}$$

Performing the gaussian integration
$$\int \frac{d^{2d}u}{(2\pi)^{2d}}\, e^{-u^2} = \frac{1}{(4\pi)^d}$$

we arrive at

$$\mathrm{index}\,(i\not{D}_E^{(2d)}) = \frac{(-1)^d}{d!\,(4\pi)^d} \int d^{2d}x\; \varepsilon^{\mu_1 \cdots \mu_{2d}}\, \mathrm{tr}_R(F_{\mu_1\mu_2} \cdots F_{\mu_{2d-1}\mu_{2d}})$$

describing the abelian anomaly in general D=2d dimensions.

So far we have discussed a U(1) anomaly related to a global symmetry: the central part of the lectures concerns instead the ~~non~~ anomaly related to a non-abelian symmetry locally coupled to field. The important point is to understand when the effective action W[A], obtained by integrating out matter fields, is invariant under gauge transformations or not (we will see how this is related to current conservation and correlation functions).

To discuss anomalies it is not necessary to develop the full machinery of BRST quantization, Slavnov-Taylor identities and so on ⇒ just the fermionic measure in the path-integral gives anomalies so instead of considering the quantum effective action

$$\Gamma[A_a, \psi_a, \bar{\psi}_a]$$

and to deal with Ward identities generated by BRST symmetry is convenient to study the effective action in presence of external gauge field

$$e^{iW[A]} = \int D\bar{\psi} D\psi \, e^{i S'(\bar{\psi}, \psi, A)}$$ ⇔ vacuum to vacuum amplitude with external fields A

Then one starts the quantization of the gauge field using

$$\tilde{S}[A] = -\frac{1}{4}\int d^4x \, F^a_{\mu\nu} F^{a,\mu\nu} + W[A]$$

If $W[A]$ is gauge invariant we do as usual (as Antonio has shown)

If $W[A]$ is NOT gauge invariant typically renormalizability breaks down and even worse unitarity

Then $A_\mu \to A'_\mu = A_\mu + D_\mu \epsilon$

$$e^{iW[A']} = \int D\bar{\psi} D\psi \, e^{i S_{mat}[\bar{\psi}, \psi, A']} = \int D\bar{\psi}' D\psi' \, e^{i S_{mat}[\bar{\psi}', \psi', A]}$$

$$= \int D\bar{\psi} D\psi \, e^{i \int d^4x \, \epsilon^a(x) A_a(x) + i S_{mat}[\,]}$$

$$= e^{i \int d^4x \, \epsilon^a A_a(x)} \, e^{iW[A]}$$

$$\Rightarrow \quad \boxed{\delta_\epsilon W[A] = \int d^4x \, \epsilon^a(x) A_a(x)}$$

Then

$$\delta_\epsilon W[A] = \int d^4x\, \delta A_\mu^a(x)\, \frac{\delta W}{\delta A_\mu^a(x)} = \int d^4x\, (D_\mu \epsilon)^a\, \frac{\delta W}{\delta A_\mu^a}$$

$$= -\int d^4x\, \epsilon^a \left(D_\mu \frac{\delta W}{\delta A_\mu}\right)_a$$

$$= \int d^4x\, \epsilon^a \mathcal{A}_a(x)$$

$$\Rightarrow \mathcal{A}_a(x) = -D_\mu \frac{\delta W}{\delta A_\mu^a} \quad \text{but} \quad \frac{\delta W}{\delta A_\mu^a(x)} = \langle J^\mu_a(x) \rangle_A$$

$$\Rightarrow \boxed{\left(D_\mu \langle J^\mu(x) \rangle_A\right)_a = -\mathcal{A}_a(x)}$$

covariant conservation violation

We ~~[illegible]~~ want to translate this characterization in terms of Feynman graphs!

From the definition of W[A] we have

$$\frac{\delta}{\delta A^{a_1}_{\mu_1}(x_1)} \cdots \frac{\delta}{\delta A^{a_n}_{\mu_n}(x_n)} W[A]\Big|_{A=0} = i^{n-1} \langle T(J^{\mu_1}_{a_1}(x_1) \cdots J^{\mu_n}_{a_n}(x_n)) \rangle_c$$

This is nothing but the coefficient

$$\boxed{\Gamma^{\mu_1 \ldots \mu_n}_{a_1 \ldots a_n}(x_1, \ldots x_n)}$$

appearing in the expansion of $\Gamma[A]$ of ~~1PI~~ generating functional of 1PI diagrams)

Let us see this for the abelian anomaly first

Take for simplicity a single U(1) fields ⇒ eigenvalues of $F = q_\gamma$ ⑲ bis

$\Rightarrow \mathrm{tr}_R\, F F_a t_b = \Sigma_\gamma q_\gamma^3$

Observe that we can take functional derivatives on the anomaly relation

$$\langle \partial_\mu J^\mu_5 \rangle = -a(x) \qquad \text{and take } A_\mu = 0$$

$$\frac{\delta}{\delta A_\nu(y)} \frac{\delta}{\delta A_\rho(z)} \left(\langle \partial_\mu J^\mu_5 \rangle \right) = - \frac{\delta}{\delta A_\nu(y)} \frac{\delta}{\delta A_\rho(z)} \left(a(x) \right)$$

↓ ↓ explicit relation

$$\partial_{x_\mu} \langle T(J^\mu_5(x) J^\nu(y) J^\rho(z)) \rangle = -\frac{1}{2\pi^2} \left(\Sigma_\gamma q_\gamma^3 \right) \varepsilon^{\nu\rho\lambda\sigma} \left(\partial_{y_\lambda} \delta^4(y-x) \right) \left(\partial_{z_\sigma} \delta^4(z-x) \right)$$

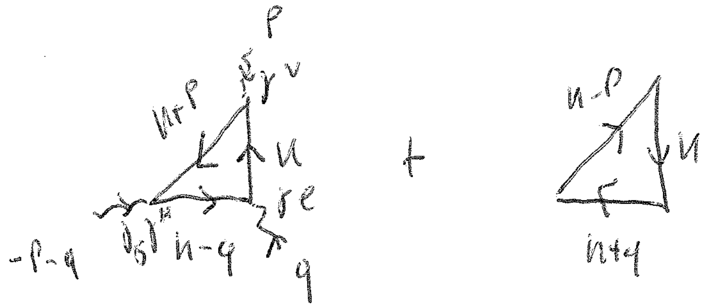
↓

Three-point function of currents $: -\Gamma_5^{\mu\nu\rho}(x,y,z)$

Transform everything in momentum space (take incoming momenta $\sim e^{ikx}$ in Fourier) and we get →

$$\boxed{i(p+q)_\mu \Gamma_5^{\mu\nu\rho}(-p-q,p,q) = \frac{1}{2\pi^2} (\Sigma_i q_i^3) \varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma}$$

↓ Feynman diagrams,



Exercise

Derive the relation for non-abelian fermions in representation R ~~[illegible]~~

Remark - If only U(1) fields are present anomaly is quadratic in A_μ ⇒ box, pentagon etc are not affected by anomaly

$$ i\, p_\mu^{(1)}\, \Gamma_5^{\mu\nu_2\ldots\nu_N}\left(p^{(1)}, p^{(2)}, \ldots p^{(N)}\right) = 0 \qquad N \geq 4 $$

- If non-abelian gauge fields are present anomaly is quartic in the gauge fields ⇒ also box and pentagon could be anomalous

Exercise Prove that the pentagon is not anomalous

$$ i\, p_\mu^{(1)}\, \Gamma_{5\,abcd}^{\mu\nu_1\nu_2\nu_3\nu_4} = 0 \quad !! $$

Hint: use that $[T, T_a] = 0$ (remember the relevant factor is $\mathrm{Tr}[T\, T_a T_b]$) and the eigenvalue of T are q_f!

and $\mathrm{Tr}_R\, T_a T_b = C_R\, \delta_{ab}$

and Jacobi identity

Let us turn now to the more interesting case of

NON ABELIAN ANOMALY

We have to use

$W[A]$ now obtained by integrating (chiral) fermions interacting with non-abelian gauge fields A_μ^a

We start from the anomaly relation

$$\boxed{D_\mu \langle J^\mu(x)\rangle_a = -\mathcal{A}_a(x)}$$

and take again two derivatives respect to $A_\nu^a(x)$ to derive a relation for three-point current functions

Remark: We have ~~correct~~ covariant derivatives $\Rightarrow$ gauge field dependence has to be taken into account

Defining $\mathcal{A}^{\nu\rho}_{a,bc}(x;y,z) = \left.\frac{\delta^2 \mathcal{A}_a(A,x)}{\delta A_\nu^b(y)\,\delta A_\rho^c(z)}\right|_{A=0}$

$$\Gamma^{\mu\nu\rho}_{abc}(x,y,z) = -\langle T(J^\mu_a(x) J^\nu_b(y) J^\rho_c(z))\rangle$$

and $\Pi^{\rho\nu}_{ab}(x,y) = i\langle T(J^\rho_a(x) J^\nu_b(y))\rangle$ ✓ Vacuum polarisation due to matter

$= \delta_{ab}\,\Pi^{\rho\nu}_{(i)}(x,y)$ $(i) \rightsquigarrow (i)$ C_{R_i} of different factors.

Anomalous Ward identity

$$\boxed{\frac{\partial}{\partial x^\mu}\Gamma^{\mu\nu\rho}_{abc}(x,y,z) + f_{abc}\left[\delta^4(x-y)\,\Pi^{\nu\rho}_{(i)}(y,z) - \delta^4(x-z)\,\Pi^{\nu\rho}_{(i)}(y,z)\right] = -\mathcal{A}^{\nu\rho}_{a,bc}(x;y,z)}$$

Fourier transform again:

$$i(p+q)_\mu \Gamma^{\mu\nu\rho}_{abc}(-p-q,p,q) = -\overset{\text{anomalous part!}}{\mathcal{A}^{\nu\rho}_{a,bc}}(-p-q;p,q)$$

$$\underbrace{-f_{abc}}_{\text{classical part!}}\left[\underbrace{\Pi^{\nu\rho}_{(c)}}_{\text{two-point function}}(p,-p)-\Pi^{\nu\rho}_{(c)}(-q,q)\right]$$

Important

We could also take 4 derivatives and 5 derivatives and so on but we will see that NON-ABELIAN GAUGE ANOMALY is cubic in A_μ^a!! ⇒ box is anomalous but $n \geq 5$ not

Nevertheless Ward ~~ideas~~ identity being non-linear in A_μ relates polygon to box!!

Remark We will also find that

$$\mathcal{A}_a(x) = 4\overset{\text{const}}{c}\,\mathrm{Tr}_R(T_aT_bT_c)\left(\partial_\mu A^b_\nu(x)\right)\left(\partial_\rho A^c_\sigma\right)\varepsilon^{\mu\nu\rho\sigma} + O(A^3)$$

That is ~~[illegible]~~ proportional to $\varepsilon^{\mu\nu\rho\sigma}$ ⇒ the classical part is ε-independent!

Then ~~[illegible]~~ concerning the three-point function we can safely discard $O(A^3)$ ($A=0$ at the end) and

$$\mathcal{A}^{\nu\rho}_{abc}(-p-q;p;q) = 8c\,\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma D^R_{abc}$$

$$D^R_{abc} = \mathrm{Tr}_R(T_{\{a}T_bT_{c\}})$$ totally symmetrized

and we calculate that

$$\boxed{-i(p+q)_\mu \Gamma^{\mu\nu\rho}_{abc}(-p-q,p,q)\Big|_{\varepsilon\,\text{part}} = 8c\,\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma D^R_{abc}}$$

II Lecture

We have now to explicitly evaluate the anomaly by means of Feynman graphs technique. Firstly we rederive the abelian anomaly by computing the equation

$$\boxed{i(p+q)_\mu \Gamma_5^{\mu\nu\rho}(-p-q,p,q) = \frac{1}{2\pi^2}\left(\Sigma_i q_i^3\right)\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma}$$

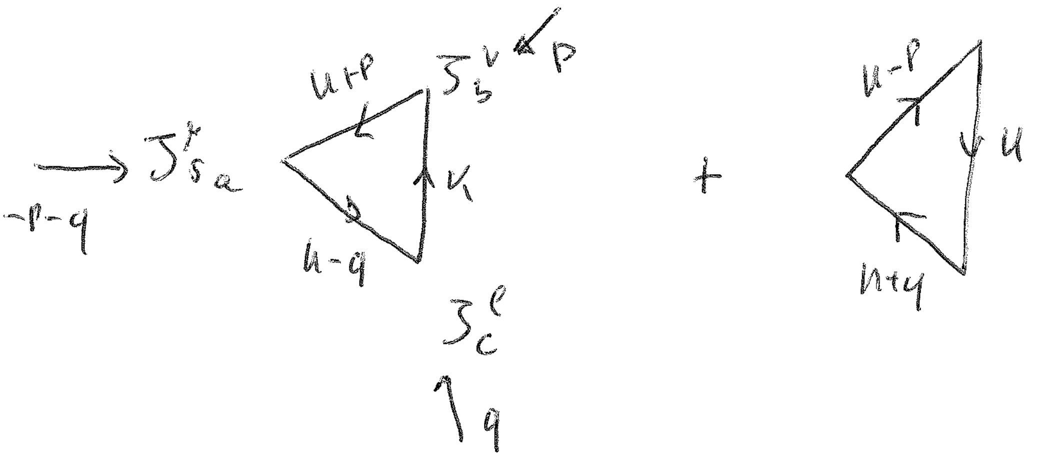
and then the non-abelian anomaly (at quadratic level) by means of

$$\boxed{-i(p+q)_\mu \Gamma^{\mu\nu\rho}_{abc}(-p-q,p,q)\Big|_{\text{G part}} = 8c\,\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma D^R_{abc}}$$

and we reconstruct the cubic part later by means of algebraic technique.

Abelian anomaly: we need the correlator of one $J^\mu_{a\,5}$ and two J^μ_b (we compute for general non-abelian current and specify later to $U(1)$ or $SU(N)$)

$$\begin{cases} J^\mu_{5a} = i\bar\psi\gamma^\mu\gamma_5 T_a\psi \\ J^\mu_a = i\bar\psi\gamma^\mu T_a\psi \end{cases}$$



2 diagrams (corresponding different contractions)

The vertices are $-\gamma^\mu\gamma_5 T_a$ and $-\gamma^\rho T_a$

The momentum space expression is

$$\Gamma^{\mu\nu\rho}_{5\,abc}(-p-q,p,q) = -i\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma_5\gamma^\mu\frac{(\not k+\not p)}{(k+p)^2-i\epsilon}\gamma^\nu\frac{\not k}{k^2-i\epsilon}\gamma^\rho\frac{(\not k-\not q)}{(k-q)^2-i\epsilon}\right)\mathrm{tr}_R(T_aT_bT_c),$$

$$+\ (p\leftrightarrow q,\ \nu\leftrightarrow\rho,\ a\leftrightarrow c)$$

The integral is <u>divergent</u> ⇒ we use <u>Pauli-Villars regularization</u>

(dimensional regularization needs the tricky extension of γ_5)

For each fermion we add another one with mass M and opposite statistics

Therefore

$$\left(\Gamma^{\mu\nu\rho}_{5\,abc}\right)_{reg} = \left(\Gamma^{\mu\nu\rho}_{5\,abc} - \Gamma^{\mu\nu\rho}_{5\,abc}(M)\right)$$

$$= -i\int\frac{d^4k}{(2\pi)^4}\left(I_0^{\mu\nu\rho}(k,p,q) - I_M^{\mu\nu\rho}(k,p,q)\right) tr_R(T_aT_bT_c)$$

$$+ \quad (p\leftrightarrow q,\ \nu\leftrightarrow\rho,\ b\leftrightarrow c)$$

$$I_M^{\mu\nu\rho} = tr\left[\gamma_5\gamma^\mu \frac{\not k+\not p+iM}{(k+p)^2+M^2-i\epsilon}\,\gamma^\nu\,\frac{\not k+iM}{k^2+M^2-i\epsilon}\,\gamma^\rho\,\frac{\not k-\not q+iM}{(k-q)^2+M^2-i\epsilon}\right]$$

Remark: for finite M (not $M\to\infty$) we have a broken chiral symmetry

Let us compute $(p+q)_\mu\left[\Gamma^{\mu\nu\rho}_{5\,abc}(-p-q,p,q)\right]_{reg}$.

