Characteristic classes

$$\{E_i \xrightarrow{\pi_i} M \,,\; D_i\} = \{E^*, D\}$$

$$\Gamma(M,E_0) \xrightarrow{D_0} \Gamma(M,E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{k-2}} \Gamma(M,E_{k-1}) \xrightarrow{D_{k-1}} \Gamma(M,E_k) \xrightarrow{D_k} 0$$

$$D_i D_{i+1} = 0 \quad \forall i = 0..k$$

D_i = diff. op. of elliptic type with finite kernel $\boxed{\dim(\ker D_i) < +\infty}$

$$H^i(E^*, M) = \frac{\ker D_i}{\operatorname{im} D_{i+1}}$$

Adjoint

$$D_i^\dagger : \Gamma(M, E_{i+1}) \to \Gamma(M, E_i)$$

$$\boxed{(s', D_i s)_{E_{i+1}} = (D_i^\dagger s', s)_{E_i}}$$

$s \in \Gamma(M, E_i)$

$s' \in \Gamma(M, E'_i)$

$(\cdot,\cdot)_E$ = fiber metric.

Laplace operator

$$\Delta_i : \Gamma(M,E_i) \to \Gamma(M,E_i)$$

$$\boxed{\Delta_i = D_{i-1} D_{i-1}^\dagger + D_i^\dagger D_i}$$

$$\Rightarrow \quad \dim H^i(E^*,D) = \dim \operatorname{Harm}^i(E^*,D)$$

$$\operatorname{Harm}^i(E^*,D) = \{\omega \in \Gamma(M,E^i) \,,\; \Delta_i \omega = (D_{i-1} D_{i-1}^\dagger + D_i^\dagger D_i)\omega\}$$

$\dim H^i(E, D_i) = \dim H^i(E, D_i^\dagger)$

infact: $\forall s \in H^i(E, D_i) \Rightarrow \exists s' \in H^i(E, D_i^\dagger)$

$s', s \in E_i \quad \rightarrow \quad (s, s')$

$s \in H(E, D_i)$

I would like to show that if $D_i^\dagger s' = 0$ then the product (s, s') can be written as $([s], [s'])$.

$$(s + D_i\eta, [s']) = (s, [s']) + (\eta, D_i^\dagger [s']) = (s, [s']) =$$

$$= (s, s' + D_i^\dagger \Omega) = (s, s') + (D_i s, \Omega) = (s, s')$$

so if $s, s' \in H(D), H(D^\dagger)$ then $([s], [s'])$ is defined only for cohomological classes

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Hodge decomposition (Riemannian manifolds)

$\forall p \in [0, n] \subset \mathbb{N}$. H^p is finite dim, $\dim H^p < +\infty$.

$$\boxed{\Omega^p(M) = \underbrace{\Delta(M)}_{\text{harmonic forms}} \oplus H^p = d\,d^\dagger (E^p) \oplus d^\dagger d (E^p) \oplus H^p(M)}$$

$$= d(E^{p-1}) \oplus d^\dagger (E^{p+1}) \oplus H^p.$$

$\Rightarrow \Delta\omega = \alpha$ has solution iff $(\alpha, \tilde\omega) = 0 \;\; \forall \tilde\omega \in \Delta(M)$

Proof: $\omega_1 \dots \omega_\ell$ o.n. basis of $H^p(M)$.

$$\forall \alpha \in \Omega^p(M) \rightarrow \alpha = \beta + \sum_{i=1}^{\ell} \langle \alpha, \omega_i \rangle \omega_i$$

$$\boxed{\int_M \alpha \wedge * \omega = \langle \alpha, \omega \rangle}$$

$\beta \in (H^p)^\perp \subset \Omega^p(M)$

$$\Omega^p = (H^p)^\perp \oplus H^p$$

now we want to prove: $(H^p)^\perp \simeq \Delta(M)$:

$$H : \Omega^p(M) \to H^p(M)$$

$$\alpha \longmapsto H(\alpha) \quad \text{s.t. } \Delta H(\alpha) = 0.$$

$\Delta(M) \subset (H^p)^\perp$:

$$\langle \Delta\omega, \alpha \rangle = \langle \omega, \Delta^+\alpha \rangle = \langle \omega, \Delta\alpha \rangle = 0$$

$\omega \in \Omega^p(M)$. if $\alpha \in H^p(M) \to \Delta\alpha = 0$

$$\Rightarrow \Delta\omega \perp \alpha \quad \forall \alpha \in H^p(M)$$

$(H^p)^\perp \subset \Delta(M)$.

$\exists c \in \mathbb{R}$ s.c. $\langle \beta, \beta \rangle \leq c \langle \Delta\beta, \Delta\beta \rangle \quad \forall \beta \in (H^p(M))^\perp$

Proof:

Suppose the contrary: $\exists \beta_i \in (H^p(M))^\perp$ s.t. $\|\beta_i\| = 1$ and $\|\Delta\beta_i\| \to 0$.

(we can assume of Cauchy) $\Rightarrow \exists \lim_{j\to\infty} \langle \beta_j, \varphi \rangle \quad \forall \varphi \in \Omega^p(M)$

Define: $\quad l(\varphi) = \lim_{j\to\infty} \langle \beta_j, \varphi \rangle$.

l is bounded:

$$l(\Delta\varphi) = \lim_{j\to\infty} \langle \beta_j, \Delta\varphi \rangle = l \langle \Delta\beta_j, \varphi \rangle = 0$$

since $\|\Delta\beta_j\| \to 0$

$\Rightarrow$ ℓ is a weak solution of $\Delta\beta = 0$ $\Rightarrow$ $\boxed{\ell(\Delta\varphi) = \langle\alpha,\varphi\rangle}$

$\boxed{\text{given } \alpha \in \Omega^p(M), \quad \ell \text{ weak solution of } \Delta\omega = \alpha}$

$\Rightarrow \exists\, \omega \in \Omega^p(M)$ s.t.

$$\ell(\beta) = \langle\omega,\beta\rangle \quad \forall \beta \in \Omega^p(M).$$

$\ell(\Delta\varphi) = \langle\alpha,\varphi\rangle$

$\ell(\Delta\varphi) = \langle\omega,\Delta\varphi\rangle = \langle\Delta\omega,\varphi\rangle \quad \forall\varphi \Rightarrow \boxed{\Delta\omega = \alpha}$

$\Rightarrow \quad \exists\, \beta \in \Omega^p(M)$ s.t. $\ell(\psi) = \langle\beta,\psi\rangle$

$\Rightarrow \quad \lim_{j\to\infty} \beta_j \to \beta$

but $\|\beta_j\| = 1$, and $\beta_j \in (H^p)^\perp \Rightarrow$

$\|\beta\| = 1, \quad \beta \in (H^p)^\perp \quad$ but $\Delta\beta = 0 \Rightarrow$

so it is a contradiction.

