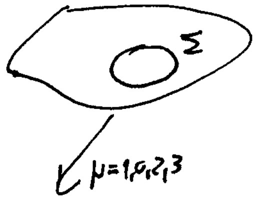
Lezione 7 D-branes on CY – Orbifolds

Alternative way to obtain gauge theories is to wrap Dp-branes on cycles in CY. Consider CY_3 and space-time filling D-branes

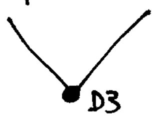


	dim Σ	type
type IIB	2n	holomorphic
type IIA	3	special Lagrangian

$\hookrightarrow$ sLag $= \begin{cases} J|_\Sigma = 0 \\ \mathrm{Im}\,\Omega|_\Sigma = 0 \end{cases}$

To decouple gravity, CY will be non compact. Here is a summary of the following construction:

- Fractional branes. CY_3 is singular and D3-branes are put at the singular point



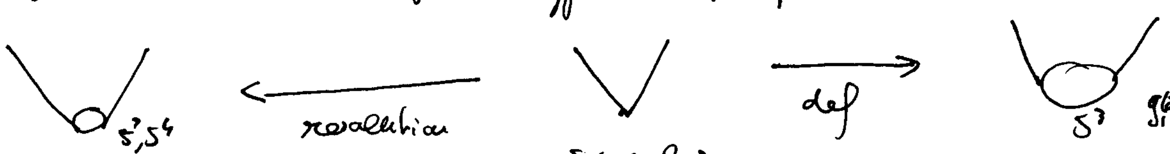
$N=1$ theories with $\prod_{i=1}^{k} U(N)$ gauge groups

we can also put D5-D7 wrapped over collapsed cycles: "effective" D3-branes

- Wrapped branes. A regular CY_3 with non vanishing 2-3-4 cycles where we wrap D5-D6-D7 branes. At low energies we obtain $N=1$ gauge theories

(Σ) Dp $\qquad \frac{1}{g^2} \cong \mathrm{Vol}(\Sigma)$

We will often encounter situations where the same family of CY's can be used for different purposes:



birational equivalent "crepant" resolutions

singular point $f_i(z)=0$

deformations $g_i(z)+f_i(z)=0$ (change in complex structure)

: We can put fractional branes on the singular point: D5-D7-D6 wrapped over collapsed cycles which effectively are D3-branes; follow their fate under resolution and deformation or put wrapped branes on regular CY

We can study quite explicitly the case where $CY_3 = \mathbb{C}^3/\Gamma$.

Recall that an orbifold is a quotient of a manifold by a discrete subgroup of its group of symmetries. We will look explicitly to $\mathbb{C}^3/\Gamma$ where $\Gamma \subset SO(6)$, the isometry group of the flat metric. Points in $\mathbb{C}^3$ are identified and D-branes, if present, also

<u>OSS:</u> there is a well defined construction for the string propagating on a orbifold. The spectrum is obtained as the set of invariant states plus twisted states localized at the orbifold projection and required by modular invariance.

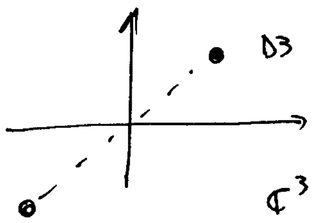
Γ preserves $N = 1$ susy if $\Gamma \subset SU(3)$ and $N = 2$ susy if $\Gamma \subset SU(2)$.
In fact, $\epsilon_{LR} = \xi \otimes \eta + c.c$ where η is a 4 of $SO(6)$

$$SU(4) = SO(6) \to SU(3) \times U(1) \qquad \eta \to 1 + \cancel{3} \qquad N=1 \text{ susy for each } \epsilon_L, \epsilon_R$$
$$SO(6) \to SU(2) \times U(1)^2 \qquad \eta \to 1 + 1 + 2 \qquad N=2 \text{ susy for each } \epsilon_L, \epsilon_R$$

In fact if $\Gamma \subset SU(3)$, $\mathbb{C}^3/\Gamma$ is a Calabi-Yau which preserves $N=2$ susy in type IIB.

$$\begin{cases} \Omega = dz_1 dz_2 dz_3 \\ J = \frac{i}{2} dz_i \wedge d\bar{z}_i \end{cases} \quad \text{are invariant under } z_i \to R_{ij} z_j \;\; R_{ij} \in SU(3))$$

When we add D3 branes, susy is further broken of 1/2.
D3 branes transverse to $\mathbb{C}^3/\Gamma$, $\Gamma \subset SU(3)$ have $N=1$ susy.
Let us take D3 in 0123 and $z_1 = x_4 + i x_5$, $z_2 = x_6 + i x_7$, $z_3 = x_8 + i x_9$.