First step

Exercise: to show that

$$(p+q)_\mu I_M^{\mu\nu\rho}(k,p,q) = \frac{8iM^2\varepsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma}{[(k+p)^2+M^2-i\epsilon][k^2+M^2-i\epsilon][(k-q)^2+M^2-i\epsilon]}$$

up terms that vanish after integration

Remark

$I_0 = 0$!

All the contribution to the anomaly comes from regulator !

Now calling

$$I(p,q,M) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+p)^2+M^2-i\epsilon][k^2+M^2-i\epsilon][(k-q)^2+M^2-i\epsilon]}$$

$$\boxed{-i(p+q)_\mu \left[\Gamma^{\mu\nu\rho}_{5abc}\right]_{reg} = \Big\{ 8iM^2 p_\lambda q_\sigma I(p,q,M)\, \mathrm{tr}_R(T_aT_bT_c) + (p\leftrightarrow q,\ \nu\leftrightarrow\rho,\ b\leftrightarrow c \Big\}}$$

We want at the end to send $M\to\infty$ so we need the leading term in M coming from $I(p,q,M)$

$$I(p,q,M) \sim \frac{1}{M^2}\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2+1-i\epsilon)^3} = \frac{i}{32\pi^2}\cdot\frac{1}{M^2}$$

$$\Rightarrow -i(p+q)_\mu \Gamma^{\mu\nu\rho}_{5abc}(-p-q,p,q) = -\frac{1}{4\pi^2}\varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma\, \mathrm{tr}_R(T_aT_bT_c)$$

$$+ (p\leftrightarrow q,\ \nu\leftrightarrow\rho,\ b\leftrightarrow c)$$

$$= -\frac{1}{2\pi^2}\varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma\, \mathrm{tr}_R\left[T_a\{T_b,T_c\}\right]$$

$T_a \to T$ we reproduce the result of path integral

Exercise

Take also $T_b = T_c = T$ ~~eg~~ and check that within our Pauli-Villars regularization vector currents are conserved

$$p_\nu \Gamma_5^{\mu\nu\rho}(-p-q,p,q) = q_\rho \Gamma_5^{\mu\nu\rho}(-p-q,p,q) = 0$$

Non-abelian anomaly

Let us now try to compute at Feynman diagrams level the quadratic part of the non-abelian anomaly: we have to study the correlator of 3 chiral currents because we want to study chiral matter

$$\begin{cases} \psi_L = P_L \psi_L \\ \psi_R = P_R \psi_R \\ \\ \psi = \psi_L + \psi_R \end{cases} \qquad P_{L,R} = \left(\frac{1 \pm \gamma_5}{2}\right)$$

The chiral gauge theory is ~~[illegible]~~ obtained using

$$\bar\psi_L \not{D} \psi_L \quad \text{or} \quad \bar\psi_R \not{D} \psi_R$$

We have two types of mass

Dirac $\quad \bar\psi\psi = \bar\psi_R \psi_L + \bar\psi_L \psi_R \quad$ ⇒ chiral theory has no Dirac mass

Majorana $\quad \bar\psi_L \psi_L^c + \bar\psi_L^c \psi_L \qquad \psi^c = i\gamma^0 C \psi^*$

Remark ~~[illegible]~~ The existence of this mass term depends on the representation R (we will discuss later)

Take massless fermions in a representation R

$$\mathcal{L}_{matter} = -\bar{\Psi}_L \not{D} \Psi_L \quad \to \text{propagators} \quad P_L \frac{i\not{k}}{k^2 - i\epsilon}$$

in terms of non chiral fermions $\to$ $$= -\bar{\Psi} \not{D} P_L \Psi \quad \to \text{vertices} \quad i\,(i\gamma^\mu T_a) P_L$$

Because $P_L^2 = P_L$ in the fermion loop

$$\sim \mathrm{Tr}\ (i\gamma^\mu T_a) \underbrace{P_L \, P_L}_{\sim P_L} \frac{-i(\not{k}+\not{p})}{(k+p)^2 - i\epsilon} (i\gamma^\nu T_a) \underbrace{P_L \, P_L}_{P_L} \frac{-i\not{k}}{k^2 - i\epsilon}$$

$\Rightarrow$ is like usual propagator with <u>chiral vertices</u> $\Rightarrow$

$$\mathcal{L}'_{matter} = -\bar{\Psi} \not{\partial} \Psi - \bar{\Psi}(-i\not{A}) P_L \Psi$$
$$= -\bar{\Psi}(\not{\partial} - i\not{A} P_L)\Psi \qquad \Psi_R \text{ is free}$$

$\Rightarrow$ we understand why generally chiral fermions have anomaly

$\xrightarrow{}$ 1) The determinant $\det[\not{\partial} - i\not{A} P_L] = e^{iW[A]}$ is not gauge invariant because $\hat{\not{D}}_L = \not{\partial} - i\not{A} P_L$ does not transf. covariantly under gauge $\Rightarrow$ the eigenvalue <u>are not</u> invariant

$\xrightarrow{}$ 2) Suppose we use $\not{D}_L = \not{D} P_L$ $\Rightarrow$ we are again in trouble because $\not{D}_L : H_L \to H_R$, two different Hilbert space

Let us describe this problem in Euclidean space $(\not{D}_L)_E$

We could consider

$$(\not{D}_L)_E^\dagger (\not{D}_L)_E \quad \text{that maps } H_L \to H_L$$

⇒ well defined determinant and gauge invariant

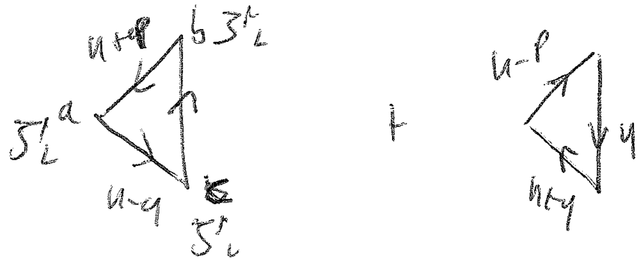
$$\det\left((\not{D}_L)_E^\dagger (\not{D}_L)_E\right) = |\det(\not{D}_L)_E|^2$$

⇒ the modulus of the determinant <u>is</u> well defined

The phase is undetermined ⇒ anomaly can only appear in the phase

⇒ <u>imaginary part of</u> $\underline{W[A]_E}$

Feynman diagrams



<u>Exercise</u> Write down $\Gamma^{\lambda\nu\rho}_{L,abc}(-p-q,p,q)$

⇒ regularize by Pauli-Villars but chiral fermion have no Dirac Mass

⇒ use <u>conventional propagator</u> and <u>chiral vertices</u> ⇒ gauge invariance is lost

The computation is boring: let us display the various steps for interested readers

Step 1

Introduce 2 ~~fe~~ bosonic, 1 fermionic regulator

M_1, M_3 bosonic M_2 fermionic

$M_0 = 0$
original fermion

$$\boxed{M_2^2 = M_1^2 + M_3^2} \longrightarrow \boxed{\text{Vanishing of log. divergence}}$$

$\eta_1 = \eta_3 = -1 \quad \eta_2 = 1 \quad \Rightarrow \sum_{s=0}^{3} \eta_s = 0$ ↳ Vanishing of quadratic

$$\left[\Gamma^{\mu\nu\rho}_{L,abc}\right]_{Reg} = \int \frac{d^4k}{(2\pi)^4} \left(\sum_{s=0}^{3} \eta_s \mathcal{J}^{\mu\nu\rho}_{M_s}(k,p,q) \right) \mathrm{Tr}_R T_a T_b T_c$$
$$+ \; (p \leftrightarrow q, \nu \leftrightarrow \rho, b \leftrightarrow c)$$

$$\mathcal{J}^{\mu\nu\rho}_{M}(k,p,q) = \mathrm{tr}\left[\gamma^\mu P_L \frac{\not k + \not p + iM}{(k+p)^2 + M^2 - i\epsilon} \gamma^\nu P_L \frac{\not k + iM}{k^2 + M^2 - i\epsilon} \gamma^\rho P_L \frac{\not k - \not q + iM}{(k-q)^2 + M^2 - i\epsilon} \right]$$

Step 2

Compute

$$-i(p+q)_\mu \left[\Gamma^{\mu\nu\rho}_{L,abc}\right]_{reg} = \left(I^{\nu\rho}(-q) - I^{\nu\rho}(p) + \sum_{s=1}^{3} \eta_s M_s^2 \, \mathcal{J}^{\nu\rho}_{M_s}(p,q) \right)$$
$$\mathrm{Tr}_R(T_a T_b T_c) \quad + (p \leftrightarrow q, \nu \leftrightarrow \rho, b \leftrightarrow c)$$

with

$$\begin{cases} I^{\nu\rho}(p) = \displaystyle\int \frac{d^4k}{(2\pi)^4} \sum_{s=0}^{3} \eta_s \frac{\mathrm{tr}\left((\not k + \not p)\gamma^\nu \not k \gamma^\rho P_L\right)}{[(k+p)^2 + M_s^2 - i\epsilon][k^2 + M_s^2 - i\epsilon]} \\[2ex] \mathcal{J}^{\nu\rho}_{M_s}(p,q) = \displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}\left[(\not p + \not q)\gamma^\nu \not k \gamma^\rho P_L\right]}{[(k+p)^2 + M_s^2 - i\epsilon][k^2 + M_s^2 - i\epsilon][(k-q)^2 + M_s^2 - i\epsilon]} \end{cases}$$

Step 3

Take $M_S \to \infty$ and take the ε part

$I^{\nu\rho}$ have no ϵ term and they are good as before

$$\mathop{L}_{M_S\to\infty} M_S^2 J^{\nu\rho}_{M_S} = \frac{i}{24\pi^2}\left[q^\nu q^\rho - p^\nu p^\rho + \frac{p^2-q^2}{2}\eta^{\nu\rho} + i\epsilon^{\lambda\nu\sigma\rho}p_\lambda q_\sigma \right]$$

$\Rightarrow$

$$-i(p+q)_\mu \Gamma^{\mu\nu\rho}_{L,abc}(-p-q,p,q)\Big|_{\epsilon\text{ term}} = \sum_{S=1}^{3} \frac{i}{24\pi^2}\, i\varepsilon^{\lambda\nu\sigma\rho} p_\lambda q_\sigma \mathrm{Tr}_R T_a T_b T_c$$

$$+ (p\leftrightarrow q, \nu\leftrightarrow\rho, b\leftrightarrow c)$$

$$\equiv \frac{-1}{12\pi^2}\varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma D^R_{\alpha\beta\gamma}$$

Therefore we have ~~[illegible]~~ determined the coefficient of the quadratic term

$$\boxed{\mathcal{A}^L_a(x) = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\lambda}\mathrm{Tr}_R\left(T_a \partial_\mu A_\nu \partial_\rho A_\sigma\right) + O(A^3)}$$

We will show that

$$\mathcal{A}^L_a(x) = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\lambda}\mathrm{Tr}_R\left[T_a \partial_\mu\left(A_\nu \partial_\rho A_\sigma - \frac{i}{4}A_\nu[A_\rho, A_\sigma]\right)\right]$$

Exercise

To show that $\mathcal{A}^L_a(x) = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^b_\nu \partial_\rho A^c_\sigma - \frac{i}{4}A^b_\nu [A_\rho, A_\sigma]^c\right) D^R_{abc}$

We want now to discuss two fundamental properties of gauge anomalies

|| Anomalies are local functional in the gauge fields with coefficients that are finite (in general the one-loop effective action is expected to be non-local and divergent)

Locality   Notice that only divergent diagrams could produce anomalies (Regularization needed) => in $D=4$ we have $n \geq 5$

Ward identity are among convergent Green functions $n=5$ would be involved with $n=4$ but actually anomaly is just cubic! See for the divergent anomalous diagram

The argument is the following : taking enough derivative with respect to external momenta the anomalous part of the vertex function is zero

Be concrete, take the 3-point function in $D=4$ => degree of divergence 1

$$\Rightarrow \; p_\mu \Gamma_5^{\mu\nu\rho} \sim \text{degree of divergence } 2$$

Take 3 derivatives with respect external momenta : remember

$$\frac{\partial}{\partial p_\mu}\left(\frac{1}{k\!\!\!/+p\!\!\!/+\dots}\right) = -\frac{(k\!\!\!/+p\!\!\!/\dots)\gamma^\mu(k\!\!\!/+p\!\!\!/+\dots)}{((k+p+\dots)^2)^2}$$

it improves convergence => 3 derivatives give convergence

$$\Rightarrow \underbrace{\frac{\partial}{\partial p}\cdots\frac{\partial}{\partial p}}_{3}\left(p\Gamma\big|_{an}\right)=0 \Rightarrow \underbrace{\partial_p\cdots\partial_p}_{3} A(p^i)=0$$

=> the anomaly $A(p_i)$ must be a polynomial of degree 2 in the external momenta => local functional of the gauge fields

Finiteness: consider

$$(p_\mu+q_\mu)\,\Gamma^{\mu\nu\rho}_{abc}(-p-q,p,q)\Big|_{\text{an. part}}$$

because we need a $\varepsilon^{\mu\nu\rho\lambda}$ we just can obtain

$$\simeq \varepsilon^{\mu\nu\rho\lambda}_{abc}\,p_\rho q_\lambda$$

(Γ has degree of divergence 1 so $p\Gamma \sim 2$)

So ε has scaling dimension zero => no dependence on momenta (locality) => no dependence on M or cut-off Λ

=> this argument generalizes in $D=2d$ dim

In any dimension $D=2d$ the anomaly is finite and local. It shows up firstly in the divergence of $d+1$ vertex function and it is given by

$$\left(\sum_{i=1}^{d} p^{(i)}_\mu\right)\Gamma^{\mu\nu_1\ldots\nu_d}_{a b_1\ldots b_d}\left(-\sum_{i=1}^{d} p^{(i)}, p^{(1)}\ldots p^{(d)}\right)\Big|_{\text{anom}} =$$

$$C\,\varepsilon^{\nu_1\ldots\nu_d\lambda_1\ldots\lambda_d}\,p^{(1)}_{\lambda_1}\cdots p^{(d)}_{\lambda_d}\,D^R_{a b_1\ldots b_d}$$

C ↓ constant (relevant notion for the anomalies)

D^R ↓ completely symmetrized trace $T_a T_{b_1}\cdots T_{b_d}$

$D=6$ → box

$D=8$ → pentagon

$D=10$ exagon

Last important point

~~Rel~~ Relevant and irrelevant anomalies and the ambiguity in their form

Remark : we can always add local counterterms to the effective actions, according general principles of renormalization

Suppose we add in $D=4$

$$\boxed{\frac{1}{6}\int d^4q\, d^4p\, \Delta\Gamma^{\mu\nu\rho}_{abc}(-p-q,p,q)\, A^a_\mu(-p-q)\, A^b_\nu(p)\, A^c_\rho(q)}$$

This modifies the three-point

$$\Gamma^{\mu\nu\rho}_{abc} \to \Gamma^{\mu\nu\rho}_{abc} + \Delta\Gamma^{\mu\nu\rho}_{abc} = \Gamma'^{\mu\nu\rho}_{abc}$$

is Γ' still anomalous for any choice of $\Delta\Gamma$? If yes i.e.

$$-i(p_\mu+q_\mu)\,\Delta\Gamma^{\mu\nu\rho}_{abc}\Big|_G = \frac{i}{12\pi^2}\,\varepsilon^{\nu\rho\lambda\sigma} p_\lambda q_\sigma D^R_{abc}$$

We cancel the anomaly by means of a local counterterm ! $\Rightarrow$ the anomaly is called trivial or irrelevant.