$\Rightarrow \exists\, c$ s.t. $\quad \|\beta_j\| \leq c\,\|\Delta\beta\|. \quad \forall \beta \in (H^p)^\perp$

$\alpha \in (H^p)^\perp, \quad \ell(\Delta\varphi) = \langle\alpha,\varphi\rangle$

$$\begin{cases} \ell \in (\Delta(\Omega^p(M)))^* \\ \forall\varphi \in \Omega^p(M) \end{cases}$$

$\Delta\varphi_1 = \Delta\varphi_2 \Rightarrow \Delta(\varphi_1 - \varphi_2) = 0 \quad \varphi_1 - \varphi_2 \in H^p \Rightarrow$

$$\langle\alpha, \varphi_1 - \varphi_2\rangle = 0$$

$\psi = \varphi - H(\varphi) \quad \Rightarrow \quad |\ell(\Delta\varphi)| = |\ell(\Delta\psi)| = |\langle\alpha,\psi\rangle| \leq \|\alpha\|\|\psi\|$

$\leq c\|\alpha\| \, \|\Delta\psi\| = c\|\alpha\| \, \|\Delta\varphi\|$

$|l(\Delta\varphi)| \leq c\|\alpha\| \, \|\Delta\varphi\|$.

is bounded on $\Delta(M) = \Delta(\Omega^p(M))$.

By Hahn-Banach theorem $\Rightarrow$ it extend on $\Omega^p(M)$

$\Rightarrow$ l is a weak solution of $\Delta\omega = \alpha$.

$\Rightarrow$ $\forall \alpha \in (H^p)^\perp$ $\exists$ $\omega \in \Omega^p(M)$ s.t.

$\Delta\omega = \alpha$.

Given (E^*, D).

$$\mathrm{ind}(E^*,D) = \sum (-)^i \dim H^i(E^*,D) =$$

$$= \sum (-)^i \dim_{\mathbb{R}} H^i(M,\mathbb{R}) =$$

$$= \sum (-)^i \dim \ker \Delta_i$$

if $D = d$ on M:

$$\mathrm{ind}(d) = \sum (-)^i \dim H^i(E^*, d) =$$

$$= \chi(M) = \sum (-)^i b^i$$

Analytical index of $D : \Gamma(M,E) \to \Gamma(M,F)$

$$\mathrm{ind}\, D = \dim \ker D - \dim \operatorname{coker} D.$$

$$\begin{cases} \ker D = \{ s \in \Gamma(M,E) \mid Ds = 0 \} \\ \operatorname{coker} D = \Gamma(M,F) / \operatorname{im} D. \end{cases}$$

$s \in \Gamma(M,F)$

$[s] \in \operatorname{coker} D$ if is defined up to $D\eta$

(Fredholm if $\dim \ker D, \dim \operatorname{coker} D < +\infty$)

$$\boxed{\operatorname{coker} D \simeq \ker D^+}$$

$$\langle \varphi, [s] \rangle = \langle \varphi, s + D\eta \rangle = \langle D^+\varphi, \eta \rangle + \langle \varphi, s \rangle$$

$$\operatorname{ind} D = \dim \ker D - \dim \ker D^{+}$$

$$0 \xrightarrow{i} \Gamma(M,E) \xrightarrow{D} \Gamma(M,F) \xrightarrow{\phi} 0.$$

$$\dim \ker D - (\dim \Gamma(M,F) - \dim \operatorname{im} D)$$

Given a complex vector bundle E, with a connection Θ (with fiber $\mathbb{C}^n$)

This is the curvature

$$\boxed{c(E,\Theta) = \det\left(1 + \frac{i}{2\pi}\Theta\right) =} \det\left(1+\frac{i}{2\pi}R\right)$$

(Chern class)

$$\boxed{= \sum_{k=0}^{n} c_k(E,\Theta)}$$

Using $\quad 1 + \frac{i}{2\pi}\Theta = O^{-1}(1+\mathcal{D})O \qquad \mathcal{D} = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$

we get

$$\boxed{= \det(1+\mathcal{D}) = \prod_{k=1}^{n}(1+x_k)}$$

$c_0 = 1$

$c_1 = \frac{i}{2\pi}\operatorname{tr}\Theta = \frac{i}{2\pi}\sum_{i=1}^{n} x_i = \frac{i}{2\pi}\operatorname{tr} R = \frac{i}{2\pi}\sum_{I} R^{I}{}_{I}$

$c_2 = \frac{1}{8\pi}\left(\operatorname{tr}\Theta^2 - (\operatorname{tr}\Theta)^2\right)$

$\vdots$

$$\boxed{c(E \oplus F) = c(E) \wedge c(F)}$$

$$\begin{cases} E \xrightarrow{\pi} M \\ F \xrightarrow{\pi'} M \end{cases}$$

Total Chern character

$$E \xrightarrow{\pi} M.$$

$$ch(E, \Theta) = tr \exp\left(\frac{i}{2\pi}\Theta\right) = tr \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{i}{2\pi}\Theta\right)^l =$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} tr\left(\frac{i}{2\pi}\Theta\right)^l =$$

$$= \sum_{j=1}^{r} \left(1 + x_j + \frac{1}{2}x_i^2 + \cdots\right) =$$

$$= r + \sum_{i=1}^{r} x_i + \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \cdots$$

$$ch_0(\Theta) = r$$

$$ch_1(\Theta) = c_1(\Theta)$$

$$ch_2(\Theta) = \frac{1}{2}\left(c_1^2(\Theta) - 2c_2(\Theta)\right)$$

$$\vdots$$

$$\begin{cases} ch(E \oplus F) = ch(E) + ch(F) \\ ch(E \otimes F) = ch(E) \wedge ch(F) \end{cases}$$

Chern class:

$$c(E,\Theta) = \det\left(1 + \frac{i}{2\pi}\Theta\right) = \sum_{k=0}^{n} c_k(E,\Theta) = \prod_{i=1}^{n}(1+x_i)$$

Chern character

$$\mathrm{ch}(E,\Theta) = \mathrm{tr}\exp\left(\frac{i\Theta}{2\pi}\right) = \sum_{j=1}^{n}\left(1 + x_j + \frac{1}{2}x_j^2 + \dots\right) =$$

$$= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(\sum_{j=1}^{n} x_j^{\ell}\right)$$