In the covering space $\mathbb{C}^3$, every D3 brane has $|\Gamma|$ images. We can start with $U(N|\Gamma|)$ D3-branes ($N=4$ SYM) and quotient the open string spectrum with Γ.

Let us consider massless bosons $A_\mu, \underset{z_1}{\phi_1}, \underset{z_2}{\phi_2}, \underset{z_3}{\phi_3}$

We can send, by allowing Γ to act on Chan-Paton factors,

$$A_\mu \to \gamma A_\mu \gamma^{-1}$$
$$\phi_i \to R_{ij} \gamma \phi_j \gamma^{-1}$$

γ = gauge symmetry in $U(N|\Gamma|)$

The form of γ is found by considering that the $N|\Gamma|$ D3-branes are obtained by each other by permuting N blocks of D3-branes using Γ : this has a name in group theory : is the regular representation of Γ, R^{EG}

$$\gamma = R^{EG} \otimes I_{N\times N}$$

R^{EG} has dimension $|\Gamma|$ and decomposes into each irreps of Γ a number of times equal to its dimension

The final Lagrangian for D3 is obtained by restricting the $N=4$ Lagrangian to invariant configurations.

<u>Example</u> $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ $\quad \Gamma = \{ I, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \} \subset SU(2)$ acting on z_1, z_2

~~First the background: invariants are $z_1^2 = w$, $z_2^2 = t$, $z_1 z_2 = s$ and z_3. $\mathbb{C}^2/\mathbb{Z}_2$ is also the quadric $wt = s^2$ in $\mathbb{C}^3$.~~

A D3 brane has an image. Start with $U(2N)$ and take

$$\gamma = \begin{pmatrix} I_{N\times N} & 0 \\ 0 & -I_{N\times N} \end{pmatrix}$$

1 and -1 are the two irreps of $\mathbb{Z}_2$

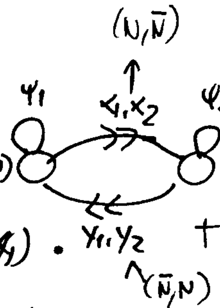
$$A_\mu = \gamma A_\mu \gamma^{-1} \qquad \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} A^{11} & -A^{12} \\ -A^{21} & A^{22} \end{pmatrix} \longrightarrow A_\mu = \begin{pmatrix} A^{11} & 0 \\ 0 & A^{22} \end{pmatrix}$$

$$\phi_3 = \gamma \phi_3 \gamma^{-1} \qquad \phi_3 = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$$

$$\phi_{1,2} = -\gamma \phi_{1,2} \gamma^{-1} \qquad \phi_1 = \begin{pmatrix} 0 & X_1 \\ Y_1 & 0 \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} 0 & X_2 \\ Y_2 & 0 \end{pmatrix}$$

bi-fundamental fields

the field content is



and gives $N=2$ multiplets (A^i_μ, ϕ^i), (X_i, Y_i). The Lagrangian is

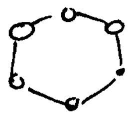
$$W = \operatorname{tr}_{2N\times 2N} \phi_3 [\phi_1, \phi_2] \longrightarrow \operatorname{tr}_{N\times N} \left[\psi_1 (X_1 Y_2 - X_2 Y_1) + \psi_2 (Y_1 X_2 - Y_2 X_1) \right]$$

which is the right form for $N=2$ susy $\sum \phi_i Q^i \tilde{Q}^i$

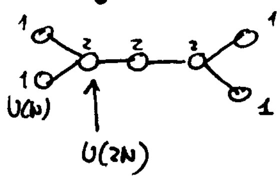
<u>Exercise I</u> $\mathbb{C}^3/\mathbb{Z}_3$:

$$W = \varepsilon_{ijk} U_i V_j W_k$$

<u>Exercise II</u> $\mathbb{C}^2/\mathbb{Z}_K \times \mathbb{C}$



<u>Exercise III</u> $\mathbb{C}^2/D_K \times \mathbb{C}$



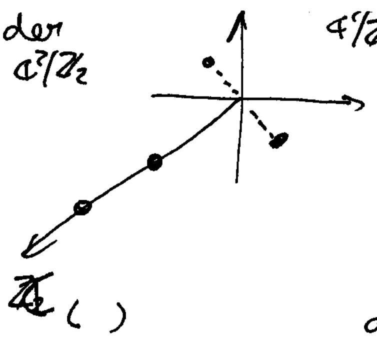
ADE classification

$\mathbb{C}^2/\Gamma \times \mathbb{C}$ $\quad \Gamma = \mathbb{Z}_N, D_K, E_{6,7,8}$ are the only subgroups of $SU(2)$