$\Delta\Gamma$ have to be local $\Rightarrow$ linear in p e q by dimension on $\varepsilon^{\mu\nu\rho\lambda}$

$$\Delta\Gamma\Big|_\varepsilon = \varepsilon^{\mu\nu\rho\lambda}(a\,k_\lambda + b\,p_\lambda + c\,q_\lambda)\,D^R_{abc} \qquad k=-p-q$$

$\Rightarrow \boxed{c-b=\frac{i}{12\pi^2}}$ but we have that Γ is completely symmetric in

$(k,\mu,a)\ (p,\nu,b)\ (q,\rho,c)$

=> Bose symmetry

but D^R_{abc} is totally symmetric, ε totally antisymmetric

$$\Rightarrow \left.\begin{array}{l} b=-a \\ c=-a \\ c=-b \end{array}\right\} \Rightarrow \text{Bose forbidden the choice } \; b=a=c=0$$

Therefore a anomaly is relevant $\Leftrightarrow \int \epsilon^a A_a(x) \neq \delta_G F$

where F is a local functional!

Otherwise we can always add to effective action a local counterterm F that can modify the form of anomaly but not put it to zero if relevant.

Fermion representations and cancellation of anomalies

Important : global anomaly (anomaly on global symmetry) simply changes the selection rules of theory at quantum level => no inconsistency of the quantum theory

local anomaly (anomaly on local gauge symmetry) => unitarity is affected => inconsistency of the quantum theory

Chiral fermions coupled on non-abelian gauge field generically implies anomaly (we have seen) => chiral fermions are common in physical model (Standard model!) => consistency requires cancellation among different chiral contributions

Let us discuss representations R of gauge groups with non-vanishing D^R_{abc} and the contributions of left handed and right handed particles.

Then cancellation in Standard model $\underline{SU(3)\otimes SU(2)\otimes U(1)}$

Let us start by considering the total contribution to the anomaly from left-handed and right-handed fermions

$$\mathcal{A}_a = \sum_i A^L_a(\psi_i^L) + \sum_j A^R_a(\psi_j^R)$$

$$= -\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^b_\nu\partial_\rho A^c_\sigma - \frac{i}{4}A^b_\nu[A_\rho, A_\sigma]^c\right)\left(\sum_i D^{R_i^L}_{abc} - \sum_j D^{R_j^R}_{abc}\right)$$

It is convenient to group $\sum_i \oplus R_i^L = R^L$ (reducible)

$\sum_i \oplus R_i^R = R^R$

Remark: We can include also Dirac ($R^L = R^R$) they cancel

To efficiently describe the dependence on the representation we make the following consideration: in $D=4$

- be ψ a left-handed particle in some representation R_L with $T^L_a = T_a^{R_L}$

- $\psi^c = i\gamma^0 C\psi^*$ describing the antiparticle transforms as $-(T_a^{R_L})^T$

(exercise) and is right-handed and therefore transforms in a right-representation $T_a^{R_R} = -(T_a^{R_L})^T$

Moreover $-D^{R_R}_{abc} = D^{R_L}_{abc} \Rightarrow$ we can describe the anomaly as ψ left-handed or the ψ^c right-handed

This holds in $D=4+4n$ dimensions (where ψ^c is right-handed if ψ left-handed)

In $D=2+4n$ dimension ψ^c is left-handed as well (particle and antiparticle have the same chirality) and

$$D^{R^c}_{a_1\ldots a_{d+1}} = D^{R}_{a_1\ldots a_{d+1}}$$

In $2+8n$ dimensions Majorana-Weyl $\psi=\psi^c \Rightarrow \frac{1}{2}$ coefficient of the anomaly

We will use later these facts

Now we need the following:

$$\boxed{\text{When } D^R_{abc} \text{ is nonvanishing ?}}$$

- R_1 and R_2 are equivalent if exists S fixed so that

$$T_a^{R_1} = S\, T_a^{R_2} S^{-1} \quad \forall a$$

- R and $\bar{R}$ complex conjugate when $\begin{cases} T_a^{\bar{R}} = -(T_a^R)^* \Rightarrow (\text{hermitian}) \\ T_a^{\bar{R}} = -(T_a^R)^T \end{cases}$

- When $(T_a^R)^T = -S\, T_a^R S^{-1}$ R equivalent to $\bar{R}$ and the representation is called pseudoreal. If S can be taken 1 it is real

Exercise $D^R_{abc}=0$ if R real or pseudoreal

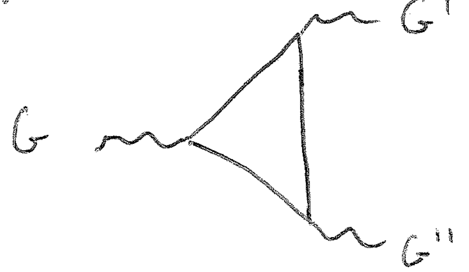
$\Rightarrow$ if a group G has only (pseudo) real reps the anomaly is always zero!!

Note $SO(4n)$, $USp(2n)$

- $SO(2n+1)^{\checkmark}$, G_2, F_4, E_7, E_8 only pseudo-real (and $SU(2) = SO(3)$)
- $SO(4n+2)$ $(n \geq 2)$, E_6 $D_{abc} = 0$ for all reps (not only pseudoreal)
- $SU(N)$ $N \geq 3$, $U(1)$ and product groups have $D^R_{abc} \neq 0$

$\Rightarrow$ in general $G = G_1 \times G_2 \cdots G_n$ at least one of G_i is SU(N) or U(1) to have $D^R_{abc} \neq 0$

The more general situation is the following



G, G', G'' are different groups in the product

$\rightarrow$ $U(1) - U(1) - U(1)$ $\quad D_{abc} \rightarrow T_r TTT = \sum_i q_i^3$

$\rightarrow$ $U(1) - G_S - G_S$ $\quad U(1) \times G_S \Rightarrow R = \oplus_j (q_j, R_j)$

charge

$\Rightarrow D^R_{abc} \Rightarrow T_{r_R} T T_a T_b = \sum_j q_j T_{r_{R_j}} T_a T_b = g^2 \sum_j q_j C_{R_j} \delta_{ab}$

$\Rightarrow$ non-vanishing anomaly for any simple group if the reps have non-vanishing U(1) charge

$\rightarrow$ $U(1) - G_S - G_{S'}$ or $G_S -$ ~~G_S~~ $- G_{S'}$ for $G_S \neq G_{S'}$

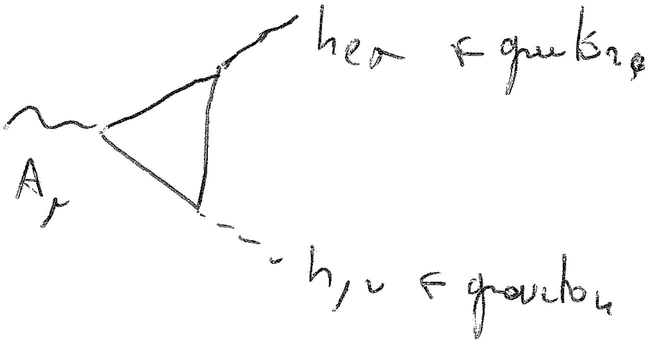
$D = 0 \quad (T_r T_a = 0)$

→ $G_S - G_S - G_S$ $\quad D_{abc} \neq 0 \quad G_S = SU(N)\ N \geq 3$

One should also consider for Standard model gravitational anomalies that we will study later. For the moment it's sufficient think of them as gauge anomaly for SO(3,1) or SO(4) ⇒ no pure anomaly

but we have $U(1) - SO(4) - SO(4)$

mixed U(1)-gravitational anomaly



$\sim \langle J_\mu \, T_{\rho\sigma} \, T_{\mu\nu} \rangle$

energy momentum tensor

<u>Important</u> <u>property</u> (very important!!)

only massless particles contribute to anomalies

- Dirac mass doesn't anomaly ($\psi = \psi_L + \psi_R$!!)
- Majorana mass term $\quad \bar\psi_L \psi_L^c + \bar\psi_L^c \psi$

⇒ this term can be <u>not compatible</u> with gauge invariance

Exercise Given the representation R in which transforms ψ_L

Show that

$$\delta(\bar\psi_L \psi_L^c + \bar\psi_L^c \psi_L) = 0 \overset{\text{only}}{\vee} \text{ if } \underline{R \text{ is real}}$$

=> if Majorana mass is gauge invariant R is real so no contribution to D^R_{abc}!!

This generalizes in any dimension!

>> anomalies do not depend on heavy (not observed) particles appearing at high-energy may be.

Anomaly cancellation in SM

The gauge group is SU(3)×SU(2)×U(1) so we have to discuss

- SU(3) × SU(3) × SU(3)
- SU(3)×SU(3) × U(1)
- SU(2) × SU(2) × U(1)
- U(1) × U(1) × U(1)
- mixed U(1) - gravitational

Look at one generation

	SU(3)	SU(2)	U(1) hypercharge
$\binom{\nu_e}{e}_L$	1	2	$\frac{1}{2}$
$(e_R)^c$	1	1	-1
$\binom{u}{d}_L$	3	2	$-\frac{1}{6}$
$(u_R)^c$	$\bar 3$	1	$2/3$
$(d_R)^c$	$\bar 3$	1	$-1/3$

we describe everything in terms of left-handed field through the conjugate field (the antiparticle)

$$(e_R)^c \sim (e^c)_L \ldots$$

<u>Unbroken phase</u>

Consider the high-energy theory where everything is massless. Be g_3 ~ SU(3), g for SU(2) and g' for U(1): we have

- SU(3) × SU(3) × SU(3) :

$$R = (1+2) + 2 \overset{\to \text{twice}}{+} (3+3) + \bar{3} + \bar{3} \quad \text{that is real} \Rightarrow D^R_{abc} = 0$$

- SU(3) × SU(3) × U(1)

we have to consider

$$\mathrm{Tr}(T_a^R T_b^R T^R) = 2\, \mathrm{tr}(T_a^3 T_b^3)\overset{(-\frac{1}{6})g'}{\downarrow} + \mathrm{tr}(T_a^{\bar{3}} T_b^{\bar{3}}) \cdot \frac{2}{3} g'$$
$$+ \mathrm{tr}(T_a^{\bar{3}} T_b^{\bar{3}}) \left(-\frac{1}{3} g'\right)$$

$$C_3 = C_{\bar{3}} \qquad = g_3 g' C_3 \delta_{ab} \left(2\left(-\frac{1}{6}\right) + \frac{2}{3} - \frac{1}{3} \right) = 0$$

- SU(2) × SU(2) × U(1)

$$\mathrm{Tr}\, T_a^R T_b^R T^R = \mathrm{Tr}(T_a^2 T_b^2)\left(\frac{1}{2} g'\right) + 3\, \mathrm{tr}(T_a^2 T_b^2)\left(-\frac{1}{6} g'\right)$$
$$= g g' C_2 \delta_{ab} \left(\frac{1}{2} + 3\left(-\frac{1}{6}\right)\right) = 0$$

- U(1) × U(1) × U(1)

$$\mathrm{Tr}(T^R T^R T^R) = 2 \left(\frac{g'}{2}\right)^3 + (-g')^3 + 3 \cdot 2 \cdot \left(-\frac{1}{6} g'\right)^3 + 3\left(-\frac{1}{3} g'\right)^3 = 0$$

Mixed

$U(1)$ - gravitational $\Rightarrow$ sum of $U(1)$ charges

$$\mathrm{tr}\, Q = 2\cdot\frac{1}{2}g' + (-g') + 3\times 2\cdot\left(-\frac{1}{6}g'\right) + 3\left(\frac{2}{3}g'\right) + 3\cdot\left(-\frac{1}{3}g'\right)$$
$$= 0 \quad !!$$

Lecture III

In the last lectures we have derived the form of the non abelian anomaly (and in particular the precise coefficient in front) up to order A^3

$$A^L{}_a(x) = -\frac{1}{24\pi^2}\,\varepsilon^{\mu\nu\rho\lambda}\,\mathrm{Tr}_R\left[T_a(\partial_\mu A_\nu\,\partial_\rho A_\lambda)\right] + O(A^3)$$

that is sufficient to discuss the anomaly cancellation. We want now to derive the full expression and extend the computation in general dimension and for gravity theory. To this aim we develop a general method based on algebraic techniques

First of all we derive an equation constraining the form of a general gauge anomaly in $D = 2d$ dimensions $\Rightarrow$ WZ consistency condition and we give the general solution in terms of descent equations $\Rightarrow$ better done in BRST formalism.

Remark We will always assume that the bundle is trivial

Let us recall the WZ consistency condition

They are a "triviality"

Gauge variation of the effective actions

$$\boxed{\delta_\epsilon \Gamma(A) = \int d^D x \; \epsilon^a(x)\, \mathcal{A}_a(x)}$$

Γ is a functional of A_μ !!

we obtain, recalling $(\delta A_\mu)^a = (D_\mu \epsilon)^a$

$$\mathcal{A}_a(x) = -\left(D_\mu \frac{\delta}{\delta A_\mu(x)}\right)_a \Gamma(A)$$

The observation is that the functional differential operator

$$\boxed{G_a(x) = -\left(D_\mu \frac{\delta}{\delta A_\mu(x)}\right)_a = -\frac{\partial}{\partial x^\mu}\frac{\delta}{\delta A_\mu^a} - f_{abc} A_\mu^b(x) \frac{\delta}{\delta A_\mu^c(x)}}$$

acting on a functional of A_μ generates the Lie algebra of G

Exercise

$$\boxed{\begin{aligned} &[G_a(x), G_b(y)] = \delta^4(x-y)\, f_{abc}\, G_c(x) \\ &\left[\int f_a G_a , \int g_b G_b\right] = \int f_{abc}\, g_b f_a G_c \end{aligned}}$$

Acting on Γ with the commutator we have the WZ c.c that anomaly must satisfy !

$\Rightarrow$ but the coefficient is not determined (linear condition)

$$\boxed{G_a(x)\, \mathcal{A}_b(y) - G_b(y)\, \mathcal{A}_a(x) = f_{abc}\, \delta^4(x-y)\, \mathcal{A}_c(x)}$$

Strategy We show that Feynman graphs + WZ are enough $\Rightarrow$ the WZ consistency condition determines uniquely the lowest order anomaly !!