Todd class

$$\mathrm{Td}(E,\Theta) = \prod_{j=1}^{r}\frac{x_j}{1-e^{-x_j}}$$

Pontryagin class

$$p_k(E,\Theta) = (-)^k c_{2k}(E^{\mathbb{C}},\Theta)$$

Euler class

$$\boxed{\chi_{Eul} = \int_M c_{top}(E)}$$

$$e^2(E) = p_{[n/2]}, \quad e(E) = c_n(E).$$

Atiyah-Singer: (FAC (non fare a lezione))

$$\mathrm{ind}(E, D) = (-)^{\frac{m(m+1)}{2}}\int_M \mathrm{ch}\left(\oplus_j (-)^j E_j\right)\frac{\mathrm{Td}(TM^{\mathbb{C}})}{e(TM)}$$

Ex:

$$E: \xrightarrow{d} \Omega^{r-1}(M)^{\mathbb{C}} \xrightarrow{d} \Omega^r(M)^{\mathbb{C}} \to \cdots$$

$$\Omega^r(M)^{\mathbb{C}} = \Gamma(M, \Lambda_r T^*M^{\mathbb{C}})$$

$$\operatorname{ind} d = \sum_{r=0}^{m} (-)^r \dim_{\mathbb{C}} H^r(M, \mathbb{C}) = \sum_{r=0}^{m} (-)^r \dim_{\mathbb{R}} H^r(M, \mathbb{R}) =$$

$$= \chi(M) \quad \text{(Euler characteristic)}.$$

$m = 2l$

$$\operatorname{ind} = (-)^{l(2l+1)} \int_M \operatorname{ch}\left(\oplus_{r=0}^{2l} (-)^r \Lambda^r T^*M^{\mathbb{C}}\right) \frac{Td(TM^{\mathbb{C}})}{e(TM)}$$

Using the splitting principle

$$\Lambda^p F = \oplus_{1 \le i_1 \le \cdots \le i_p} (L_{i_1} \otimes \cdots \otimes L_{i_p})$$

$$\operatorname{ch}(\Lambda^p F) = \sum_{1 \le i_1 \le \cdots \le i_p} \operatorname{ch}(L_{i_1}) \wedge \cdots \wedge \operatorname{ch}(L_{i_p}) =$$

$$= \sum e^{x_{i_1} + \cdots + x_{i_p}}$$

this comes from T^*M

$$\operatorname{ch}\left(\oplus_{r=0}^{2l} (-)^r \Lambda^r T^*M^{\mathbb{C}}\right) = \prod_{i=1}^{2l} \left(1 - e^{-x_i}\right)(T^*M^{\mathbb{C}})$$

this comes from $(-)^r$

$$\mathrm{ind} = (-)^{\ell} \int_M \frac{\prod_{i=1}^{m} x_i(TM^{\mathbb{C}})}{\underbrace{\cancel{\prod (1 - e^{x_i})}}_{\mathrm{Td}(TM^{\mathbb{C}})}} \quad \frac{\prod_{i=1}^{m} \cancel{(1 - e^{x_i})}(TM^{\mathbb{C}})}{e(TM)} =$$

$$= (-)^{\ell} \int_M \frac{\prod_{i=1}^{m} x_i(TM^{\mathbb{C}})}{e(TM)} = (-)^{\ell} \int_M \frac{c_m(TM^{\mathbb{C}})}{e(TM)} =$$

$$c_m(TM^{\mathbb{C}}) \neq c_m(TM \oplus \overline{TM}) = (-)^{\frac{m}{2}} e\,(TM \oplus TM) =$$

$$= (-)^{\ell} e^2(TM)$$

$$= \boxed{\int_M e(TM) = \chi(M)}$$

Kähler metric

1) it appears in the formulation of supergravity $N=1$ with matter. (scalar field metric).
2) metric on internal manifold (CY manifolds).

$$g : \Gamma(M, TM) \otimes \Gamma(M, TM) \longrightarrow C^\infty(M)$$
$$X, Y \longmapsto g(X,Y) = g_{\alpha\beta} X^\alpha Y^\beta$$

Def. M $2n$-dim manifold w. an almost complex structure J. A metric on M is called hermitian w.r.t. J if

$$g(J(X), J(Y)) = g(X,Y)$$

Fundamental form:

$$k(X,Y) = \frac{1}{2\pi} g(J(X), Y).$$

in components:

$$\boxed{\underbrace{k_{\alpha\beta}}_{\text{fundamental form}} = \underbrace{J_\alpha{}^\gamma}_{\text{a. complex structure}} \underbrace{g_{\gamma\beta}}_{\text{metric}}}$$

Notice that $k(Y,X) = \frac{1}{2\pi} g(J(Y),X) = \frac{1}{2\pi} g(X, J(Y)) =$

$$= \frac{1}{2\pi} g\big(J(x), J(J(y))\big) = \frac{1}{2\pi} g\big(J(x), J^2(y)\big) =$$

$$= -\frac{1}{2\pi} g(J(x), y) = -k(x,y) \quad \text{so} \quad \boxed{k \in \Lambda^2(M)}$$

Them: g is hermitian if and only if k is anti-symmetric.

$$\left(\begin{array}{lll} k = Jg & k^T = g^T J^T = g J^T = J^{-1} g = -Jg = k \\ JgJ^T = g & \Rightarrow \quad gJ^T = J^{-1}g & J^{-1} = -J \end{array} \right).$$

Def A hermitian almost complex manifold M is a complex mfld w. a hermitian metric
(In the same way we p

$$g(x,y) = g_{IJ} x^I y^J + g_{I\bar{J}} x^I y^{\bar{J}} + g_{\bar{I}J} x^{\bar{I}} y^J + g_{\bar{I}\bar{J}} x^{\bar{I}} y^{\bar{J}}$$

Reality of g $\Rightarrow$ $g_{IJ} = \overline{g_{\bar{I}\bar{J}}}$, $g_{I\bar{J}} = \overline{g_{\bar{I}J}}$

Symmetry $\Rightarrow$ $g_{IJ} = g_{JI}$, $g_{\bar{I}\bar{J}} = g_{\bar{J}\bar{I}}$

Hermiticity $\Rightarrow$ $g_{I\bar{J}} = \overline{g_{\bar{I}J}}$, and $g_{IJ} = g_{\bar{I}\bar{J}} = 0$.