Group is $\prod U(n_i N)$ $\quad n_i$ Dynkin labels and matter is given by links

Spacetime configurations

Consider again $\mathbb{C}^2/\mathbb{Z}_2$



Each brane at $x_i^{(\alpha)}$ $i =$, ... has an image in $-x_i^{(\alpha)}$. A brane and its image make a "physical" brane which can be moved in an arbitrary point in $\mathbb{C}^2/\mathbb{Z}_2$

For $x^i_{(\alpha)} = 0$, a physical brane can be seen as a "composite" object and split in the (4,5) plane into two fractional branes

The factors $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ in γ refer to the two types of branes

- If we take $\gamma = I_{N\times N}$ and no images the projection of $N=4$ SYM is just $N=2$ pure $U(N)$
- If we take $\gamma = \begin{pmatrix} I_{N+M\times N+M} & 0 \\ 0 & I_{N\times N} \end{pmatrix}$ we have $N+M$ branes of one type and N branes of another

and the gauge theory is $U(N+M)\times U(N)$

Set of (minimal energy) BPS configurations in space-time

- branes moving in $\mathbb{C}$ are parameterized by two complex number φ_1, φ_2 each : $2N$ complex numbers

- branes moving in $\mathbb{C}^2/\mathbb{Z}_2$ can be placed in arbitrary position in $\mathbb{C}^2/\mathbb{Z}_2$. Moduli space for 1 brane is $\mathbb{C}^2/\mathbb{Z}_2$:

$$z_1 \to -z_1 \qquad \mathbb{C}^2/\mathbb{Z}_2$$
$$z_2 \to -z_2$$

Alternative description with invariants

$$w = z_1^2 \qquad t = z_2^2 \qquad s = z_1 z_2$$

satisfying

$$s^2 = wt^2 \qquad \text{quadric in } \mathbb{C}^3$$

N branes, since there is no force between D-branes, can be put in general positions : moduli space of configurations: $\mathrm{Sym}\,(\mathbb{C}^2/\mathbb{Z}_2)^N$

Field Theory configurations:

All the BPS states should be reflected by field theory vacua:
consider first one brane: $N=1$

F terms
$$\begin{cases} x_1 Y_2 - x_2 Y_1 = 0 \\ \psi_1 Y_2 = \psi_1 X_1 = \;--- \;= 0 \\ \psi_2 Y_2 = \;--- \;= 0 \end{cases}$$

and give two branches: ($N=2$)

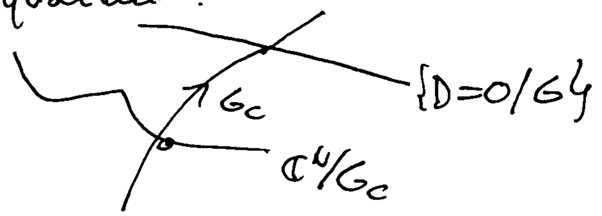
HIGGS	$\psi_1 = \psi_2 = 0$	$x_1 Y_2 = x_2 Y_1$
COULOMB	$x_i = Y_i = 0$	ψ_1, ψ_2 arbitrary

- In the Coulomb branch we have two parameters ψ_1, ψ_2 parametrizing positions of the fractional branes on $\mathbb{C}: z_3 = x_8 + i x_9$.

- In the Higgs branch we have to solve $\{D=0, F=0\}$ and mod by the gauge group

THEOREM: (symplectic quotient / $N=1$ susy) the space of solutions of
$$\{D = 0 / G\}$$
in $\mathbb{C}^N$, $N =$ number of fields, is a Kähler manifold isomorphic to $\mathbb{C}^N /\!/ G_c$, where G_c is the complexified gauge group.