WZ can be recasted into a BRST operator

BRST

$s, \; s^2=0$

$s A_\mu = D_\mu \omega$ $\Rightarrow$

ω is a ghost

action on a functional $F(A)$

$$s F[A] = \int d^4x \, \omega^a_{(x)} G_a(x) F[A]$$

$$s \Gamma(A) = \mathcal{A}(\omega, A)$$

$$s \mathcal{A}(\omega, A) = s^2 \Gamma(A) = 0 \; !!$$

$$\boxed{WZ \Leftrightarrow \quad s\mathcal{A}(\omega, A) = 0}$$

Exercise: To show that $s\mathcal{A}(\omega,A)=0$ is equivalent explicitly to WZ

- $s\omega^a = -\frac{1}{2} f^{abc} \omega_b \omega_c$
- $s(\omega^a A_a) = s\omega^a A_a - \omega^a s A_a$

Statement: The anomaly $\mathcal{A}(\omega, A)$ is a BRST closed functional of ghost number one $\Rightarrow$ it is defined up a BRST exact functional $sF(A)$ ($F(A)$ of ghost number zero) local

$$\begin{cases} \mathcal{A}'(\omega,A) = \mathcal{A}(\omega,A) + s F(A) \\ s\mathcal{A}' = s\mathcal{A} = 0 \end{cases}$$

$sF(A)$ is a "irrelevant piece" $\Rightarrow$ relevant anomaly $\begin{cases} s\mathcal{A} = 0 \\ \mathcal{A} \neq sF \end{cases}$

Non-trivial BRST cohomology classes at ghost number one on the space of local functional

Let us determine now explicitly the higher orders A_μ of the non-abelian anomaly in $D=4$

We switch now on describing YM fields in terms of differential form

<u>Short resumé</u>

$$\begin{cases} A = A_\mu dx^\mu = A_\mu^a T_a dx^\mu \\ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu}^{\ \ a} T_a dx^\mu dx^\nu \end{cases}$$

Then $A^2 = A \wedge A = \frac{1}{2} (A_\mu A_\nu - A_\nu A_\mu) dx^\mu dx^\nu = \frac{1}{2} [A_\mu , A_\nu] dx^\mu dx^\nu$

$$\Rightarrow \boxed{F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA - iA^2}$$

In this language gauge transformation are $\begin{cases} \delta A = d\epsilon - i[A,\epsilon] \\ \delta\psi = i\epsilon\psi \end{cases}$

where $\epsilon = \epsilon^a T_a$, T_a hermitian and $\boxed{D = dx^\mu D_\mu = d - iA}$

Taking T_a <u>antihermitian</u>

$$\begin{cases} T_a^\dagger = -T_a \\ [T_a , T_b] = f_{abc} T_c \end{cases} \qquad \begin{cases} A, F = dA + dA^2 \quad D = d + A \\ v = -i\epsilon \quad , \delta\psi = -v\psi \quad \delta A = dv + [A,v] \end{cases}$$

Bianchi identity $\boxed{DF = dF + AF - FA = 0}$

Anomaly : $\underline{D=4}$

$$\delta_\epsilon \Gamma(A) = \int d^4x \sqrt{g}\, \epsilon^a \mathcal{A}_a(A)$$

$$= \frac{1}{24\pi^2}\int d^4x \sqrt{g}\, \epsilon^a(x)\, \varepsilon^{\mu\nu\rho\lambda}\, \mathrm{Tr}\left[T^a(\partial_\mu A_\nu \partial_\rho A_\lambda)\right] + o(A$$

$$= -\frac{1}{24\pi^2}\int \mathrm{Tr}_R(\epsilon\, dA\, dA) + o(A^3)$$

$$= \frac{1}{24\pi^2}\int \mathrm{Tr}_R(v\, dA\, dA) + o(A^3)$$

→ starting point

Turn v to be a ghost ω $(dx^\mu \omega = -\omega dx^\mu)$

$$\Rightarrow \begin{cases} sA = -d\omega - \{A,\omega\} \\ sF = [F,\omega] \end{cases} \qquad sd = -ds$$

$$\begin{cases} s(dA) = [dA,\omega] - [A,d\omega] \\ sA^2 = -d\omega A + A d\omega + [A^2,\omega] \\ sA^3 = -d\omega A^2 + A d\omega A - A^2 d\omega - \{A^3,\omega\} \\ sA^4 = -d\omega A^3 + A d\omega A^2 - A^2 d\omega A + A^3 d\omega + [A^4,\omega] \end{cases}$$

Strategy: we impose $sA = 0$

$\Rightarrow$ $s\int \mathrm{Tr}(\omega\, dA\, dA)$ a scaling dimension 6

$\Rightarrow$ it can be cancelled by $s\int \mathrm{tr}(\omega F(A))$
↘ dimension 4

$\Rightarrow \quad A^4, \; AdAA, \; A^2dA, \; dAA^2$

Ansatz

$$A(\omega, A) = \frac{i}{24\pi^2}\int \mathrm{tr}\left[\omega\left(dAdA + \alpha A^2dA + \beta AdAA + \gamma dAA^2 + \lambda A^4\right)\right]$$

Observation $\quad s\,\mathrm{tr}(\omega A^4) \sim \mathrm{tr}\,\omega^2A^4 + O(\omega, d\omega)$

No other terms $A^4\omega^2$ can be produced $\Rightarrow \lambda = 0$

Exercise $\quad$ Compute explicitly

$s \; \mathrm{tr}(\omega A^2 dA)$

$s \; \mathrm{tr}(\omega\, AdAA)$

$s \; \mathrm{tr}(\omega\, dAA^2)$

Remark $\quad$ We get terms with $\rightarrow$ 1 derivative and 3 A's

$\rightarrow$ 2 derivatives and 2 A's

Terms with one derivatives ($\sim \omega^2 dAA^2$ or $d\omega A\omega A^2$ etc)

cannot come from $\omega dAdA$ $\Rightarrow \alpha, \beta, \gamma$ must be to cancel this terms

or to sum up total derivatives

$\alpha = -\beta = \gamma = b \qquad$ (up to $b\, d\, \mathrm{tr}(\omega^2A^3)$)

$\Rightarrow$ implies also that the new terms sum to $\; b\, \mathrm{tr}(\omega\, d(A^3))$

Then we have to cancel the terms with two-derivatives

from the b-term $\rightarrow b\,\mathrm{tr}[-2d\omega\, d\omega\, A^2 + d\omega\, A\, d\omega\, A] + d\,\mathrm{tr}(\;)$

from $dA\,dA$ $\rightarrow \mathrm{tr}[d\omega\, d\omega\, A^2 - d\omega\, A\, d\omega\, A]$

Now $\mathrm{tr}[d\omega A\; d\omega A] = -\mathrm{tr}[d\omega A\; d\omega A] = 0$

~~Of~~ finally $\boxed{b = \frac{1}{2}}$

$$\Rightarrow \boxed{\mathcal{A}(\omega, A) = \frac{i}{24\pi^2}\int \mathrm{Tr}_R\, \omega\left(A\,dA + \frac{1}{2}A^3\right)}$$

coming back to "physical" variables

$$\boxed{\mathcal{A}_a = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A_\nu^b\,\partial_\rho A_\sigma^c - \frac{i}{4}A_\nu^b[A_\rho, A_\sigma]^c\right)D^R_{abc}}$$

Now we would like to solve in general the equation $s\mathcal{A} = 0$ $\mathcal{A} \neq sF$ in the space of local functionals $\Rightarrow$ of course the coefficient in front is not determined and it has to be fixed by Feynman diagrams (or other) considerations

Let us start by considering a gauge invariant polynomial in field strength F

$$P_m(F) = \mathrm{tr}F^m$$

any invariant polynomial is done by sums or products of P_m

P_m has two properties

- P_m is closed
- Integrals of P_m <u>are topological invariants</u> $\Rightarrow$ i.e. they are invariant against <u>deformations</u> of A that preserve the transition functions of the bund

Closure

$$dP_m = d\,\mathrm{Tr}\,F^m = m\,\mathrm{Tr}\,dF\,F^{m-1} \overset{\text{Bianchi}}{=} m\,\mathrm{Tr}\left\{[FA-AF]F^{m-1}\right\}=0$$

Invariance

Take two A', A_1 and A_0 with the same transition functions g_{ij}

$$P_m(F_1)-P_m(F_0)=dR \qquad R \text{ globally defined } 2m-1 \text{ form}$$

$$\int P_m(F_1)-\int P_m(F_0)=0$$

Simple : define $\begin{cases} A_t = A_0 + t(A_1-A_0) \\ F_t = dA_t + A_t^2 \end{cases}$ $t\in[0,1]$

check $\dfrac{dF_t}{dt} = D_t(A_1-A_0)$ D_t constructed with A_t

Then F_t has Bianchi $D_t F_t = 0$

$$\Rightarrow \frac{dP_m(F_t)}{dt} = m\,\mathrm{Tr}\left[D_t(A_1-A_0)F_t^{m-1}\right] = m\,d\,\mathrm{Tr}\left[(A_1-A_0)F_t^{m-1}\right]$$

Now $\mathrm{Tr}\left[(A_1-A_0)F_t^{m-1}\right]$ is globally defined being invariant when change chart ($F_t \to g^{-1}F_t g$ and also does (A_1-A_0)))

$$\Rightarrow R = m\int_0^1 dt\,\mathrm{Tr}\left[(A_1-A_0)F_t^{m-1}\right]$$

Definition of CS forms

Because $P_m(F)$ is closed locally

$$P_m(F) = d\varphi_{2m-1}(A,F) \qquad \text{(in a patch } U_i\text{)}$$

taking $A_0 = 0$

$$\varphi_{2m-1}(A,F) = m\int_0^1 dt\, t^{m-1}\, \mathrm{Tr}\left[A\left(F+(t-1)A^2\right)^{m-1}\right]$$

⇒ this is called CS $(2m-1)$ form !!

Of course it is defined up to ~~the~~ <u>exact forms</u>!

Exercise $\begin{cases} -\ \varphi_3, \varphi_5 \Rightarrow \\ -\ \text{check} \begin{cases} d\varphi_3 = P_2 \\ d\varphi_5 = P_3 \end{cases} \end{cases}$

Descent equations

Central point φ_{2m-1} <u>is not</u> gauge invariant but

$$\delta\varphi_{2m-1}(A,F) = d\varphi^1_{2m-2}(v,A,F)$$

In fact the variation (under gauge) of φ_{2m-1} is locally exact

$$d(\delta\varphi_{2m-1}) = \delta(d\varphi_{2m-1}) = \delta(P_m) = 0 \qquad !$$

⇒ $\delta\varphi_{2m-1}$ <u>is closed</u>

<u>Exercise</u> $\begin{cases} \text{given} \quad \delta A = dv + [A,v], \quad \delta F = [F,v] \\ \text{A)} \quad \varphi^1_2 = \mathrm{Tr}\, v\, dA + d\Omega \\ \text{B)} \quad \varphi^1_4 = \mathrm{Tr}\, v\left(A dA + \frac{1}{2}A^3\right) \end{cases}$

Remark δq_{2m-1} is linear a dv (commutator terms cancel)

$$\delta q_{2m-1}(A,F) = q_{2m-1}(A+dv,F) - q_{2m-1}(A,F)$$

and there is the useful formula (up exact terms)

$$q'_{2m-2} = m(m-1)\int_0^1 dt\,(1-t)\,\mathrm{tr}_2\, v\, d\left(A F_t^{m-2}\right)$$

$$= \mathrm{tr}_2\, v\, d\, q_{2m-2}$$

Exercise Prove that

$$\begin{cases} q_{2m-1}(A+\hat{v},F) - q_{2m-1}(A,F) = q^1_{2m-2}(\hat{v},A,F) \\ \hat{v} = dv \end{cases}$$

Ambiguity on q_{2m-2}

Defined up exact form $d\beta_{2m-3}$ but also up to a gauge variation of ~~of~~ a form α_{2m-2}

In fact $q_{2m-1} \sim q_{2m-1} + d\alpha_{2m-2}$

$$\delta q_{2m-1} \sim \delta q_{2m-1} + \delta d\,\alpha_{2m-2}$$
$$+ d(\delta\alpha_{2m-2})$$

Therefore

Take a characteristic class and its polynomial $P(F^n)$

$$\left(dP_m = \delta P_m = 0 \right.$$

Descent equations $\left\{ \begin{array}{l} P_m = dq_{2m-1} \\ \delta q_{2m-1} = dq^1_{2m-2} \end{array} \right.$

with $q'_{2m-2} \simeq q^1_{2m-2} + \delta\alpha_{2m-2} + d\beta_{2m-3}$

We have seen that in $D=4$ the anomaly is given by $\int q^1_4 \Rightarrow$

We ~~would~~ need the characteristic polynomial in six dimension $\Rightarrow$ 2 dimension more

We want to show that this is a general fact

Starts from $2d+2$ and take

$$I_{2d+2} = c\, P(F^{d+1}) \rightarrow \int q'_{2d} \quad \text{is the consistent anomaly}$$

Technically $\Rightarrow$ extend the space in $2d$ with two "fake" dimensions (ρ, θ), a disk s. that

$$A_\mu(x,\rho,\theta) \in M \times D \qquad A \text{ is a } 2d+2 \text{ form}$$

Descent equations provides a solution of $st=0$ $t \neq F$

$$\begin{array}{ll} P_{d+1} = \mathrm{Tr}\, F^{d+1} & \\ dP_{d+1} = s P_{d+1} & \Rightarrow \end{array} \qquad \begin{array}{l} P_{d+1} = dq_{2d+1} \\ s q_{2d+1} = dq'_{2d} \end{array}$$

Claim $\int q'_{2d}(\omega, A)$ is a representative of BRST cohomology class at ghost number one

$\Rightarrow \quad s\left(\int \varphi'_{2d}\right) = 0$ with $\int \varphi'_{2d} \neq s \int \alpha_{2d}$

<u>Proof</u> $\quad \boxed{s\varphi_{2d+1} = d\varphi'_{2d}}$

$$\Rightarrow \quad 0 = s(s\varphi_{2d+1}) = s\,d\varphi'_{2d} = -d(s\varphi'_{2d})$$

$\Rightarrow$ $s\varphi'_{2d}$ is a <u>closed form</u> $\Rightarrow$ in $2d+2$ $M \times D$

every form <u>closed is exact</u> !

$$\Rightarrow \quad s\varphi'_{2d} = d\alpha'_{2d-1} \quad \Rightarrow s\int \varphi'_{2d} = \int d\alpha'_{2d-1} = 0 \;!!$$

It can be also proved that

$$\boxed{\varphi'_{2d} \neq s\alpha_{2d}}$$

<u>At the end</u>

$$\mathcal{A}(\omega, A) = c\int \varphi'_{2d}(\omega, A) + sF$$ that follows from the descen

$$\cancel{I^{d+1}} \quad I_{2d+2} = c\, P_{d+1}(F^{d+1}) \quad \text{if } c=0 \;\; \mathcal{A} = sF$$

<u>Remark</u> <u>c is fixed</u> by the <u>quadratic terms</u> of the anomaly !!