For example

$$JgJ^T = g \quad g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} \Rightarrow \begin{cases} g_{00} = g_{11} \\ g_{01} = -g_{10} \end{cases} \quad g = \begin{pmatrix} g_{00} & g_{01} \\ -g_{01} & g_{00} \end{pmatrix}$$

and $g_{zz} = g_{00} + ig_{01} + ig_{10} - g_{11} = 0$ $\quad g_{\bar{z}z} = 2(g_{00} + ig_{01}) = \overline{g_{z\bar{z}}}$,

$g_{z\bar{z}} = g_{00} + ig_{10} - ig_{01} + g_{11} = 2(g_{00} - ig_{01})$ $\quad g_{\bar{z}\bar{z}} = 0$.

Def (M, J, g) a complx mfld, g Hermitian on M.
$\omega \to$ Hermitian (fundamental form).
It is kähler if $d\omega = 0$ (ω = kähler form).

Thm: (M, J, g) J is an almost complx structure.
g Hermitian metric.
ω " form.
∇ Levi-Civita connection of g ($T_\nabla = 0$, $\nabla g = 0$)
(Riemannian)

the following conditions are equivalent:
$\Rightarrow$
1) J is a complex structure and g is kähler.
2) $\nabla J = 0$
3) $\nabla \omega = 0$
4) $Hol(\nabla) \subseteq U(m)$, J is the corr. $U(m)$-structure.

The equation $d\omega = 0$ imply locally (on any chart)
that $\boxed{\omega = dd^c k = \partial\bar\partial k}$
$d^c = i(\partial - \bar\partial)$, using the decomposition. $d = \partial + \bar\partial$.

(local / global dd^c-lemma
M (must be compact))

k is a real function.

Lemma:

(M, J, g) compact Kähler manifold.

g, g' metric on (M, J)
ω, ω' Kähler forms.

$[\omega] = [\omega'] \in H^2(M, \mathbb{R})$

$\Rightarrow$ $\exists$ a smooth, real function ϕ on M. such that

$$\omega' = \omega + dd^c \phi.$$

ϕ is unique up to $\phi + c$. (compactness).

If M is not compact $\Rightarrow$

$$\phi \to \phi + \underbrace{f(z) + \bar{f}(\bar{z})}_{\text{Kähler transformation.}}$$

Definitions

Ricci form

$$\mathcal{R} = \mathrm{tr}\, R^I{}_J = R^I{}_{I J \bar{J}}\, dz^J \wedge d\bar{z}^{\bar{J}}$$

$$\mathrm{tr}\ \bar{\partial}\left[h^{\bar{M}I}\partial h_{\bar{M}J}\right] = \bar{\partial}\left[h^{\bar{M}I}\partial h_{\bar{M}I}\right] =$$

$$\boxed{\mathcal{R} = -i\,\bar{\partial}\partial \ln\det h_{\bar{M}I}}$$

This is in order to be real

$$\boxed{\overline{\mathcal{R}} = \mathcal{R}}$$

$$\overline{h_{\bar{M}I}} = h_{\bar{I}M}$$

$$\overline{\det h_{\bar{M}I}} = \det h_{\bar{I}M}$$

Computation of first Chern class of a projective variety obtained as a complete intersection.

Given $\mathcal{L}$ (line bundle) rank $= 1$

$$c_1(\mathcal{L}) = \frac{i}{2\pi} \operatorname{tr} R = \frac{i}{2\pi} \bar{\partial}\left(h^{-1}\partial h\right) =$$

$\mathcal{L} \xrightarrow{\mathbb{C}} M$ $\quad \mathbb{C}$ is the structure group.

$$\boxed{c_1(\mathcal{L}) = \frac{i}{2\pi} \bar{\partial}\partial \ln h}$$

$h(z,\bar{z})$ is the metric on $\mathcal{L}$.

(namely given $v, w \in T\mathcal{L}$ $\quad v = v(z)\,\partial_z$ $\quad$ $z, \bar{z}$ are the coord. of $\mathcal{L}$.)
$\quad w = w(z)\,\partial_z$

$$(w, v) = \overline{w}(\bar{z})\, h(z,\bar{z})\, v(z)$$

$$c_1(\mathcal{L}) = \frac{i}{2\pi} \bar{\partial}\partial \ln \|\xi\|^2.$$

with : $\|\xi\|^2 = \langle \xi, \xi \rangle = \bar{\xi}(\bar{z})\, h(z,\bar{z})\, \xi(z)$

(since $\bar{\partial}\xi = \partial\bar{\xi} = 0$)

A $\boxed{\text{kähler mfold } M \text{ is a } \underline{\text{HODGE}} \text{ mfold}}$ iff $\exists\ \mathcal{L} \to M$ such that

$$c_1(\mathcal{L}) = [k]$$

where $[k]$ is the cohomology class of the kähler (1,1) form on M. $\Longleftrightarrow$

$$\boxed{k = \frac{i}{2\pi}\, g_{I\bar{J}}\, dz^I \wedge d\bar{z}^{\bar{J}} = \frac{i}{2\pi} \bar{\partial}\partial \ln \|W(z)\|^2}$$

where $W(t)$ is a holomorphic section

$$W(z) : M \to \mathcal{L}.$$

$$g_{I\bar{J}}\, dz^I \wedge d\bar{z}^{\bar{J}} = dz^I \wedge d\bar{z}^{\bar{J}}\, \partial_I \partial_{\bar{J}} \ln h$$

$\ln h = K$ (K is the kähler potential)

$$\Rightarrow \boxed{h(z,\bar{z}) = \exp(K(z,\bar{z}))}$$

$\Rightarrow$ [For compact kähler Manifold the Hodge condition has deep top. implications!]

It can be shown that

$$\int_{\Delta\ \text{compact}} c_1(M) \in \mathbb{Z} \quad \text{or equivalently:}$$

$$c_1(M) = \sum \alpha_i \omega_i$$

$$\omega_i \in H^2(M), \ \alpha_i \in \mathbb{Z}.$$

$\Rightarrow$ Also the kähler class has the same properties.