[This means that in each orbit of G_c, which includes a real non-compact part, there is one solution of the D terms equations:



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In practice, in our case we have 4 fields x_i, Y_i and a gauge action $U(1)$ acting on (x_1, x_2, Y_1, Y_2) with charges $(1,1,-1,-1)$. The second $U(1)$ is decoupled, since it acts trivially on (x_1, Y_1).
The moduli space is

$$\left.\begin{matrix} F: & x_1 Y_2 = x_2 Y_1 \\ D: & \sum |x_i|^2 - |Y_i|^2 = 0 \end{matrix}\right/ U(1) \implies \{x_1 Y_2 = x_2 Y_1 / \underset{U(1)_c}{\underset{\parallel}{\mathbb{C}^*}}\}$$

INVARIANTS ARE $w = x_1 Y_1$, $t = x_2 Y_2$

$$\mathbb{C}^*: \begin{cases} x_i \to \lambda x_i \\ Y_i \to \frac{1}{\lambda} Y_i \end{cases} \quad \lambda \in \mathbb{C}^* \longrightarrow \quad s = \underbrace{x_1 Y_2 = x_2 Y_1}_{\text{F term}}$$

satisfying $wt = s^2 \Rightarrow \mathbb{C}^2/\mathbb{Z}_2$

For N branes, in the Coulomb branch $\psi_i = \begin{pmatrix} \psi_i^{(1)} & \\ & \ddots \\ & & \psi_i^{(N)} \end{pmatrix}$ by a gauge transformation and we obtain 2N parameters for the fractional branes. More complicated the Higgs branch, where however we find $\mathrm{Sym}\,(\mathbb{C}^2/\mathbb{Z}_2)^N$

Exercise I: take a gauge group as a base point — say 1 and construct gauge invariants under group 2: $(X_1Y_1)_{\alpha\beta}$, $(X_1Y_2)_{\alpha\beta}$, $(X_2Y_1)_{\alpha\beta}$ are independent adjoint fields for group 1. Show that they commute by F terms and diagonalize them simultaneously.
~~Repeat the~~

Exercise II: repeat the exercise for $\mathbb{C}^2/\mathbb{Z}_4$ and $\mathbb{C}^2/\mathbb{Z}_3$

Since the theory is $N=2$ there is an "$N=2$" version of the symplectic quotient. For $N=2$ $W = \Phi Q\tilde{Q}$ and $F_Q = \frac{\partial W}{\partial Q} = \Phi\tilde{Q} = 0$ is satisfied by $\phi = 0$ in the Higgs branch. Then for each gauge generator T^a we have

$$\left\{ \begin{array}{l} D^a = Q^\dagger Q - \tilde{Q}\tilde{Q}^\dagger = 0 \\ F^a_\Phi = Q\tilde{Q} = 0 \end{array} \right. \Big/ \; G$$

which can be combined in a triplet of D terms

$$\vec{D} = \mathrm{tr}\, \mathcal{Q}^\dagger \vec{\sigma}\, \mathcal{Q}\, \sigma_3$$

where the "quaternion" $\mathcal{Q} = \begin{pmatrix} Q & \bar{\tilde{Q}} \\ -\tilde{Q} & \bar{Q} \end{pmatrix}$. We have, for each T^a, 3 D terms and 1 gauge conditions: 4 real conditions. The dimension of moduli space is

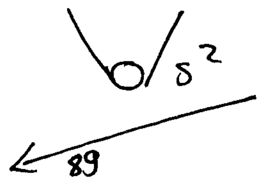
$$4(\#\,\text{hyper}) - 4(\#\,\text{vectors})$$

which is multiple of 4

THEOREM (hyperkahler quotient / $N=2$ susy) the Higgs moduli space of an $N=2$ theory is an hyperKahler manifold obtained by

$$\{\vec{D} = 0 / G\} \qquad \text{hyperkahler quotient}$$

Wrapped branes : we can also take a different approach and, starting with the smooth ALE space, wrap a D5 brane on S^2



the effective theory is pure $N=2$ YM with group $U(N)$:

- Fields (A_μ, ϕ_8, ϕ_9). No motion for S^2 in ALE

- Susy is preserved with a twist : on S^2 there are no cov. constant spinors $\delta\lambda = D_\mu \varepsilon \neq 0$. However we can turn on a background gauge field for the global symmetry (twisting)

$$\delta\lambda = \partial_\mu \varepsilon + \omega_\mu^{ab}\Gamma^{ab}\varepsilon + A_\mu \varepsilon = 0$$

$$\text{if } (A_\mu = -\omega_\mu)$$

satisfied by constant ε

Global simmetry for D5 in $SO(4)$ but broken to $SO(2)_{ALE} \times SO(2)_{89}$ in the background. $SO(2)_{ALE}$ acts on the normal bundle of S^2 and it is used to cancel the spin connection. Same mechanism goes thru to scalar fields.