Given a set of chiral fields ψ_i

$$I^{\text{total}}_{2d+2} = \sum_{\psi_i} I^{(i)}_{2d+2} \Rightarrow \quad I^{(i)}_{2d+2} = c(\psi_i)\, P_{d+1}(F^{d+1})$$

$$c = \frac{i}{24\pi^2} \text{ for positive chiral fermions} \quad I_6 = \frac{i}{24\pi^2}\mathrm{Tr}\, F^3$$

Remark

$\delta W = c\int Q'_{2d}$ $\rightarrow$ $d\beta_{2d-1}$ no effect

$\rightarrow$ $\delta\alpha_{2d}$ is a redefinition of W by local polynomial

Important

The characteristic polynomial you start is defined with no ambiguity $\Rightarrow$ different Q'_{2d} corresponds to the same polynomial $P_{d+1}(F)$ $\Rightarrow$ the existence and the properties of P_{d+1} characterises invariantly the anomalies

Relation between consistent anomalies and index theorem

We have seen that the solution of the consistency conditions is obtained starting from a characteristic polynomial (solution D=2d dimension)

$$\mathrm{Tr}\left[F^{d+1}\right]$$

⇒ this is the abelian anomaly in 2d+2 dimensions:

is there a relation between consistent anomaly in 2d and abelian anomaly in 2d+2

Yes! (Alvarez-Gaumé and Ginsparg)

The following discussion is based on a crucial observation

- The anomalous part of the Euclidean effective action for chiral fermions is contained in its immaginary part

Consider a positive chirality fermion $\gamma_5 \Psi_+ = \Psi_+$

In Euclidean space $\bar{\Psi}$ is not related to the hermitian conjugate of Ψ (in D=4 $SL(2,\mathbb{C}) \to SU(2)_R \otimes SU(2)_L$) and it is an independent negative chirality spinor

$$\begin{cases} S_E = \int d^d x \, \bar{\Psi} \, i \, D\!\!\!/_+ \Psi \\ D\!\!\!/_+ = \gamma^i (\partial_i - i A_i) \left(\frac{1+\gamma_5}{2} \right) \end{cases}$$

The functional

$$e^{-\Gamma[A]} = \int D\bar{\Psi} \, D\Psi \, e^{-\int d^d x \, \bar{\Psi} \, i D\!\!\!/_+ \Psi}$$

is actually ill-defined!

Be $H_\pm$ the Hilbert spaces of positive/negative chirality spinors

$\not{D}_+$ maps the Hilbert space H_+ in a different Hilbert space H_-

=> there is no canonical way to define a determinant (eigenvalue problem is ill-defined)

We can safely construct instead

$$(i\not{D}_+)^\dagger = i\left(\frac{1+\gamma_5}{2}\right)\not{D} = i\not{D}\left(\frac{1-\gamma_5}{2}\right) = \not{D}_- \qquad \not{D}_- : H_- \to H_+$$

and obtain (formally)

$$\det\left[(i\not{D}_-)(i\not{D}_+)\right] = \det\left[(i\not{D}_+)^\dagger \, i\not{D}_+\right] = \left|\det(i\not{D}_+)\right|^2$$

=> this is well defined, related to a good eigenvalue problem and gives the absolute value of $e^{-\Gamma[A]}$

=> the real part of $\Gamma[A]$

Alternatively => add a free negative chirality spinor and construct an operator mapping $H = H_+ \oplus H_-$ into itself

$$\hat{\not{D}} = \gamma^i\left(\partial_i + A_i\left(\frac{1+\gamma_5}{2}\right)\right) \Rightarrow e^{-\Gamma[A]} = \det(i\hat{\not{D}})$$

=> we have good eigenvalue problem but gauge invariance is lost! (The eigenvalue problem is not gauge invariant). Then

$$\hat{\not{D}} = \begin{pmatrix} 0 & \not{D}_+ \\ \not{D}_- & 0 \end{pmatrix} \Rightarrow (i\hat{\not{D}})^\dagger (i\hat{\not{D}}) = \begin{pmatrix} i\not{D}_+ i\not{D}_- & 0 \\ 0 & i\not{D}_- i\not{D}_+ \end{pmatrix}$$

$$\Rightarrow \left|\det(i\hat{\not{D}})\right|^2 = \det(i\not{D}_+ i\not{D}_-)\det(i\not{D}_- i\not{D}_+) = \text{const.}\det(i\not{D})$$

$\not{D}$ is the old good self-adjoint <u>Dirac</u> operator!

$$\Rightarrow \left|\det i\hat{\not{D}}\right| = (\det i\not{D})^{1/2}$$

Therefore we get

$$e^{-\Gamma[A]} = \det(i\hat{\not{D}}) = \left[\det(i\not{D})\right]^{1/2} e^{i\Phi(A)}$$

$\Phi(A)$ is the imaginary part of the euclidean effective action $\Rightarrow$ the anomaly resides there!

We could have expected this from euclidean continuation

$$i x^0 = z^1,\ x^1 = z^2, \dots, x^{p-1} = z^p$$

a p-form $\quad \xi = \frac{1}{p!}\xi_{\mu_1\dots\mu_p}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p} = \frac{1}{p!}\xi^E_{i_1\dots i_p}dz^{i_1}\wedge\dots\wedge dz^{i_p}$

because $\quad \partial_0 = i\partial_0^E = \partial_{z_1} \Rightarrow$ no i or change of sign

$\Rightarrow p = d \quad \displaystyle\int_{M_M}\xi = \int_{M_E}\xi^E$ $\quad$ Anomalies are written purely in terms of differential forms!

$$\Rightarrow e^{i\Gamma_M^{An.}} = e^{i\Gamma_E^{An}}$$

With this observation in mind we can derive a beautiful equation for $\Phi(A)$

Take our euclidean space-time to be a <u>sphere</u> S^{2d} (compactify $\mathbb{R}^{2d}$ to S^{2d})

The starting point is to extend the dependence of the gauge transformation on another parameter $\underline{\theta}$

$\theta \in [0, 2\pi]$ $\quad g(x) \to g(x,\theta)$ for $g \in G$ with a periodic dependence

$$\boxed{g(x,0) = g(x,2\pi)}$$

Basically we extend our gauge element from [illegible] maps $\underline{g : S^{2d} \to G}$ to $\underline{g : S^{2d+1} \to G}$

Then "gauge" transform the vector potential

$$A^{\theta} = g^{-1}(x,\theta)\,(d + A(x))\,g(x,\theta) \Rightarrow \quad d = dx^{\mu}\partial_{\mu} \text{ with no } \theta \text{ derivatives}$$

calculate the Dirac operator $\Rightarrow$ $\hat{\not{D}}(A^{\theta})$ that is <u>still</u> an operator on S^{2d}

Now we assume that $\not{D}(A)$ has no zero modes so we can write a non-trivial relation

$$\boxed{\det\left(i\hat{\not{D}}(A^{\theta})\right) = \det\left(i\not{D}(A)\right)^{1/2} e^{i\Phi(A^{\theta})}}$$

↑ no θ dependence from gauge <u>invariance</u>

The fact that $A(x,0) = A(x,2\pi)$ (we introduce $A(x,\theta)$) implies that the determinant has to be the same at 0 and 2π so that we have

$\phi(A^{\theta}) = \phi(A,\theta))$ $\qquad \Phi(A,2\pi) = \phi(A,0) + 2\pi m \qquad m \in \mathbb{Z}$

Clearly if Φ is gauge invariant $\Rightarrow$ $m=0$

anomaly $\Leftarrow$ $m \neq 0$

55

Therefore
$$\int_0^{2\pi} d\theta \, \frac{d\Phi}{d\theta}(1,\theta) = 2\pi m$$

and using the exact expr- of anomaly as functional variation

$$i\frac{d\phi[1,\theta]}{d\theta} = -\frac{\partial\Gamma}{\partial\theta} = -\int d^{2d}x \underset{\substack{\downarrow \\ \text{now we are} \\ \text{in curved space}}}{\sqrt{g}} \frac{\delta\Gamma(A^\theta)}{(\delta A^\theta)^a_\mu(x)} \frac{\partial (A^\theta)^a_\mu(x)}{\partial\theta}$$

Now using the fact that θ is a gauge variation

$$\begin{cases} \dfrac{\partial (A^\theta)^a_\mu}{\partial\theta} = \left(\partial_\mu v + [A^\theta_\mu, v]\right)^a = (D_\mu v)^a \\ v(\theta,x) = g^{-1}(\theta,x)\dfrac{\partial}{\partial\theta} g(\theta,x) \end{cases}$$

$$\Rightarrow \boxed{m = \frac{1}{2\pi i}\int_0^{2\pi} d\theta \int d^{2d}x \sqrt{g} \; v^a(x) \, D_\mu^a \frac{\delta\Gamma(A^\theta)}{\delta(A^\theta)^a_\mu}}$$

this is basically the anomaly for the test function v

$\Downarrow$

integrated appropriately is an integer m !!

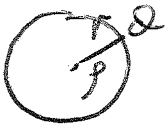
We would like now to compute the integer m : let us notice that it is -invariant under continuous deformations of $g(x,\theta)$ $\left(\underset{\text{same } m}{g_1(x,\theta) \to g_2(x,\theta)}\right)$

$m \simeq$ characterizes the $\Pi_{2d+1}(G)$

m is given also by a winding number around S^1 parametrized by ϑ: how to compute it

Introduce one more "fake" dimension

$A^\vartheta(x) \to A(x,\vartheta,\rho)$ $\quad \rho \in [0,1]$ and now ϕ depends on two-parameters describing a disk



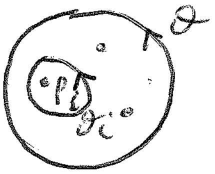
$A(x,\vartheta,1) = A^\vartheta$

Two results can be stated

i) $m = \int_0^{2\pi} d\vartheta \, \frac{\partial\phi(A,\vartheta)}{\partial\vartheta} \equiv \sum m_i$

where $\quad m_i = \oint_{P_i} d\vartheta^i \, \frac{\partial\phi(A,\vartheta_i,\rho_i)}{\partial\vartheta_i}$

where P_i are the points into the disk where $\det(i\not{D}(A(x,\vartheta,\rho)))$ vanishes!!



2) The zeros of $\det[i\hat{\not{D}}(A(x,\vartheta,\rho))]$ (point P_i) are in one to one correspondence with the zeros of a $(2d+2)$ Dirac operator such ~~the~~ that the winding number ± 1 equal ± 1 chirality modes.

$$\Rightarrow \boxed{\frac{1}{2\pi}\int_0^{2\pi} d\theta \, \frac{d\phi}{d\theta} = m = \mathrm{ind}\left(i \not{D}_{2d+2}\right)}$$

$$\left(\mathrm{ind}\, i\not{D}_{2d+2}\right) = \frac{1}{2\pi i}\int_0^{2\pi} d\vartheta \int d^{2d}x \sqrt{g} \;\; v(\vartheta)^a \left(D_\gamma^{\vartheta} \frac{\delta \Gamma(A^\vartheta)}{(\delta A^\vartheta)_\gamma}\right)_a$$

Technical points

Which kind of Dirac operator?
Which kind of 2d+2 manifold?

The choice of the manifold is $S^2 \times S^{2d}$ with

$$\begin{tikzcd}[column sep=small] S^2 = & S_+^2 \arrow[d] & + & S_-^2 \arrow[d] \\ & \mathrm{Disk}(\rho_+, \vartheta) & & \mathrm{Disk}(\rho_-, \vartheta) \end{tikzcd}$$

The choice of the gauge field

- on $S_+^2 \times S^{2d}$ $\quad A(x, \rho_+, \vartheta) = A_+(x, \rho_+, \vartheta) = f(\rho_+)\, g^{-1}(x,\vartheta)\left(A + d + d\vartheta \frac{\partial}{\partial\vartheta}\right) g$
- on $S_-^2 \times S^{2d}$ $\quad A(x, \rho_-, \vartheta) = A_-(x, \rho_-, \vartheta) = A(x)$

$$\text{with} \begin{cases} f(\rho_+) \sim \rho_+ & \rho_+ \to 0 \\ f(\rho_+) \sim 1 & \rho_+ \to 1 \end{cases} \Rightarrow \begin{array}{l} \text{overlap} \\ \text{transition} \\ \text{function} \end{array} \begin{array}{l} F_+ = g^{-1} F_- g \\ F_- = dA + A^2 \end{array}$$

Then index theorem

$$\mathrm{Ind}\left(i\not{D}_{2d+2}(A)\right) = \frac{(-i)^{d+1}}{(d+1)!\,(2\pi)^{d+1}} \int_{S^2 \times S^{2d}} \mathrm{tr}\left(F^{d+1}\right) \rightarrow \begin{array}{l}\text{characteristic} \\ \text{polynomial}\end{array}$$

Remark : We are in curved space and in general (we will see) there is another factor contributing to the index, related to the geometry ($\hat{A}(M)$)
For $S^2 \times S^{2d} \Rightarrow \hat{A}(M) = 1$

Then let us use the descent and Stoke's theorem

$$\mathrm{In}(i \not{D}_{2d+2}(A)) = \frac{(-i)^{d+1}}{(d+1)!(2\pi)^{d+1}} \left[\int_{S_+^2 \times S^{2d}} d\varphi_{2d+1}(A_+, F_+) + \int_{S_-^2 \times S^{2d}} d\varphi_{2d+1}(A_-, F_-) \right]$$

$\Rightarrow$
go to the boundary
the second vanishes (no $d\theta$!!)

$$= \frac{(-i)^{d+1}}{(d+1)!(2\pi)^{d+1}} \int_{S^1 \times S^{2d}} \varphi_{2d+1}(A_+, F_+)$$

but on S^1 A_+ and F_+ are the gauge transformed of A, F by $g(x,\theta)$!

$\Rightarrow$
descent

$$\int_{S^1 \times S^{2d}} \varphi_{2d+1}(A_+, F_+) = \int_{S^1 \times S^{2d}} \left[\varphi_{2d+1}(A^\theta + d\theta\, v^\theta, F^\theta) - \varphi_{2d+1}(A^\theta, F^\theta) \right]$$

added but vanishes again, no $d\theta$

Then ~~$\varphi^1_{2d}(A^\theta + d\theta\, v\theta, F^\theta)$~~

$$\varphi_{2d+1}(A^\theta + v^\theta d\theta, F^\theta) - \varphi_{2d+1}(A^\theta, F^\theta) =$$

$$\varphi^1_{2d}(v_\theta d\theta, A^\theta, F^\theta) = d\theta\, \varphi^1_{2d}(v^\theta, A^\theta, F^\theta)$$

$$\mathrm{Ind}(i\not{D}_{2d+2}) = \frac{(-1)^{d+1}}{(d+1)!\,(2\pi)^{d+1}} \int_{S^1} d\theta \int_{S^{2d}} \varphi^1_{2d}(v^\theta, A^\theta, F^\theta)$$

$$= \int_{S^1} d\theta \int d^{2d}x \sqrt{g}\, (v^\theta)_a \left(D^\theta_\gamma \frac{\delta\Gamma(A^\theta)}{\delta A^\theta_\gamma} \right)_a$$

The last relation is valid for any θ dependence ⇒ no integral over θ

$\Rightarrow A^\theta \to A$

$$\boxed{\delta_v \Gamma_E[A] = \int d^{2d}x \sqrt{g}\, v^a \left(D_\gamma \frac{\delta\Gamma[A]}{\delta A_\gamma^a} \right) = \frac{(i)^d}{(d+1)!} \frac{1}{(2\pi)^d} \int \varphi^1_{2d}(v, A, F)}$$

the solution of the cohomological problem!!