$\Rightarrow$ By Kodaira Theorem.:

The integrality of the kähler class (class of the kähler 2-form) is a necessary and sufficient condition for $\underline{M}$ to be projective algebraic $\Leftrightarrow$ vanishing locus of hom. polynomials in $\bigotimes_{i=1}^{p} \mathbb{C}P^i$

Based on the fact that.

$$\mathbb{C}P^N = \frac{SU(N+1)}{SU(N) \times U(1)} \Rightarrow C_{tot}(\mathbb{C}P^N) = (1+k)^{N+1} =$$

$$= \sum_{l=0}^{N+1} \binom{N+1}{l} \underbrace{k_\wedge \cdots \wedge k}_{l}$$

Proof)

$\mathbb{C}P^N$: $\{X^A, \ A = 0, 1 \ldots N\} \sim$

$\sim \{\lambda X^A\} \quad \lambda \in \mathbb{C}^*$

$\underline{U} = \{U_\Lambda\}_{\Lambda = 1 \ldots N} \qquad U_\Lambda = \{X^\Lambda \neq 0\}$

$$\begin{cases} z_\Lambda^a = \dfrac{X^a}{X^\Lambda}, & z_\Lambda^\Lambda = 1 \\ a = 0, \ldots \hat{\Lambda}, \ldots N & \end{cases}$$

$$\boxed{K_{FS} = \ln\left(1 + \sum_{a=1}^{N} \bar{z}^a z^a\right)}$$

This metric is gauge invariant under $SU(N) \times U(1) =$ $= U(N)$

kähler potential.

$$\Rightarrow \begin{cases} g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K_{FS} = \dfrac{1}{(1+|z|^2)^2}\left((1+|z|^2)\delta^{a\bar{b}} - \bar{z}^a z^b\right) \\ g^{a\bar{b}} = (1+|z|^2)\left(\delta^{a\bar{b}} + \bar{z}^a z^b\right) \end{cases}$$

The connection can be seen as the connection of the gauge group $SU(N) \times U(1)$.

Viewing $\mathbb{C}P^N$ as a coset manifold we have:

$$L(z) \in G = SU(N+1)$$
$$z \in G/H \quad (\text{coords of } \mathbb{C}P^N)$$

$$\Omega^A{}_B = (L^+ dL)^A{}_B = (L^+)^A{}_{B'} dL^{B'}{}_B$$

which satisfies:

$$\boxed{d\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B = 0.}$$

Using the parametrization $Z = \begin{pmatrix} z^1 \\ \vdots \\ z^N \end{pmatrix}$

$$\boxed{L(z) = \begin{pmatrix} \frac{1}{\sqrt{1+zz^+}} & z\frac{1}{\sqrt{1+z^+z}} \\ -z^+\frac{1}{\sqrt{1+zz^+}} & \frac{1}{\sqrt{1+z^+z}} \end{pmatrix}}$$

$$(zz^+)^+ = zz^+$$

$$L^+L = \begin{pmatrix} \frac{1}{\sqrt{\ }} & -\frac{1}{\sqrt{\ }}z \\ +\frac{1}{\sqrt{\ }}z^+ & \frac{1}{\sqrt{\ }} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\ }} & z\frac{1}{\sqrt{\ }} \\ -z^+\frac{1}{\sqrt{\ }} & \frac{1}{\sqrt{\ }} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{(1+zz^+)} + \frac{1}{(1+zz^+)^{\frac{1}{2}}} zz^+ \frac{1}{(1+zz^+)} & \frac{1}{\sqrt{\ }}z\frac{1}{\sqrt{\ }} - \frac{1}{\sqrt{\ }} + \frac{1}{\sqrt{\ }} = 0 \\ 0 & \frac{1}{\sqrt{1+z^+z}} z^+z \frac{1}{\sqrt{1+z^+z}} + \frac{1}{(1+z^+z)} \end{pmatrix}$$

$$(1+zz^+)zz^+ = zz^+(1+zz^+)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Omega^A{}_B = \begin{pmatrix} \Omega^a{}_b & E^a \\ -\bar{E}_b & \Omega^0{}_0 \end{pmatrix} = (L^\dagger dL)^A{}_B$$

$(\Omega^a{}_b, \Omega^0{}_0)$ connections of $\boxed{SU(N) \times U(1)}$

$(E^a, \bar{E}_{\bar{a}})$ vielbeins of $\mathbb{C}P^N$.

$$\boxed{E^a = \frac{1}{\sqrt{1 + z z^\dagger}} \, dz^a \, \frac{1}{\sqrt{1 + z^\dagger z}}}$$

and we have:

$$\begin{cases} ds^2 = & E^a \wedge \bar{E}_a = g_{a\bar{b}} \, dz^a \wedge d\bar{z}^{\bar{b}} \\ K = \frac{i}{2\pi} & E^a \wedge \bar{E}_a = \frac{i}{2\pi} g_{a\bar{b}} \, dz^a \wedge d\bar{z}^{\bar{b}} \end{cases}$$

and these expressions are equal to the expressions computed with

$$K = \frac{i}{2\pi} \partial \bar{\partial} \ln \left(1 + \sum_{a=1}^N z^a \bar{z}_a\right)$$

or $g_{a\bar{b}} = \partial_a \bar{\partial}_b \ln \left(1 + \sum_{a=1}^N z^a \bar{z}_a\right)$.

$$\boxed{\omega^a{}_b = -\Omega^a{}_b + \Omega^0{}_0 \delta^a{}_b}$$

spin connection of $SU(N) \times U(1)$

and by selecting from the MC eqs:

$$T^a = \boxed{dE^a - \omega^a{}_b \wedge E^b = 0}$$

torsion equations $\boxed{T^a = 0}$

and finally:

$$\boxed{R^a{}_b = d\omega^a{}_b - \omega^c{}_a\,\omega^c{}_b}$$

$$R^a{}_b = E^a \wedge \bar{E}_b + \delta^a_b\, E^c \wedge \bar{E}_c$$

and therefore

$$c_0 = 1$$

$$\hat{c}_1(\mathbb{C}P^N) = \mathrm{tr}\, R^a{}_b = E^a \wedge \bar{E}_a + \delta^a_a\, E^a \wedge \bar{E}_a =$$

$$= (N+1)\, E^a \wedge \bar{E}_a =$$

$$= -2\pi i\, (N+1) \qquad k$$

$$\boxed{c_1(\mathbb{C}P^N) = (N+1)\, k} = \frac{i}{2\pi}\, \mathrm{tr}_{U(N)}\, R$$

$$c_2(\mathbb{C}P^N) = + \frac{1}{8\pi}\left(\mathrm{tr}_{U(N)} R^2 - (\mathrm{tr}_{U(N)} R)^2\right) =$$

$$= \frac{(N+1)N}{2}\, k \wedge k$$

$$c_3(\mathbb{C}P^N) = \frac{1}{3!}\left(\frac{i}{2\pi}\right)^3\left(\mathrm{tr}_{U(N)} R^3 - \mathrm{tr}_{U(N)} R^2\, \mathrm{tr}_{U(N)} R + \dots\right)$$

$$= \frac{(N+1)N(N-1)}{3!}\, k \wedge k \wedge k$$

$$\vdots$$

$$\boxed{c_{tot}(\mathbb{C}P^N) = \det\left(1 + \frac{i}{2\pi} R\right) = (1+k)^{N+1}}$$

Summary

$\boxed{\dim M = 2n}$

Almost Complex Manifolds (M, J)

$\exists\ J: TM \to TM\ ,\ J^2 = -1$

Nijenhuis Tensor

$N(x,y) = 2\{[J(x), J(y)] - [x,y] - J[x, J(y)] - J[J(x), y]\}$

$x, y \in T(M)$

Complex Manifolds (M, J)

$N(x,y) = 0 \quad \forall x, y \in T(M).$

Riemann Complex Manifold. (M, J, g)

the holonomy is reduced from $SO(2n) \to U(n)$.

and J is covariantly constant $\nabla J = 0$.