- the gauge coupling is

$$\frac{1}{g^2} \cong Vol(S^2)$$

Lecture 8 D-branes on resolved orbifold

- We can also study resolution of singularities

Consider again $\mathbb{C}^2/\mathbb{Z}_n$. There is a smooth family of CY_2 that include $\mathbb{C}^2/\mathbb{Z}_n$. There are Ricciflat hyperkahler manifolds asymptotic to $\mathbb{C}^2/\mathbb{Z}_n$; they are called ALE spaces. CY_2 is equivalent to

$$dJ = 0 \qquad d\omega = 0$$

which give indeed 3 different Kahler forms. The ALE spaces are obtained by $\mathbb{C}^2/\mathbb{Z}_n$ by replacing the singular point with a ~~[illegible]~~ set of 2-spheres which intersects like the Dynkin diagram of $SU(n)$

Example: $\mathbb{C}^2/\mathbb{Z}_2$ in complex coordinates

$$wt = z^2 \longrightarrow \underset{\substack{\| \\ x_1^2+x_2^2}}{wt} = z(z-\lambda)$$

$z=0$ $\quad$ $z=\lambda$

$O(-2)\to\mathbb{P}^1$

λ is a complex parameter. ALE_2 has 3 parameters: the complex coordinates obscure 1 of the 3!

For $\mathbb{C}^2/\mathbb{Z}_n$ things are similar $\quad wt = z^n \to \prod(z-z_i)$

type IIB compactified on ALE_n has new fields with zero mass obtained by reducing the type IIB sugra on S^2:

$$A^+_{(4)} \longrightarrow C^T_{(2)} = \int_{S^2} A_{(4)} \qquad A^+_4(x,y) = C^{(+)}_2(x)\,\omega_{S^2}$$

$$C_{(2)}, B_{(2)} \longrightarrow c, b = \int_{S^2} C_{(2)}, B_{(2)}$$

$$g_{\mu\nu} \xrightarrow[J,\omega]{} \vec{\xi}$$

Altogether this is a multiplet of (2,0) 6d supersymmetry. In the limit where $S^2 \to 0$ and $ALE_n \to \mathbb{C}^2/\mathbb{Z}_n$ these are fields localized at the singularity. In the closed string description are the twisted sectors. The 5 scalars $\vec{\xi}$, c, b are parameters controlling the "stringy resolution" of the singularity

- Role of $\vec{\xi}$: the geometrical moduli of $\mathbb{C}^2/\mathbb{Z}_n$.

From the point of view of the probe D3 brane are FI terms. the $N=1$ moduli space is given by

$$D = 0,\quad F = 0 \;/\; G$$

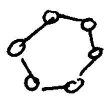
in the N=2 case $W = \phi_i Q\tilde{Q}$ and $F_Q = 0$ is satisfied in the Higgs branch by $\phi = 0$. Then

$$D^a = Q^\dagger Q - \tilde{Q}\tilde{Q}^\dagger = 0$$
$$F_\phi^a = Q\tilde{Q} = 0 \Bigg/ G$$

which can combined into $\vec{D} = \mathrm{tr}\, Q^\dagger \vec{\sigma} Q \sigma_3 = 0 / G$ where the quaternion $Q = \begin{pmatrix} Q & \tilde{Q} \\ -\tilde{Q} & Q \end{pmatrix}$. 3 D terms + 1 gauge condition for each generator of the gauge group. Dimension of the moduli space $= 4(\# H) - 4(\# V)$.