Lecture 4

We pass now to study chiral theory coupled to gravity and we want discuss anomalies in this context. Generically they are called gravitational anomalies

To convince you they exist let us study the simplest situation as possible ⇒ spin-$\frac{1}{2}$ Weyl fermions in $D=2$

$$\boxed{\mathcal{L} = \int d^2x\, (\det e)\, E^\nu{}_a \left(\frac{1}{2} \bar{\Psi} i \gamma^a \overleftrightarrow{\partial}_\nu \Psi \right)}$$

Randall Corlo's lectures

The space-time metric is $g_{\mu\nu}$ described in terms of local orthonormal frames $e^a{}_\mu$ $\quad g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$

E^μ_a is the inverse: $\begin{cases} E_a^{\ \mu}\, e^b_{\ \mu} = \delta_a^{\ b} \\ e_\mu^{\ a}\, E_a^{\ \nu} = \delta_\mu^{\ \nu} \end{cases}$

We consider weak-gravitational field $g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu} \Rightarrow$

we choose a gauge for which $\boxed{e_\mu^{\ a} = \eta_\mu^{\ a} + \frac{1}{2} h_\mu^{\ a}}$

$\Rightarrow$ the Lagrangian is $\mathcal{L} = \mathcal{L}_{free} + \Delta\mathcal{L}$

$$\Delta\mathcal{L} = -\frac{1}{2} h^{\mu\nu} T_{\mu\nu}$$

where $T_{\mu\nu} = \frac{1}{4} i \bar\psi (\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu)\psi$ energy-momentum tensor

Next impose Weyl $\gamma_5 \psi = -\psi$, $\gamma^5 = \gamma^0\gamma^1$

Light-cone coordinates: $x^\pm = \sqrt{\frac{1}{2}}(x^0 \pm x^1) \Rightarrow \gamma^\pm = \sqrt{\frac{1}{2}}(\gamma^0 \pm \gamma^1) \begin{cases} (\gamma^+)^2 = (\gamma^-)^2 = 0 \\ \gamma^+\gamma^- + \gamma^-\gamma^+ = 2 \end{cases}$

$\begin{cases} V^+ = V_- \quad V^- = V^+ \\ W^\mu W_\mu = W^+ V_+ + W^- V_- \end{cases}$

Fermions $\gamma_5\psi = -\psi \Rightarrow 0 = \gamma^+\psi = \gamma_-\psi \Rightarrow \overset{\text{eq. of motion}\ \downarrow}{0 = (\gamma_+\partial_- + \gamma_-\partial_+)\psi}$

$= \partial_-\psi = 0$

The only non vanishing component is

$$\boxed{T_{++} = \frac{1}{2} i \bar\psi \gamma_+ \overleftrightarrow{\partial}_+ \psi}$$

$$\boxed{\Delta\mathcal{L} = -h_{--} \frac{1}{4} i \bar\psi \gamma_+ \overleftrightarrow{\partial}_+ \psi}$$

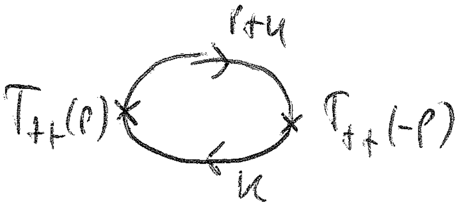
Effective action for the metric => two-point function

$$W[p] = \int d^2x\, e^{ipx} \langle 0 | T(T_{++}(x)\, T_{++}(0)) | 0 \rangle$$

We see immediately there should be an anomaly => conservation law for T_{++}

$$\partial_- T_{++} = 0 \quad \Rightarrow \quad \underset{\forall p_-}{p_- W = 0} \quad \Rightarrow W[p] = 0$$

Not possible (two-point function)



Compute $W[p]$ (exercise)

1) Performing Dirac algebra obtains

$$\boxed{W[p] = -\frac{1}{4} \int \frac{dk_+ dk_-}{(2\pi)^2} (2k+p)_+^2 \frac{1}{\left(k_- + i\frac{\epsilon}{k_+}\right)} \frac{1}{(k+p)_- + \frac{i\epsilon}{(k+p)_+}}}$$

2) Perform integral by residues: perform k_- integral that gives zero unless the poles are opposite $\Rightarrow p_+ > 0 \Rightarrow 0 < k_+ < -p_+$

3) $$W[p] = \frac{i}{24\pi} \frac{p_+^3}{p_-} \quad \Rightarrow \quad p_- W[p] = \frac{i}{24\pi} p_+^3 \;!!$$

4) The effective action at this order is ($-\frac{1}{2}h_{--}$ at vertices, $\frac{1}{2}$ bose)

$$\Gamma(h_{\mu\nu}) = -\frac{1}{192\pi} \int d^2p\, \frac{p_+^3}{p_-} h_{--}(p)\, h_{--}(-p)$$

5) Check general covariance ($\delta x^\mu = \epsilon^\mu$)

$$\delta h_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$$

$$\Rightarrow \text{momentum space} \quad \begin{cases} \delta h_{++} = -2ip_+\epsilon_+ \\ \delta h_{--} = -2ip_-\epsilon_- \\ \delta h_{+-} = -ip_-\epsilon_+ - ip_+\epsilon_- \end{cases}$$

Our expression is not invariant! The most general local counterterm we can add is

$$\Delta\Gamma = \int d^2p \, [A p_+^2 h_{--}(p) h_{+-}(-p) + B p_+ p_- h_{+-}(p) h_{+-}(-p)$$
$$+ C p_+ p_- h_{++}(p) h_{--}(-p) + D p_-^2 h_{++}(p) h_{+-}(-p)]$$

⇒ Prove that no choice of A, B, C, D cure the anomaly that is relevant !!

To discuss anomalies for chiral matter coupled to gravity we need some formalism, especially to relate the algebraic methods to gravity

We shall consider different type of tensor

coordinate tensor $A^{\mu\nu}{}_\rho$, $g_{\mu\nu}$, …

frame tensor $A^{ab}{}_c$, η_{ab}

$$\begin{cases} g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \\ E_a{}^\mu e_\nu{}^b = \delta_a{}^b \\ e^a_\mu E_a{}^\nu = \delta_\mu{}^\nu \end{cases}$$

related by the vielbein $A^{ab}{}_c = e^a_\mu e^b_\nu E_c{}^\rho A^{\mu\nu}{}_\rho$

Consider the relevant covariant derivatives

- frame tensor $\boxed{D = d + [\omega, \cdot]}$

where ω is a connection one-form matrix-valued

$$\omega^a{}_b = (\omega_\mu)^a{}_b \, dX^\mu$$

if p is the form degree of $A^{ab}{}_c$ $\quad A^{ab}{}_c = A^{ab}{}_{c\mu_1\ldots\mu_p} dx^{\mu_1}\ldots dx^{\mu_p}$

$$\boxed{(DA)^{ab}{}_c = dA^{ab}{}_c + \omega^a{}_d A^{db}{}_c + \omega^b{}_d A^{ad}{}_c - (-1)^p A^{ab}{}_d \, \omega^d{}_c}$$

ω is the analog of $(A^a T_a)^i{}_e$ (as a matrix is an element of $SO(D-1,1)$ or $SO(D)$ algebra)

- Coordinate tensor

$$\nabla = d + [\Gamma, \cdot]$$

where $\Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu\lambda} dx^\lambda$

$$\boxed{(\nabla A)^{\mu\nu}{}_\rho = dA^{\mu\nu}_\rho + \Gamma^\mu{}_\sigma A^{\sigma\nu}{}_\rho + \Gamma^\nu{}_\sigma A^{\mu\sigma}{}_\rho - (-1)^p A^{\mu\nu}{}_\sigma \Gamma^\sigma{}_\rho}$$

Compatibility of D and ∇ means that

$$(DA)^{ab}{}_c = e^a{}_\mu e^b{}_\nu E^\lambda{}_c (\nabla A)^{\mu\nu}{}_\lambda$$

$\Rightarrow \quad de^a{}_\mu + \omega^a{}_b e^b{}_\mu - e^a{}_\rho \Gamma^\rho{}_\mu = 0$

$\quad \Rightarrow \quad \omega^a{}_b = -E_b{}^\mu \nabla e^a{}_\mu$

zero-torsion condition $\Rightarrow$
Γ is Levi-Civita connect. -
$\mu\nu$ symmetric
$\omega^a{}_{b\mu}$ is the spin-connection

$\Rightarrow \nabla g_{\mu\nu} = 0$

- Definition of curvature two-form (from Riemann Tensor)

$$\begin{cases} \text{frame:} & R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b \Rightarrow R = d\omega + \omega^2 \ \text{(matrix form)} \\ \text{Coordinate:} & R^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\lambda \Gamma^\lambda{}_\nu \Rightarrow R^\mu{}_\nu = E^\mu_a e^b_\nu R^a{}_b \end{cases}$$

Relevant Symmetries

Diffeomorphism group

$$x^\alpha \to x'^\alpha = x^\alpha - \xi^\alpha(x)$$

We want to describe the transformation on tensors

$$T(x) \to T'(x)$$

$\Rightarrow$ We need Lie derivative on tensor p-valued forms

$$\mathcal{L}_\xi T^{\alpha\ldots}_{\beta\ldots} = \Big[\xi^\nu \partial_\nu T^{\alpha\ldots}_{\beta\ldots,\mu_1\ldots\mu_p} - \partial_\nu \xi^\alpha T^{\nu\ldots}_{\beta\ldots,\mu_1\ldots\mu_p} - \text{all upper indexes}$$

$$+ T^{\alpha\ldots}_{\nu\ldots,\mu_1\ldots\mu_p} \partial_\beta \xi^\nu + \text{all lower index}$$

$$+ T^{\alpha\ldots}_{\beta\ldots,\nu\ldots\mu_p} \partial_{\mu_1} \xi^\nu + \text{all lower index} \Big] \frac{1}{p!} dx^{\mu_1} \ldots dx^{\mu_p}$$

"Matrix" notation $\mathcal{L}_\xi T = \xi \cdot \partial T - [\partial \xi, T]$

and $[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1, \xi_2]}$ $\boxed{[\xi_1, \xi_2]^\nu = \xi_1^\mu \partial_\mu \xi_2^\nu - \xi_2^\mu \partial_\mu \xi_1^\nu}$

Einstein transformation

$$T'^{\alpha\ldots}_{\beta\ldots\|\mu_1\ldots\mu_p}(x) - T^{\alpha\ldots}_{\beta\ldots\|\mu_1\ldots\mu_p}(x) = \mathcal{L}_\xi T^{\alpha\ldots}_{\beta\ldots\|\mu_1\ldots\mu_p}(x)$$

$$\delta^E_\xi T(x) = \mathcal{L}_\xi T(x)$$

We need the transformation on the two-form R

$$\delta^E_\xi R = \mathcal{L}_\xi R$$

The Levi-Civita connection transform instead as

$$\delta^E_\xi \Gamma = \mathcal{L}_\xi \Gamma + dv_\xi \qquad (v_\xi)^\alpha{}_\beta = \frac{\partial \xi^\alpha}{\partial x^\beta}$$

Lorentz transformation

The vielbein transforms under local Lorentz transformations

$$e^a_\mu \to (L^{-1})^a{}_b(x)\, e^b_\mu \qquad L^a{}_b(x) \in \begin{matrix} SO(1,D-1) \\ SO(D) \end{matrix}$$

Matrix notation

$$\left\{ \begin{matrix} e \to L^{-1} e & \omega \to L^{-1}(\omega + d)L \\ E \to E L & D \to L^{-1} D L \\ & R \to L^{-1} R L \end{matrix} \right.$$

Infinitesimal

$L = 1 + v \qquad L^a{}_b = \delta^a{}_b + v^a{}_b(x) \qquad v^a{}_b$ antisymmetric

$$\delta^L_v e^a_\mu = -v^a{}_b e^b_\mu \Rightarrow \begin{matrix} \delta^L_v e = -v e \\ \delta^L_v E = E v \\ \delta^L_v \omega = D v \to \end{matrix} \qquad \begin{matrix} \delta^L_v R = [R, v] \\ \\ dv^a{}_b + v^a{}_b \omega^b{}_c - \\ \omega^a{}_c v^c{}_b \end{matrix}$$

Consider now the consequences of Einstein and Local Lorentz symmetries on the action of matter coupled to the background geometry ⑥⑥

$$S_m = \int d^d x \sqrt{|g|} \mathcal{L}_M$$

The variation with respect $g_{\mu\nu}$ gives the energy-momentum tensor of the matter system

$$\frac{\delta S_m}{\delta g_{\mu\nu}} = \frac{1}{2} \sqrt{g} T^{\mu\nu} \quad \Rightarrow \quad \begin{array}{ll} 1) \ T^{\mu\nu} = T^{\nu\mu} & \text{symmetry} \\ 2) \ \nabla_\mu T^{\mu\nu} = 0 & \text{conservation} \end{array}$$

In the frame formalism we have instead

$$\frac{\delta S_m}{\delta e^a_\mu} = \det e \ T^\mu{}_a$$

$$\Rightarrow T^{\mu\nu} = \frac{1}{2} \left(T^\mu{}_a E^{a\nu} + T^\nu{}_a E^{a\mu} \right)$$

Let us discuss a fermionic action

We need fermions to get anomalies (not only fermions, also other fields can have anomalies ... chiral fields): fermions are represented by spinors

$\Rightarrow$ Representations of Lorentz group

$\Rightarrow$ we have to refer to a Dirac action in the tangent frame $\Rightarrow$ we need vielbein

$\Rightarrow$ Fermionic action is both invariant under diffeomorphisms and Local Lorentz transformations

Dirac operator on curved manifold

$$\not{D} = \gamma^\mu(x) D_\mu = E_a{}^\mu \gamma^a D_\mu$$

The covariant derivative is $D_\mu = \partial_\mu + \omega_\mu$ where

$$\boxed{\omega_\mu = \omega^a{}_{b\mu} \frac{\sigma_a{}^b}{2}}$$

$\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$

generators of spin representation of Lorentz group

Spinors transform as

$$\begin{cases} \psi(x) \to \rho^{-1}(L(x))\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}(x)\rho(L(x)) \\ \rho(L) \simeq 1 + \frac{1}{2}\alpha_{ab}\sigma^{ab} \end{cases}$$

$$\omega_\mu \to \rho^{-1}\omega_\mu\rho + \rho^{-1}\partial_\mu\rho \Rightarrow \not{D} \to \rho^{-1}\not{D}\rho$$

Therefore

$$\int d^Dx \, \det(e) \, \bar{\psi} \, i \not{D} \psi$$

is invariant under Local Lorentz. Moreover ψ and $\bar{\psi}$ are scalars under coordinate transformations and $\not{D}$ too

Chiral spinors: just introduce $P_\pm$ into $\not{D}$ $\quad \not{D} \to \not{D} P_\pm \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$

We have now <u>classical invariance</u>

Let us write now

$$\det(e)\mathcal{L} = \frac{1}{2}\det(e)\bar{\Psi} i\left(\gamma^\mu D_\mu - \overleftarrow{D}_\mu \gamma^\mu\right)\Psi$$

with the left covariant derivative

We have the energy momentum tensor for the chiral theory

$$\det(e)T^\mu{}_a = \frac{\delta S}{\delta e^a_\mu} = \frac{1}{2}\det(e)\bar{\Psi} i\left(\gamma_a D^\mu - \overleftarrow{D}^\mu \gamma_a\right)P_+\Psi$$

$$\Rightarrow \boxed{T^{\mu\nu} = \frac{1}{4}\bar{\Psi}\left(\gamma^\mu D^\nu - \overleftarrow{D}^\nu \gamma^\mu\right)P_+\Psi + \mu \leftrightarrow \nu}$$

This is the symmetric part. The antisymmetric will vanish as a consequence of local Lorentz symmetry

$$\delta^L_\alpha S = \int d^Dx \det e\, T^\mu{}_a\, \delta^L_\alpha e^a_\mu$$

$$= \int d^Dx \det e\, T^\mu{}_a\, \alpha^a{}_b\, e^b_\mu \qquad \alpha_{ab} = -\alpha_{ba}$$

$$= \int d^Dx \det e\, \alpha_{ab} T^{ab} = 0 \Rightarrow T^{ab} = T^{ba}$$

Then the fermionic action is invariant under Einstein transformations

$$\delta^E_\xi e^a_\mu = \xi^\nu \nabla_\nu e^a_\mu + e^a_\nu \nabla_\mu \xi^\nu$$

$$\delta^E_\xi \omega^a{}_{b\mu} = \xi^\nu \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{b\nu}\partial_\mu \xi^\nu$$

$$\delta^E_\xi \Psi = \xi^\nu \partial_\nu \Psi \qquad \delta^E_\xi \bar{\Psi} = \xi^\nu \partial_\nu \bar{\Psi}$$

$$\Rightarrow \quad \delta^E_\xi S = \int d^dx \left(e^a_\nu \nabla_\nu \xi^\nu + \xi^\nu \nabla_\mu e^a_\mu\right)(\det e)T^\mu{}_a$$

Integrating by parts and using $\omega^a{}_b = E_b{}^\mu D e_\mu{}^a$

$$\Rightarrow \quad \delta^E_\xi S = -\int d^dx (\det e)\, \xi^\nu \left(D_\mu T^{\mu\nu} - \omega_{ab\nu} T^{ab} \right)$$

Lorentz $\Rightarrow$ T^{ab} symmetric

$$\boxed{D_\mu T^{\mu\nu} = 0}$$

Remark: conservation needs both Einstein and Lorentz!!