(J is invariant under the holonomy group. $UJ = J$).

Kähler Manifolds (M, J, g)

$k(x,y) = \frac{1}{2\pi} g(J(x), y)$

$dk = 0 \quad \Rightarrow \quad \nabla J = 0$

$\nabla g = 0$, (or $\nabla k = 0$)

Hodge Manifolds $(M, J, g, \mathcal{L})$

$c_1(\mathcal{L}) = [k] \qquad \mathcal{L} =$ is a line bundle over M.

Holonomy

Riemannian Manifold (M_m, g)

$$\mathrm{Hol}(\nabla) = SO(m).$$

$U = \mathrm{Pexp} \oint_\gamma \omega^a_b$

ω^a_b is the spin-conn.

Kähler M (M_m, g, J)

$\Rightarrow \omega^a_b \in u(m)$

$$\mathrm{Hol}(\nabla) = U(m)$$

---

holomorphic m-form

CY (M_m, g, J, Ω)

$$\mathrm{Hol}(\nabla) = SU(m).$$

$$\begin{cases} \omega^a_b \in su(m) \\ \omega^a_b \delta^b_a = 0 \end{cases}$$

Hodge theory on Complex manifolds

1) $\boxed{g \text{ on } M \text{ (hermitian structure) on } M \text{ if.}}$

$g_x(X,Y)$ on M, $\forall Y \in TM$, is compatible with the complex structure.

$$g_x(J_x(X),Y) + g_x(X, J_x(Y)) = 0$$

$\omega = g(J_x(X),Y) = \omega(X,Y) \rightarrow$ fundamental form

locally:
$$\omega = \frac{i}{2}\sum_{I,J=1}^{n} h_{I\bar{J}}\, dz^I \wedge d\bar{z}^J$$

$\forall x \in M$. $h_{I\bar{J}}(x)$ define a (+ve) hermitian metric.

2) Lefschetz operator.

$$L: \Lambda^k M \longrightarrow \Lambda^{k+2} X$$
$$\alpha \longmapsto \omega \wedge \alpha$$

$\boxed{\text{Depends on the Kähler form } \omega}$

3) Hodge $*$-operator (as usual).

$$*: \Lambda^k M \longrightarrow \Lambda^{2n-k} M$$

$\boxed{\text{Depends on the metric } g.}$

4) Dual Lefschetz operator

$$\Lambda: *^{-1} \circ L \circ * : \Lambda^k M \longrightarrow \Lambda^{k+2} M.$$
$$\alpha \longmapsto *^{-1}(\omega \wedge *\alpha)$$

Decompositions

(M,g) hermitian mfld.

$$\Lambda^k M = \bigoplus_{i\geq 0} L^i (P^{k-2i} M)$$

where

$$P^{k-2i} M = \ker(\Lambda : \Lambda^{k-2i}_M \to \Lambda^{k-2i-2} M)$$

(bundle of primitive forms)

Operators:
$$H = \sum_{k=0}^{2m} (k-m)\,\Pi^k \quad \text{(counting operator)}$$

$$I = \sum_{p,q}^{m} i^{p-q}\,\Pi^{p,q} \quad \text{(real operator)}.$$

$$\Pi^k : A^*(M) \to A^k(M)$$
$$\Pi^{p,q} : A^*(M) \to A^{p,q}(M).$$

Differential operators:

$$\boxed{d^\dagger = (-1)^{m(k+1)+1} * \circ d \circ * : A^k(M) \longrightarrow A^{k-1}(M).}$$

(adjoint operator).

Laplace operator

$$\boxed{\Delta = d^\dagger d + d d^\dagger : A^k(M) \to A^k(M).}$$

if $m = 2m$. (M admits a complex structure).

$$\Rightarrow d^\dagger = - * d * .$$

$$\boxed{\partial^{\dagger} = -* \circ \bar{\partial} \circ * \qquad \bar{\partial}^{\dagger} = -* \circ \partial \circ *.}$$

It can be proven that:

$$\begin{tikzcd} \Lambda^{p,q}(M) \arrow[r, "\partial^{\dagger}"] \arrow[d, "*"'] & \Lambda^{p-1,q}(M) \\ \Lambda^{n-q,n-p}(M) \arrow[r, "\bar{\partial}"] & \Lambda^{n-q,n-p+1}(M) \arrow[u, "-*"'] \end{tikzcd}$$

and $\qquad \bar{\partial}^{\dagger} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q-1}(M).$

In addition,

$$d^{\dagger} = \partial^{\dagger} + \bar{\partial}^{\dagger}, \quad (\partial^{\dagger})^2 = 0, \quad (\bar{\partial}^{\dagger})^2 = 0.$$

$$\left.\begin{array}{l} \Delta_{\partial} = \partial^{\dagger}\partial + \partial\partial^{\dagger} \\ \Delta_{\bar{\partial}} = \bar{\partial}^{\dagger}\bar{\partial} + \bar{\partial}\bar{\partial}^{\dagger} \end{array}\right\} \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q}(M).$$

A <u>Kähler structure</u> (or Kähler metric).
is an hermitian structure g for which the <u>fundamental form</u> ω is closed

$$d\omega = 0 \quad \Rightarrow \quad \partial\omega + \bar{\partial}\omega = 0 \Rightarrow \begin{cases} \partial\omega = 0 \\ \bar{\partial}\omega = 0 \end{cases}$$

ω is denoted by <u>Kähler form</u>

Hodge condition

We need a line bundle over $\mathbb{CP}^N$ such that we can implement the Hodge condition.

$$\mathcal{O}_{1,N+1} \longrightarrow \mathbb{CP}^N.$$

Hyperplane bundle.

$\boxed{\text{Homogeneous polynomial of degree 1 in the homog. coords}}$

total space $\{X^A\} \xrightarrow{\pi}$ point of $\mathbb{CP}^N$

$(A=0,\dots,N)$

The fiber is 1-d holomorphic vector bundle (line-bundle).