<u>TEO</u>: $\{\vec{D} = 0\}/G$ is hyperkahler (hyperk. quotient)

- <u>Role of $\vec{\xi}$</u>: is a FI in field theory: $\vec{D} = \vec{\xi}/U(1)$ is still hyperkahler. We are considering 1 D3-brane. We have one $\vec{\xi}$ for each $U(1)$; we have n of them but $\sum \vec{\xi} = 0$; $n-1$ independent parameters like the number of spheres



This is exactly the Kronheimer construction for the ALE_n space. The metric can be explicitly written as

$$ds^2 = V(y)\, dy_i^2 + \frac{1}{V(y)} (d\tau + A)$$
$$V(y) = \sum_{i=1}^{n} \frac{1}{|y - y_i|}$$

$i = 1,2,3$, $\mathrm{grad}\, V = -\mathrm{rot}\, A$, V harmonic in $\mathbb{R}^3$

$\dim = 4n - 4(n-1) = 4$ ↑ one $U(1)$ decoupled

Exercise: check that ds^2 is smooth if τ is periodic, and is asymptotic to $\mathbb{C}^2/\mathbb{Z}_n$, where $\tau \to \tau/n$ for $y \gg 1$

- <u>Role of b and c</u>: b and c are periodic. In suitable normalization $b, c \in [0,1]$. For $\vec{\xi} = 0$ at the perturbative point $b = 1/2$. The orbifold is a singular geometry but corresponds to a regular worldsheet description. String theory is really singular only when $\vec{\xi} = b = c = 0$: in this case branes wrapped on S^2 become tensionless. For $\vec{\xi} = 0$ but $b, c \neq 0$ we can have branes wrapped on the collapsed S_2 but with finite tension. Consider for example $\mathbb{C}^2/\mathbb{Z}_2$. The 2 brane types are explained by observing that we have 2 $A_\mu^{(4)}$ fields: $A_{(4)}$ and $A_{(4)}^+ = \int_{S^2} C_{(6)}$ and 2 D3-branes charges. $A_{(4)}^+$ measure a D5 charge:

$\Sigma = \mathbb{P}_1$ fractional brane 1 : D5 wrapped on collapsed S_2

$\Sigma = [pt] - \mathbb{P}_1$ frac brane 2 : $\overline{D5}$ " " and $F \neq 0$

$\int F = -1$ ↑ ↑ anti D5

D5, anti D5 : $\int_{S_2} F = -1$

$$-\int d^6x\, e^{-\phi}\sqrt{g+F+B} \quad \pm \int C_6 + \underset{\downarrow}{C_4}(F+B) + \frac{C_2}{2}(F+B)^2 + \frac{C_0}{6}(F+B)^3$$

D3 tension $|b|$ $\quad$ $|1-b|$ $\quad$ $\sqrt{g}=0$

D3 charge $= \begin{cases} b \\ 1-b \end{cases} \Rightarrow$ BPS objects

and the charges

	$A_{(4)}$	$A^T_{(4)}$
$D3_1$	b	1
$D3_2$	$1-b$	-1
physical	1	0

b and c give also the gauge coupling of the two factors

group 1 : $F^2 b e^{-\phi} + F\tilde{F}(b C_0 + c)$

group 2 : $F^2 (1-b) e^{-\phi} - F\tilde{F}((b-1)C_0 + c)$

$$\tau = \theta + \frac{i}{g^2} \Rightarrow \begin{cases} \tau_1 = c + b\tau \\ \tau_2 = -c + (1-b)\tau \end{cases} \qquad \boxed{\tau = C_0 + i e^{-\phi}}$$

Notice $\tau_1 + \tau_2 = \tau$

OSS : At the orbifold point $\begin{cases} b = \frac{1}{2} \\ c = 0 \end{cases}$ and $\tau_1 = \tau_2$

Exercises I : Orbifold

A) $N=1$ orbifold. Consider $\mathbb{C}^3/\mathbb{Z}_3$: $\begin{array}{l} z_1 \to \omega z_1 \\ z_2 \to \omega z_2 \\ z_3 \to \frac{1}{\omega^2} z_3 \end{array}$; $\omega^3=1$.

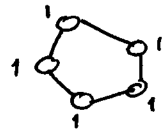
A1) Show by looking to the supersymmetries of $N=4$ SYM : $\varepsilon = \xi \otimes \eta_4 + c.c$ that $\mathbb{Z}_3$ preserves $N=1$

A2) Using $\gamma = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$ show that the theory is

$W = \varepsilon_{ijk} U_i V_j W_k$
with $SU(3)$ global symmetry

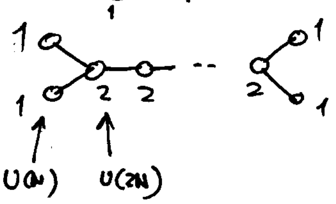
B) $N=2$ orbifolds. Discrete subgroups of $SU(2)$ are classified by ADE : $\mathbb{Z}_n$, D_n, $E_{6,7,8}$ [see form in hep-th 0608050 – section 3.1.1]
(A_n, D_n, $E_{6,7,8}$) simply-laced algebras
and the gauge theory is based on the "affine" (one node added) Dynkin diagram of $A_n, D_n, E_{6,7,8}$; nodes are gauge groups
links are bi-fundamental hypers