Gravitational anomalies

We have to consider now the quantum effective action (Riemannian manifold)

$$Z[e^a_\mu] = e^{-W[e^a_\mu]} = \int D\psi D\bar\psi \, e^{-\int d^Dx \det e \, \mathcal{L}_\psi}$$

Now we have

$$\boxed{\frac{\delta W[e^a_\mu]}{\delta e^a_\mu} = \det(e) \langle T^\mu{}_a \rangle}$$

antisymmetric part

Lorentz anomaly $\quad \boxed{\delta^L_\alpha W[e^a_\mu] = \int d^Dx \det(e)\, \alpha_{ab} \langle \hat{T}^{ab} \rangle = G^L(\alpha)}$

Einstein anomaly $\quad \boxed{\delta^E_\xi W[e^a_\mu] = \int d^Dx \det(e)\, \xi^\nu \left(\langle D_\mu T^\mu{}_\nu \rangle - \omega_{ab\nu} \langle \hat{T}^{ab} \rangle \right) = G^E(\xi)}$

Lorentz anomaly $\Rightarrow$ antisymmetric part of e-m. tensor $\langle \hat{T}^{ab} \rangle \neq 0$

Einstein anomaly $\Rightarrow$ $D_\mu \langle T^{\mu\nu} \rangle \neq 0$

Remark Even if $G^E(\xi)=0$ the e.m. tensor is *not* conserved if $G^L(\alpha)\neq 0$!! But not a signal of breakdown of general coordinate invariance (energy flow for degrees of freedom in the direction specified by the vielbein)

Gravitational anomalies can be computed as *local polynomial* in the connections and the curvature

- perturbation theory (linearizing coupling to gravity)
- Heat-kernel methods
- Fujikawa approach

Depending on the method appear or Lorentz anomaly or Einstein anomaly $\Rightarrow$ let us look at a *cohomological approach*

Consistency condition

We have to define the functional generators of Lorentz and Einstein transformations on functional of (e,ω) or (e,Γ) in matrix notation

Lorentz $$\boxed{\delta^L_\alpha = \int d^Dx \left[-\alpha e \frac{\delta}{\delta e} + D\alpha \frac{\delta}{\delta \omega} \right]}$$

Einstein $$\boxed{\delta^E_\xi = \int d^Dx \left[(i_\xi de + e v_\xi)\frac{\delta}{\delta e} + [(i_\xi d + d i_\xi)\Gamma + D v_\xi]\frac{\delta}{\delta \Gamma} \right]}$$

where $\mathcal{L}_\xi e = i_\xi de + e v_\xi$; $\delta_\xi \Gamma = \mathcal{L}_\xi \Gamma + d v_\xi = (i_\xi d + d i_\xi)\Gamma + D v_\xi$

where $(v_\xi)^\alpha{}_\beta = \partial_\beta \xi^\alpha$

on p-forms $$i_X \omega = \frac{1}{p!}\sum_{s=1}^{p} X^{\mu_s}\Omega_{\mu_1\cdots\mu_s\cdots\mu_p}(-1)^{-s-1}\, dx^{\mu_1}\underset{\uparrow}{\cdots} dx^{\mu_p}$$

tolle $d\hat{x}^{\mu_s}$

$$= \frac{1}{(p-1)!} X^\nu \omega_{\nu\mu_2\cdots\mu_p}\, dx^{\mu_2}\cdots dx^{\mu_p}$$

The important point is to derive the algebra!

$$\begin{cases} [\delta^L_{\alpha_1},\delta^L_{\alpha_2}] = \delta^L_{[\alpha_1,\alpha_2]} \\ [\delta^E_{\xi_1},\delta^E_{\xi_2}] = \delta^E_{[\xi_2,\xi_1]} \\ [\delta^L_\alpha,\delta^E_\xi] = \delta^L_{\xi\cdot\partial\alpha} \end{cases}$$

$$[\xi_1,\xi_2]_\mu = \xi_1^\nu\partial_\nu\xi_{2\mu} - \xi_2^\nu\partial_\nu\xi_{1\mu}$$

Then we get the anomalies

$$\begin{cases} \delta^L_\alpha W[e,\omega] = G^L(\alpha,\omega) \\ \delta^E_\xi W[e,\Gamma] = G^E(v_\xi,\Gamma) \end{cases}$$

consistency condition $\Rightarrow$

$$\begin{cases} \delta^L_{\alpha_1} G^L(\alpha_2,\omega) - \delta^L_{\alpha_2} G^L(\alpha,\omega) = G^L([\alpha_1,\alpha_2],\omega) \\ \delta^E_{\xi_1} G^E(\xi_2,\Gamma) - \delta^E_{\xi_2} G^E(v_\xi,\Gamma) = G^E(v_{[\xi_1,\xi_2]},\Gamma) \\ \delta^L_\alpha G^E(v_\xi,\Gamma) - \delta^E_\xi G^L(\alpha,\omega) = G^L(\xi\cdot\partial\alpha,\omega) \end{cases}$$

Remark: the algebra is interrelated : this has a far reaching consequence

=> There exists a local , non polynomial counterterm S'_{BZ} for the quantum effective action W that shifts from pure Einstein anomaly to pure Lorentz anomaly and viceversa

=> they are different aspects of the same phenomenon!

Briefly

Solution of consistency condition : in analogy with the gauge case the consistency condition is transformed into a cohomological problem by defining a suitable BRST operator => one obtains, starting from an invariant polynomial in 2d+2 dimensions a series of descent equations leading to a solution. Starting from $P(R^{d+1})$ $(D = 2d)$

$$\begin{cases} P(R^{d+1}) = d\,Q_{2d+1} \\ s\,Q_{2d+1} = d\,Q^{1}_{2d} \\ s\,Q^{1}_{2d} = d\,Q^{2}_{2d-1} \end{cases} \Rightarrow G = \int Q^{1}_{2d}$$

Then depending if we specify $P(R^{d+1})$ in terms of

- coordinate frame $\overset{\text{we get}}{\Rightarrow}$ Einstein anomaly
- Lorentz frame Lorentz anomaly

$$G^E(v_3, \Gamma) = \int \varphi^1_{2d}(v_3, \Gamma, R)$$

$$G^L(\alpha, \omega) = \int \varphi^1_{2d}(\alpha, \omega, R)$$

The important point is to know the polynomial to start with $\Rightarrow$ pert. theory or index theorem

To be concrete

CS form

$$\begin{cases} \varphi_{2d+1}(\Gamma, R) = \overset{\substack{\text{dimension!} \\ \downarrow}}{(d+1)} \int_0^1 dt \, P(\Gamma, R_t^d) \\ \varphi_{2d+1}(\omega, R) = (d+1) \int_0^1 dt \, P(\omega, R_t^d) \end{cases} \begin{cases} R_t = tR + (t^2 - t)\Gamma^2 \\ R_t = tR + (t^2 - t)\omega^2 \end{cases}$$

Anomaly

$$\begin{cases} \varphi^1_{2d}(v_3, \Gamma, R) = (d+1)d \int_0^1 dt \, (1-t) \, P(v_3, d(\Gamma, R_t^{d-1})) \\ \varphi^1_{2d}(\alpha, \omega, R) = (d+1)d \int_0^1 dt \, (1-t) \, P(\alpha, d(\omega, R_t^{d-1})) \end{cases}$$

Take $d=1$ and we observe anomalies in $D=2$ dimensions

$$P(R) = -\frac{1}{96} \operatorname{Tr} R^2$$

CS terms

$$\begin{cases} \varphi_3 = \frac{1}{96\pi} \mathrm{Tr}\left[\omega d\omega + \frac{2}{3}\omega^3\right] \\ \varphi_3 = \frac{1}{96\pi} \mathrm{Tr}\left[\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right] \end{cases}$$

Anomalies

$$\begin{cases} \varphi'_2 = \frac{1}{96\pi} \mathrm{Tr}\left[\alpha d\omega\right] \\ \varphi'_2 = \frac{1}{96\pi} \mathrm{Tr}\left[v_\xi d\Gamma\right] \end{cases}$$

BZ counterterm

Define an interpolating vielbein (consider e as a matrix of $GL(2d)$)

$$e_t = \mathbb{1} + t(e - \mathbb{1}) \quad \to \begin{array}{l} t=0 \Rightarrow \mathbb{1} \\ t=1 \Rightarrow e \end{array} \qquad I = [0,1]$$

$$S_{BZ} = \int_{M_{2d}\times I} \varphi_{2d+1}\left(\omega(e_t) + v_t\right)$$

$$\begin{array}{l} \omega(e_t) = e_t^{-1}(\omega + d)\, e_t \\ v_t \;=\; e_t^{-1} \cdot \frac{\partial e_t}{\partial t} \end{array} \Rightarrow \begin{array}{ll} t=0 & = \omega \\ t=1 & = e^{-1}(\omega+d)\, e = \Gamma \end{array}$$

$$\begin{cases} \delta_\xi^E S_{BZ} = \int_{M_{2d}} \varphi'_{2d}(v_\xi, \Gamma) = -G^E(v_\xi, \Gamma) \\ \delta_\xi^L S_{BZ} = \int_{M_{2d}} \varphi'_{2d}(\alpha, \omega) = G^L(\alpha, \omega) \end{cases}$$

Fundamental point: the choice of the invariant polynomial **to start with!**

This can be obtained by using the index theorem in $2d+2$ dimension

$$\text{Index}\left(i\not{D}_{2d+2}\right) = \int_{S^2\times M_{2d}} \hat{A}(M_{2d})_{2d+2}$$

where $\hat{A}(M)$ is called Dirac genus

$$\hat{A}(M) = \prod_a \frac{x_a/2}{\sinh(\frac{x_a}{2})} \qquad \frac{R_{ab}}{2\pi} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_d \\ & & & -x_d & 0 \end{pmatrix}$$

↘ skew eigenvalue of R_{ab}

where R_{ab} is the two-form curvature considered as a matrix in $SO(2d)$: every x_a is a two-form!

$\hat{A}(M)$ can be expanded to arbitrary order in the curvature

$$\hat{A}(M) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{Tr}R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}\left(\mathrm{Tr}R^2\right)^2 + \frac{1}{360}\mathrm{Tr}R^4\right]$$

$$+\frac{1}{(4\pi)^6}\left[\frac{1}{10368}\left(\mathrm{Tr}R^2\right)^3 + \frac{1}{4320}\mathrm{Tr}R^2\,\mathrm{Tr}R^4 + \frac{1}{5670}\mathrm{Tr}R^6\right] + \dots$$

- Traces are in the fundamental of $SO(2d)$

$$\int_{S^2\times M_{2d}} \hat{A}(M)_{2d+2} \propto \text{(pick the } 2d+2 \text{ form)}$$

$$\equiv \int_{S^2\times M_{2d}} P(R^{d+1})$$ → the correct invariant polynomial leading to Q^1_{2d} !!

Important point

Notice that $\hat{A}(M)$ is constructed only in terms of forms of degree that are multiples of 4 (in fact traces of odd numbers of R vanishes due to antisymmetry of Tas, antisymmetric for SO(2d))

⇒ the polynomial ~~vanishes~~ is not zero when $2d+2=4(k+1)$

$$\Rightarrow \boxed{2d=4k+2}$$

we have gravitational anomalies in $D=2$, $D=6$, $D=10$ ...

No gravitational anomaly in $D=4$!!

Mixed gauge-gravitational anomalies

It is important also to explore anomalies when both gauge and gravity fields are present ⇒ mixed anomalies

The anomaly is encoded into the index of the 2d+2 Dirac operator

$$\mathrm{Ind}\,(i\not{D}(A,\omega)_{2d+2}) = \int \left[\hat{A}(M)\,\mathrm{ch}(-F)\right]_{2d+2}$$

$$\mathrm{ch}(-F) = \mathrm{Tr}_R\left(\exp -\frac{iF}{2\pi}\right)$$

↳ Dirac genus and gauge characteristic polynomial generator (Chern character)

⇒ one again obtains the descent with respect the index density

Consider Lorentz anomaly

$$(\delta^G + \delta^L)\, W[A,e] = A(v,A,\alpha,\omega) \sim \int Q^1_{2d}$$

Let us discuss in details D=4

$$2\pi\left[\hat{A}(M)\,\mathrm{ch}(-F)\right]_6 = \left[\left(1+\frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{Tr}R^2\right)\left(1-\frac{i}{2\pi}\mathrm{tr}_R F - \frac{1}{2(2\pi)^2}\mathrm{Tr}_R F^2 + \frac{1}{6}\frac{i}{(2\pi)^3}\mathrm{Tr}_R F^3\right)\right]_6$$

$$= \frac{1}{24\pi^2}\mathrm{Tr}_R F^3 - \frac{1}{192\pi^2}\mathrm{Tr}_R F\,\mathrm{Tr}R^2$$

usual gauge anomaly ↖

↗ mixed anomalies

<u>Remark</u> Simple Lie algebra $\mathrm{Tr}_R F = 0$ ⇒ only U(1) part contributes

$\mathrm{Tr}_R F = i \sum_{i,\gamma} q_i^{(\gamma)} F^{(\gamma)}$ q_i are the U(1) charges of the chiral fields coupled to the $A^{(\gamma)}$ gauge fields

The polynomial is

$$-\frac{1}{192\pi^2}\sum_{i,\gamma} q_i^{(\gamma)} F^{(\gamma)}\,\mathrm{Tr}\,R^2$$

⇒ two different form of the total anomalies

$$-\frac{1}{192\pi^2}\sum_{i,\gamma} q_i^{(\gamma)} F^{\gamma}\,\mathrm{Tr}R^2 = \begin{array}{l} \nearrow\ d\left(-\frac{1}{192\pi^2}\left(\sum_{i,\gamma} q_i^{(\gamma)} A^{(\gamma)}\,\mathrm{tr}R^2\right)\right) \\ \searrow\ d\left(\frac{1}{192\pi^2}\sum_{i,\gamma} q_i^{(\gamma)} F^{(\gamma)}\,\mathrm{Tr}\left(\omega d\omega + \frac{2}{3}\omega^3\right)\right) \end{array}$$

leading to

$$\delta\left(-\frac{1}{192\pi^2}\sum_{i,\gamma} q_i^{(\gamma)} A^{(\gamma)}\,\mathrm{tr}_2R^2\right) = d\left(-\frac{1}{192\pi^2}\sum_{i,\gamma} q_i^{(\gamma)}\varepsilon^{(\gamma)}\,\mathrm{tr}R^2\right)$$

or

$$\delta\left(-\frac{1}{192\pi^2}\sum_{i,\gamma} q_i^{(\gamma)} F^{(\gamma)}\,\mathrm{tr}\left[\omega d\omega + \frac{2}{3}\omega^3\right]\right) = d\left(-\frac{1}{192\pi^2}\sum_{i,\gamma} q_i^{(\gamma)} F^{\gamma}\,\mathrm{Tr}(\alpha d\omega)\right)$$

⇒ we can obtain effective actions that are <u>Lorentz</u> or <u>gauge</u> invariant

$$\delta W^{(1)} = -\frac{1}{192\pi^2}\int \sum_{i,\gamma} q_i^{(\gamma)}\epsilon^{(\gamma)}\,\mathrm{Tr}R^2 \qquad \text{pure gauge}$$

$$\delta W^{(2)} = -\frac{1}{192\pi^2}\int \sum_{i,\gamma} q_i^{(\gamma)} F^{(\gamma)}\,\mathrm{Tr}[\alpha\, d\omega] \qquad \text{pure Lorentz}$$

We can pass from one to the other by local counterterm

$$-\frac{1}{136\pi^2}\int \sum_{i\gamma} q_i^{(\gamma)} A^{(\gamma)} \, \mathrm{tr}\left[\omega d\omega + \frac{2}{3}\omega^3\right]$$

and also interpolates

<u>Important</u> We cannot cancel the two anomalies simultaneously because the invariant polynomial is different <u>from zero</u>!