$(X^0,\dots X^N) \longrightarrow [X^0,\dots,X^N]$

$S(z) = \{X^A(z)\}$ frame of sections $\quad X^A: \mathbb{CP}^N \to \mathcal{O}_{1,N+1}$

$$s'(z) = \{X^{A'}(z)\} = \{c(z)\, X^A(z)\} = c(z)\{X^A(z)\} =$$

$$= c(z)\, S(z)$$

local coordinate on $\mathcal{L}$ in the frame $\{X^A\}$

$$\|s'(z)\|^2 = \sum_{A=0}^{N} \bar{X}^{A'}(\bar{z})\, X^{A'}(z) = \sum_{A=0}^{N} \bar{X}^A(\bar{z})\, X^A(z)\, \bar{c}(\bar{z})\, c(z) =$$

$$= \bar{c}(\bar{z})\, c(z)\, h(z,\bar{z})$$

$$\boxed{h(z,\bar{z}) = \sum_{A=0}^{N} \bar{X}^A(\bar{z})\, X^A(z)}$$

Metric on the line bundle.

Therefore:

$$\omega = h^{-1}\partial h = \frac{1}{|X|^2}\partial |X|^2 = \frac{1}{|X|^2}\sum_{A=0}^{N}\bar{X}^A \partial X_A$$

$$\begin{cases} R = d\omega + \omega_\wedge\omega = (\bar\partial + \partial)\omega + \omega_\wedge \omega = \bar\partial\omega + \underbrace{(\partial\omega + \omega_\wedge\omega)}_{=0} \\ \quad = \bar\partial(h^{-1}\partial h) = -\frac{1}{|X|^4}\left[|X|^2\delta_{A\bar B} - \bar X_A X_{\bar B}\right]\partial X^A{}_\wedge \bar\partial \bar X^{\bar B} \\ |X|^2 = \sum_{A=0}^{N} \bar X^A X_A \end{cases}$$

Notice that is well define on $\mathbb{CP}^N$.

$$R \to -\frac{1}{|c|^4|X|^4}\cancel{|c|^2}\left(|X|^2\delta_{A\bar B} - \bar X_A X_{\bar B}\right)\partial(cX^A)_\wedge \bar\partial(\bar c \bar X)$$

$$= -\frac{1}{|c|^2|X|^4}\left(|X|^2\delta_{AB} - \bar X_A X_{\bar B}\right)\Big[|c|^2\partial X^A{}_\wedge\bar\partial\bar X^B + (\bar c\,\partial c)X^A{}_\wedge\bar\partial X^B + (\bar\partial\bar c)c\,\partial X^A{}_\wedge \bar X^B\Big] =$$

but
$$\begin{cases}\left(|X|^2\delta_{AB} - \bar X_A X_{\bar B}\right)X^{\bar B} = 0 \\ |X|^2\delta^X_{AB} - X_A\bar X^A X_B\end{cases}$$

$= R$ (so it is invariant under the rescaling and therefore it survives the projections).

$$\Rightarrow \boxed{C_1(U_{1,N+1}) = \frac{i}{2\pi}\operatorname{tr}R = k}$$

is the Kähler two form computed by the FS metric.

For the hyperplane bundle (on linear combinations of the coordinates)

$$C_{tot}(\mathcal{U}_{1,N+1}) = 1 + k.$$

If we consider polynomials of degree ν_α, $W_\alpha(X)$ (homogeneous)

they transform as sections of ν_α-power of the $U_{1,N+1}$

$$N_\alpha \simeq (\mathcal{U}_{1,N+1})^{\nu_\alpha}$$

$$\Rightarrow \boxed{C_{tot}(N_\alpha) = (1 + \nu_\alpha k)}$$

Consider an <u>algebraic</u> m-dim surface $M_m \subseteq \mathbb{CP}^N$ defined as the zeros of:

$$\{ W_\alpha(X) = 0 \}_{\alpha = 1 \dots r} \qquad r = N - m.$$

$$\Rightarrow \quad T(\mathbb{CP})^N = TM_m \oplus \underbrace{N(M_m)}_{\text{Normal bundle}}.$$

$\Rightarrow$ by the Whitney formula:

$$C_{tot}(T(\mathbb{CP}^N)) = C_{tot}(T(M_m)) \oplus C_{tot}(N(M_m)).$$

$$C_{tot}(N(M_m)) = \prod_{i=1}^{r} (1 + \nu_i k)$$

$$\Rightarrow \boxed{C_{tot}(T(M_m)) = \frac{(1+k)^{m+r+1}}{\prod_{i=1}^{r}(1+\nu_i k)}}$$

$$\Rightarrow \boxed{c_1(M_n) = \left(n+r+1-\sum_{i=1}^{r}\nu_i\right)k}$$

Hodge manifold

and it vanishes if

$$\boxed{n+r+1=\sum_{i=1}^{r}\nu_i}$$

if $c_1(M_n)=0$ (for a CY).

$$\Rightarrow \quad n+r+1=\sum_{i=1}^{r}\nu_i$$

e.g. $\begin{cases} n=3 \\ r=1 \\ \nu_1=5 \end{cases}$ CY_3 (dimension of the 3-d. surface in $\mathbb{CP}^{n+r} = \mathbb{CP}^4$)

one single polynomial.

of degree 5.

e.g. $\boxed{x^5+y^5+z^5+t^5+4\,xytz=0}$

Susy and Kähler manifolds

sab 1

<u>Super vector space</u>: is a vector space V endowed with the direct sum decomposition: $V = V_0 \oplus V_1$

$(\mathrm{End})(V)$ is also a super-vector space

$$\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$$

$F \in \mathrm{End}(V)_0$ if $F(V_i) = V_i$

$F \in \mathrm{End}(V)_1$ if $F(V_i) = V_{i+1}$ $\qquad V_2 = V_0.$

<u>Super Lie algebra</u>:

$\mathbb{C}$-linear <u>even</u> homomorphism $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

$$[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \rightarrow$

i) $[a,b] = -(-)^{|a||b|}[b,a]$ $\qquad |a| = \begin{cases} 0 & a \in V_0 \\ 1 & a \in V_1 \end{cases}$

ii) $[a,[b,c]] = [[a,b],c] + (-)^{|a||b|}[b,[a,c]].$

$$\Lambda^*_{\mathbb{C}}(M) = \underbrace{\left(\bigoplus_k \Lambda^{2k}_{\mathbb{C}}(M)\right)}_{\text{even}} \oplus \underbrace{\left(\bigoplus_k \Lambda^{2k+1}_{\mathbb{C}}(M)\right)}_{\text{odd.}}$$

sub 2

Riemannian Geometry (and SUSY)

(M, g) compact, oriented, Riemann mfld.