$\omega^K=1$ $\quad \gamma = \begin{pmatrix} \omega & 0 \\ 0 & \frac{1}{\omega} \end{pmatrix}$ $\quad \mathbb{C}^2/\mathbb{Z}_K \times \mathbb{C}$



K nodes

$\mathbb{C}^2/D_K \times \mathbb{C}$



with gauge group $\prod U(n_i N)$ with n_i Dynkin labels

B1) Compute the Lagrangian for $\mathbb{C}^2/\mathbb{Z}_K \times \mathbb{C}$

B2) OPTIONAL : Compute the Lagrangian for $\mathbb{C}^2/D_4 \times \mathbb{C}$ the first non-trivial D-group.

C) Moduli space (Abelian case). For $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^2/\mathbb{Z}_K \times \mathbb{C}$

C1) Compute the algebraic equations of $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^2/\mathbb{Z}_K$

C2) Compare with the field theory moduli space

D) Moduli space (non-abelian case). Take $\mathbb{C}^3/\mathbb{Z}_3$.

D1) Choose a base-point group, say 1, and construct invariants fields under gauge groups 2 and 3

$$M_{ijk} = (U_i V_j W_k)_{\alpha\beta}$$

and show that they trasformi in the adjoint of $U(N)_1$

D2) Show that the matrices M_{ijk} satisfy the matrix form of the algebraic equation for $\mathbb{C}^3/\mathbb{Z}_3$

D3) Show that you can diagonalize M_{ijk} simultaneously.

D4) Show that in the vacuum where M_{ijk} are diagonal there is a surviving discrete gauge symmetry S_N that permutes eigenvalues (Weyl symmetry)

D5) Show that the moduli space is

$$\mathrm{Sym}\,(\mathbb{C}^3/\mathbb{Z}_3)^N$$

E) ALE space

[OPTIONAL: see the hyperkahler construction in hep-th/9608085]

Take

$$\begin{cases} ds^2 = V(\vec{y})\, d\vec{y}^{\,2} + \dfrac{1}{V(\vec{y})}\,(d\tau + A)^2 \\ A = \vec{A}(y)\, d\vec{y} \\ V(\vec{y}) = \displaystyle\sum_{i=1}^{n} \frac{1}{|\vec{y}-\vec{y}_i|} \end{cases} \qquad \text{with } dV = * dA$$

E1) Write the metric on S^3 using

$$\begin{cases} z_1 = \cos\frac{\theta}{2}\, e^{i\phi_1} & \theta \in [0,\pi] \\ z_2 = \sin\frac{\theta}{2}\, e^{i\phi_2} & \phi_i \in [0,2\pi] \\ |z_1|^2 + |z_2|^2 = 1 \end{cases}$$

and obtain the Hopf fibration $ds^2 = \frac{1}{4}\left[d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi)^2\right]$ where $\phi_1 = \frac{\phi+\psi}{2}$, $\phi_2 = \frac{\phi-\psi}{2}$. The periodicity of ϕ_1 and ϕ_2 is 2π. What is the periodicity of ψ?

E2) Check that the metric is smooth at $\vec{y} = \vec{y}_i$ if τ is a periodic variable and compute the period.

E3) Study the behaviour of the metric for large $|y|$ and check that it becomes $\mathbb{C}^2/\mathbb{Z}_n$ asymptotically

[HINT: for $\vec{y} \sim \vec{y}_i$, $A = \cos\theta\, d\phi$ and the metric is $\mathbb{R}^4$ in spherical coordinates with Hopf variables for S^3. For $|y| \gg 1$, $A = n\cos\theta\, d\phi$ and $\tau \to \tau/n$]

Lezione 9 D-branes on CY — conifold

More generally a collection of physical D3-branes at a singular point of a CY_3 give rise to a quiver gauge theory with many gauge groups

Generically $N=1$: D3 $\prod_{i=1}^{G} U(n_i N)$ adjoint + bifundamental fields

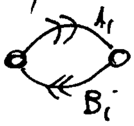
(CY breaks to $N=2$ branes to $N=1$)