<u>Unless</u> $\sum_i q_i^{(\gamma)} = 0 \quad \forall \gamma$ (no charge!!)

<u>Other fields with anomalies</u>

- → chiral spin $\frac{3}{2}$ gravitino
- ↘ self or anti self dual tensor → coupled to gravity they give anomaly in $D = 4k+2$

Chiral gravitino ok, but why the tensor? in $4k+2$ this field is constructed by the product of two spin $\frac{1}{2}$ fields of the <u>same chirality</u>

$$\begin{cases} H^{\mu_1 \dots \mu_d} = \frac{1}{d!} \varepsilon^{\mu_1 \dots \mu_d}{}_{\mu_{d+1} \dots \mu_{2d}} H^{\nu_1 \dots \nu_d} \\ \\ H^{\mu_1 \dots \mu_d} \sim \bar{\psi} \gamma^{[\mu_1} \dots \gamma^{\mu_d]} \chi \end{cases}$$

if d is odd they must have the same chirality!

Let us construct the appropriate characteristic polynomials

Positive chirality gravitino

Spin $\frac{1}{2}$ with vector index $\qquad \frac{1}{2}\otimes 1 = \underset{\text{this is}}{\frac{3}{2}} \oplus \frac{1}{2}$

The extra vector index is like an additional SO(2d) gauge symmetry

⇒ index density improvement $\qquad \mathrm{tr}\exp\left[\frac{i}{2\pi}\frac{1}{2}R_{ab}T^{ab}\right] = \mathrm{tr}\left(e^{\frac{i}{2\pi}R}\right)$

$$(T^{ab})_{cd} = \delta^a{}_c\delta^b{}_d - \delta^a{}_d\delta^b{}_c \qquad \text{(Vector reps)}$$

$$\mathrm{ind}(i\not{D}_{3/2}) = \int_M \Big[\hat{A}(M)\big(\mathrm{tr}\, e^{\frac{iR}{2\pi}} \underset{\text{subtracts of } \frac{1}{2} \text{ part}}{-\ 1}\big)\mathrm{ch}(-F)\Big]_{2d+2}$$

Self-dual rank d tensor (no gauge)

We have to consider again an extra term

$$\mathrm{tr}\left[\exp\left(\frac{i}{2\pi}\frac{1}{2}R_{ab}T^{ab}\right)\right] \qquad \text{with} \qquad T^{ab} = \frac{1}{2}\gamma^{ab}$$

(spin $\frac{1}{2}$ reps)

$$\Rightarrow \mathrm{Ind}(iD_A) = \underset{\substack{\text{chiral}\\ \text{constrain}\\ \text{or second spin}}}{\frac{1}{2}} \cdot \underset{\text{reality}}{\frac{1}{2}} \int \underset{\text{Hirzebruch Polynomial}}{\left[\hat{A}(M)\, e^{\left[\frac{i}{2\pi}\frac{1}{4}R_{ab}\gamma^{ab}\right]}\right]_{2d+2}}$$

Application in ten dimensions: IIB SUGRA and low-energy limit of type I and heterotic strings

We are going to discuss anomaly cancellation for some supergravity theories in $D=10$ $\Rightarrow$ anomaly cancellation requires an highly non-trivial matching of coefficients, because the equations determining the vanishing of the anomaly are overdetermined.

In the second example SUGRA is coupled to YM $\Rightarrow$ the choice of the gauge group also plays a peculiar role.

Surprising: anomaly free theory $\Leftrightarrow$ low-energy limit of superstrings

Let us discuss the relevant characteristic polynomials: in general

$$I_{2d+2} = \sum_j I^{(j)}_{2d+2} \qquad \text{(sum over chiral fields)}$$

Remark in $D=4+4k$ particles and antiparticles have opposite chirality $\Rightarrow$ negative chirality particles $\Rightarrow$ positive chirality antiparticles in charge conjugate reps.

in $D=2+4k$ part. and antipart $\Rightarrow$ same chirality

in $D=2+8k$ Majorana-Weyl fermions $\Rightarrow$ they are their own antiparticles

In the following we need these particular polynomials

$$I_{12}^{(grav.)} = \frac{1}{64}\frac{1}{(2\pi)^5}\left[-\frac{11}{126}\,tr_2 R^6 + \frac{5}{96}\,tr_2 R^4\, tr_2 R^2 - \frac{7}{1152}\,(tr_2 R^2)^3\right]$$

$$I_{12}^{(Ant.)} = \frac{1}{8}\frac{1}{(2\pi)^5}\left[\frac{31}{2835}\,tr_2 R^6 - \frac{7}{1080}\,tr_2 R^4\, tr_2 R^2 + \frac{1}{1296}\,(tr_2 R^2)^3\right]$$

(not coupled to gauge fields)

$$I_{12}^{(\frac{1}{2})} = \frac{1}{64}\frac{1}{(2\pi)^5}\,tr_R(1)\left[\frac{1}{5670}\,tr_2 R^6 + \frac{1}{4320}\,tr_2 R^4\, tr_2 R^2 + \frac{1}{10368}\,(tr_2 R^2)^3\right]$$

$$-\frac{1}{32\pi}\frac{1}{(2\pi)^5}\left(tr_R F^2\right)\left[\frac{1}{360}\,tr_2 R^4 + \frac{1}{288}\,(tr_2 R^2)^2\right]$$

$$+\frac{1}{1152}\frac{1}{(2\pi)^5}\left(tr_R F^4\right) tr_2 R^2 - \frac{1}{720}\frac{1}{(2\pi)^5}\,tr_R F^6$$

spin $\frac{1}{2}$ coupled with gauge.

Remark The reality condition for gravitino and spin $\frac{1}{2}$ (Majorana-Weyl)
$\Rightarrow \frac{1}{2}$ in front!

Type IIB SUGRA

Consider a theory in 10 Dimensions with Majorana-Weyl gravitini, spin-$\frac{1}{2}$ fermions and antiselfdual antisymmetric tensors. No gauge group

Be $n_{3/2}$ the number of positive chirality minus negative chirality gravitini

$n_{1/2}$ $\frac{1}{2}$ fermion

n_A antiselfdual fields

They are coupled to gravity => let us compute the total anomaly polynomial ⑻③

$$\frac{1}{128}\frac{1}{(2\pi)^5}\left[\frac{-495 n_{3/2}+n_{1/2}+992 n_A}{5670}\mathrm{Tr}_2 R^6+\frac{225 n_{3/2}+n_{1/2}-448 n_A}{4320}\mathrm{Tr}_2 R^4 \mathrm{Tr}_2 R\right.$$

$$\left.+\frac{-63 n_{3/2}+n_{1/2}+128 n_A}{10368}\left(\mathrm{Tr}_2 R^2\right)^3\right]$$

=> ~~thee~~ we have three equations that should be satisfied (we have to put to zero the coefficients of the three of the curvatures) =>

Simple solution $\boxed{n_{3/2}=2n_A\ ;\ n_{1/2}=-2n_A}$

$n_A=1$ → 1 self dual tensor
$n_{3/2}=2$ → 2 gravitini (+) (L-R)
$n_{1/2}=-2$ → 2 spin $\frac{1}{2}$ (-) (R-L)

} chiral content → of IIB SUGRA !!

Green-Schwarz mechanism

Idea: if there are other fields in the game that transforms non-trivially under gauge and Lorentz (we are going to consider Lorentz anomaly) we can construct non-gauge invariant couplings that cancels the anomaly !

Example

D=4 gauge theory U(1)

$$\delta\Gamma(A) = -\frac{1}{24\pi}\sum_i q_i^3 \int \alpha(x)\, F\wedge F$$

Suppose we have a field ϕ (axion) that under gauge shifts

$$\phi \to \phi + \alpha$$

then a term $\frac{1}{24\pi^2}\sum_i q_i^3 \int \phi\, F\wedge F$ cancels the anomaly!

In general it is not sufficient to generate a good quantum theory

- kinetic term and gauge invariance?
- Renormalizability?

Consider now D=10 N=1 SUGRA coupled to N=1 SYM with G

SUGRA

1 gravitino MW (+)

1 spin $\frac{1}{2}$ MW (−)

graviton

B-field (antisym $B_{\mu\nu}$)

SYM

1 gauge field

1 spin $\frac{1}{2}$ MW (+)

$\Big\}$ adjoint reps R

The B-field is not Lorentz invariant and gauge invariant : it is

$$H = dB - \varphi_3^{(YM)}(A,F)$$

that is gauge-invariant φ_3 is the CS-term

One has to include also a gravitational CS in H (not tree level)

$$H = dB - \varphi_3^{(YM)}(A,F) + \beta\, \varphi^L(R,\omega)$$

$$\Rightarrow \quad (\delta^G + \delta^L) B = \varphi_2^1(v,A,F) + \beta\, \varphi_2^1(\lambda,R,\omega)$$

$\Rightarrow$ gauge invariance of H

Let us discuss the anomaly in these theories $\Rightarrow$ anomaly polynomial

$$I_{12}^{tot} = I_{12}^{(gr)} - I_{12}^{(\frac{1}{2})} + I_{12}^{(\frac{1}{2})}(R = adj)$$

$$I_{12}^{tot} = \frac{1}{64}\frac{1}{(2\pi)^5}\left[\frac{\dim G - 496}{5670}\, tr R^6 + \frac{\dim G + 224}{4320}\, Tr R^4\, Tr R^2 + \frac{\dim G - 64}{10\,368}\,(tr R^2)^3\right]$$

$$- \frac{1}{32}\frac{1}{(2\pi)^5}(tr F^2)\left[\frac{1}{360}\, tr R^4 + \frac{1}{288}(tr R^2)^2\right] + \frac{1}{1152}\frac{1}{(2\pi)^3}\, Tr F^4\, Tr R^2$$

$$- \frac{1}{720}\frac{1}{(2\pi)^5}\, tr F^6$$

It has no possibility to vanish alone

Green-Schwarz mechanism:

Construct a local term

$$\Delta\Gamma = \int B \wedge X_8 \quad \text{with} \quad (\delta^G + \delta^L) X_8 = 0$$

X_8 constructed from gauge and gravitational characteristic classes

$$\Rightarrow \quad X_8 = d X_7 \quad \text{and} \quad (\delta^G + \delta^L) X_7 = d X_6$$

$$\Rightarrow \quad (\delta^G + \delta^L)\Delta\Gamma = \int \left(\varphi_2^{1(YM)} - \beta \varphi_2^{1(L)} \right) \wedge X_8$$

$$= -\int d\left(\varphi_2^{1(YM)} - \beta \varphi_2^{1(L)} \right) \wedge X_7$$

$$= \int \left(\delta_G \varphi_3^{YM} - \beta \delta_L \varphi_3^{(L)} \right) \wedge X_7$$

Notice that this corresponds to the variation of a 12-form polynomial

(prove) $$\Delta \hat{I}_{12} = \left(\mathrm{tr} F^2 - \beta \, \mathrm{tr} R^2 \right) \wedge X_8 \quad !!$$

$\Rightarrow$ this could be used to cancel anomaly? Notice that the term $\mathrm{tr} R^6$ cannot be cancelled ~~unless~~ unless $\Rightarrow$ $\boxed{\dim G = 496}$

The $\mathrm{tr} F^6$ cannot appear unless it is expressed in terms of $\mathrm{tr} F^4 \, \mathrm{tr} F^2$ or $(\mathrm{tr} F^2)^3$ $\Rightarrow$ again we have to select a gauge group

I° case

$G = SO(n)$

$\Rightarrow$ relation between $\mathrm{Tr}(\mathrm{Ad})$ and $\mathrm{tr}(\text{fundamental})$

$$\left.\begin{aligned} \mathrm{Tr}F^2 &= (n-2)\,\mathrm{tr}F^2 \\ \mathrm{Tr}F^4 &= (n-8)\,\mathrm{tr}F^4 + 3\,(\mathrm{tr}F^2)^2 \\ \mathrm{Tr}F^6 &= (n-32)\,\mathrm{tr}F^6 + 15\,\mathrm{tr}F^2\,\mathrm{tr}F^4 \end{aligned}\right\} \quad n=32 \quad \mathrm{tr}F^6 \text{ out!}$$

$\Rightarrow$ $SO(32)$ and $\dim SO(32) = 496$!!

Then for $SO(32)$

$$I^{tot} = \frac{1}{384}\frac{1}{(2\pi)^5}\left[\mathrm{tr}R^2 - \mathrm{tr}F^2\right]\left[\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2 - \mathrm{tr}F^2\,\mathrm{tr}R^2 + 8\,\mathrm{tr}F^4\right]$$

Similar to ΔI

Taking $\beta = 1$ $\quad X_8 = \frac{1}{384}\frac{1}{(2\pi)^5}\left[\mathrm{tr}R^4 + \frac{1}{4}(\mathrm{tr}R^2)^2 - \mathrm{tr}F^2\,\mathrm{tr}R^2 + 8\,\mathrm{tr}F^4\right]$

We are in business! $\quad \Delta I_{12} + I_{12} = 0$!!

II° case

$G = E_8 \times E_8 \qquad \dim E_8 \times E_8 = 496$

$$\mathrm{tr}F^4 = \frac{1}{100}(\mathrm{Tr}F^2)^2 \qquad \mathrm{Tr}F^6 = \frac{1}{7200}(\mathrm{Tr}F^2)^3$$

Defining $\frac{1}{30} \mathrm{T}_2 F^2 = t_2 F^2$

$\Rightarrow$ (F_1 and F_2 are the two field strength of the E_8's)

$$I^{tot}_{12} = \frac{1}{38} \frac{1}{(2\pi)^5} \left[t_2 R^2 - t_2 F_1^2 - t_2 F_2^2 \right] \Big[t_2 R^4 + \frac{1}{4} (t_2 R^2)^2$$
$$- t_2 R^2 (t_2 F_1^2 + t_2 F_2^2)$$
$$- 2 \, t_2 F_1^2 \, t_2 F_2^2$$
$$+ 2 (t_2 F_1^2)^2 + 2 (t_2 F_2^2)^2 \Big]$$

$\Rightarrow$

$$X_8 = \frac{1}{384} \frac{1}{(2\pi)^5} \Big[t_2 R^4 + \frac{1}{4} (t_2 R^2)^2 - t_2 R^2 (t_2 F_1^2 + t_2 F_2^2)$$
$$- 2 \, t_2 F_1^2 \, t_2 F_2^2 + 2 (t_2 F_1^2)^2 + 2 (t_2 F_2^2)^2 \Big]$$