- there is no orbifold construction. We can still think in terms of "fractional" branes: D5 and D7 branes wrapped on collapsed 4-2 cycles:

$$|6| = n_0 + n_2 + n_4$$

- For toric CY there is a "constructive" way of determining the gauge theory

Here we will consider a simple case, the "conifold" defined by the equation $xy = zw$ in $\mathbb{C}^4$. The dual theory was found by Klebanov-Witten and it is:



$U(N)\times U(N)$ $\quad$ A_i in $(N,\bar{N})$, B_i in $(\bar{N},N)$ $\quad$ $W = \epsilon_{ij}\epsilon_{pq} A_i B_p A_j B_q$

Main motivation comes from moduli space:

— Consider $N=1$, a $U(1)$ decouples: $\Longrightarrow \boxed{W\equiv 0}$

$\boxed{D:}$ $\begin{array}{cccc} A_1 & A_2 & B_1 & B_2 \\ 1 & 1 & -1 & -1 \end{array}$ $= \mathbb{C}^4/\{1,1,-1,-1\}$ symplectic quotient

Defining invariants $x = A_1B_1 \quad y = A_2B_2 \quad z = A_1B_2 \quad w = A_2B_1$

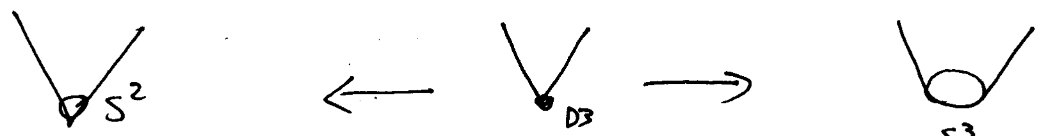
$$\Longrightarrow \quad xy = zw$$

— Consider N: F-terms important. Consider the "adjoint" mesons based on 1 $\quad M_{ij} = (A_iB_j)_{\alpha\beta}$. Exercise: show that M_{ij} commutes due to F-terms and satisfy the matrix equation for the conifold.

M_{ij} can be diagonalized and give N copies of the abelian case.

<u>OSS 1</u>: Another important check comes from AdS/CFT correspondence

The conifold has two well known "smoothings":



Deformation
$xy - zw = \varepsilon$

small resolution
"Resolved conifold"
$\mathcal{O}(-1)\times\mathcal{O}(-1) \to \mathbb{P}^1$

$U(N)\times U(N)$
$C(T^{11})$
$T^{11} \sim S^2\times S^3$

"Deformed conifold"
T^*S^3

OSS 1 : the conifold is the limit of a family of resolved conifold. Size of S^2 is a FI in field theory

$$\{ |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = \xi \} / U(1)$$

For $\xi \to 0$ branes wrapped on S^2 becomes "fractional branes" on the conifold :

$\Sigma_1 = [P_1]$ $\qquad\qquad$ $\Sigma_2 = [pt] - [P_1]$

$\int F = -1$ $\qquad$ anti D5

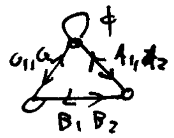
As before we have moduli b and c and $t_1 = bT + C$
$t_2 = (1-b)T + C$.

OSS 2 : Similar construction applies to all toric CY, in particular orbifolds:

Example $\mathbb{C}^3/\mathbb{Z}_3$ which can be resolved to $\mathcal{O}(-3) \to \mathbb{P}^2$ has 3 fractional branes

$$\Sigma_1 = \mathbb{P}^2 \qquad \Sigma_2 = -2\mathbb{P}^2 + \mathbb{P}^1 - \frac{[pt]}{2} \qquad \Sigma_3 = \mathbb{P}^2 - \mathbb{P}^1 - \frac{[pt]}{2}$$

Note the $\Sigma_1 + \Sigma_2 + \Sigma_3 = -[pt]$ the class of a point : a unit physical D3 - brane.

Exercise : Check that 

$$W = \mathrm{tr}\left[\phi(A_1A_2 - C_1C_2) + B_1B_2C_2C_1 - B_2B_1A_2A_1\right]$$

reproduce the moduli space of SPP : $xy = zw^2$

OSS 3 : We can also have wrapped branes :

Example 1 : N D6 branes in type IIA on deformed conifold
Example 2 : N D5 branes in type IIB on resolved conifold

In each case we have $N=1$ pure $U(N)$ theory in 4 dim with $\frac{1}{g^2} \sim \mathrm{vol}(S_{2,3})$ and a twist.