$(d, d^*, \Delta = dd^* + d^*d)$

$d^2 = 0 \quad (d^*)^2 = 0 \quad \Delta = dd^* + d^*d.$

$\Rightarrow \quad Q_1 = d + d^* \qquad Q_2 = i(d - d^*).$

$$\boxed{\begin{cases} \{Q_1, Q_1\} = 2\Delta = \{Q_2, Q_2\} \\ [Q_1, Q_2\} = 0 \end{cases}}$$

$\Rightarrow$ $N = (1,1)$ susy algebra.

$\{Q_1, \Delta\} = \{Q_2, \Delta\} = [\Delta, \Delta] = 0.$

$g_0 = \Delta \qquad g_1 = (d, d^*)$

Complex Geometry

(M, J) where J is " " a complex structure.

$d = \partial + \bar{\partial}, \Rightarrow \quad [d, \partial] = 0 \quad [d, \bar{\partial}] = 0$

$[\partial, \bar{\partial}]_+ = 0.$

(where we need the integrability of the complex structure).

Adding a Riemann structure $\Rightarrow$ $d^* = \partial^+ + \bar{\partial}^+$

so we have $\Delta, \partial, \bar{\partial}, \partial^+, \bar{\partial}^+$ but they are not closed! The super lie algebra might be infinite dim !!

For example $\left\{\begin{array}{l} \partial\partial^{+} + \partial^{+}\partial = \Delta_\partial \\ \bar\partial\bar\partial^{+} + \bar\partial^{+}\bar\partial = \Delta_{\bar\partial} \end{array}\right.$ 3

new laplacian!.

<u>Kähler Geometry</u> (then $\exists\, \omega$ such that $d\omega = 0$)

the kähler condition $d\omega = 0$.

$\Rightarrow$ kähler identities. and therefore.

$\Delta,\ \partial, \bar\partial,\ \partial^{+},\ \bar\partial^{+}$ is closed and

it forms a Lie superalgebra.

Adding also $(L, \Lambda, \mathcal{H})$ $\Rightarrow$

$\mathcal{H}$ = Counting operator.

$\mathcal{H}\, \omega^{(p,q)} = (p+q)\, \omega^{(p+q)}$

$g_0 = (\mathcal{H}, L, \Lambda, \Delta)$.

$g_1 = (\partial, \bar\partial, \partial^{+}, \bar\partial^{+})$.

$\to$ which corresponds to an $N=(2,2)$ algebra.

In addition it can be proved that $(\mathcal{H}, L, \Lambda)$ form an $SL(2)$ multiplet.

$\Delta \longleftrightarrow T$ (energy momentum tensor).

$\mathcal{H}, L, \Lambda \longleftrightarrow J_i$ ($SO(2)$ -currents)

$\begin{array}{c}\partial, \bar\partial \\ \partial^{+} \bar\partial^{+}\end{array} \longleftrightarrow$ supercharges $Q_i, \bar Q_i$.

which can be realized by a following σ-model.

Lemma: ω is a closed $(1,1)$-form on M (cmplx mfld). $\quad$ 4

ω is (+ve) definite $\quad \omega = \frac{i}{2} h_{I\bar{J}} dz^I \wedge d\bar{z}^{\bar{J}}$,

such that $h_{I\bar{J}}$ is positive definite $\forall x \in M$.

$\Rightarrow \exists$ a kähler metric g ~~st~~ s.t. ω is the associate kähler form.

Coroll: Any projective manifold is kähler.

kähler identities

i) $[\bar{\partial}, L] = [\partial, L] = 0$

$[\bar{\partial}^+, \Lambda] = [\partial^+, \Lambda] = 0.$

~~ex~~ : $[\bar{\partial}, L](\alpha) = \bar{\partial} L(\alpha) - L(\bar{\partial}\alpha) =$

$= \bar{\partial}(\omega \wedge \alpha) - \omega \wedge \bar{\partial}\alpha =$

$= \underbrace{\bar{\partial}\omega}_{0} \wedge \alpha + \omega \wedge \bar{\partial}\alpha - \omega \wedge \bar{\partial}\alpha = 0$

0 because $d\omega = 0$.

$[\bar{\partial}^+, \Lambda](\alpha) = - * \partial * *^{-1} L * (\alpha) - *^{-1} L * (- * \partial *)(\alpha) =$

$= - * \partial L * (\alpha) - (*)^{-1} (-)^k L \partial (* \alpha) =$

$= - * \partial L * (\alpha) + * L \partial * (\alpha) =$

$= - * [\partial, L] * (\alpha) = 0$

where we used $*^2 = (-)^k$ for a cmplx mfld.
(on $A^k(M)$)

ii) $[\bar\partial^+, L] = i\partial \qquad [\partial^+, L] = -i\bar\partial$

$[\Lambda, \bar\partial] = -i\partial^+ \qquad [\Lambda, \partial] = i\bar\partial^+$

(we omit the proof).

iii)

$\partial\bar\partial^+ + \bar\partial^+\partial = 0$:

indeed $i(\partial\bar\partial^+ + \bar\partial^+\partial) =$

$= \cancel{i}\,\partial[\Lambda,\partial] + [\Lambda,\partial]\partial =$

$= \partial\Lambda\partial - \Lambda\partial^2 + \Lambda\partial^2 - \partial\Lambda\partial = 0.$

$$\Delta_\partial = \partial^+\partial + \partial\partial^+ = i[\Lambda,\bar\partial]\partial + \partial\, i[\Lambda,\bar\partial] =$$

$$= i\left(\Lambda\bar\partial\partial - \bar\partial\Lambda\partial + \partial\Lambda\bar\partial - \partial\bar\partial\Lambda\right) =$$

$$= i\left(\Lambda\cancel{\bar\partial\partial} - (\bar\partial[\Lambda,\partial] + \bar\partial\cancel{\partial}\Lambda) + ([\partial,\Lambda]\bar\partial + \Lambda\cancel{\partial\bar\partial}) - \partial\cancel{\bar\partial}\Lambda\right) =$$

$$= \bar\partial\bar\partial^+ + \bar\partial^+\bar\partial = \Delta_{\bar\partial}$$

$$\Delta = (\partial+\bar\partial)(\partial^+ + \bar\partial^+) + (\partial^+ + \bar\partial^+)(\partial+\bar\partial) = \partial\partial^+ + \cancel{\partial\bar\partial^+} + \cancel{\bar\partial\partial^+} + \bar\partial\bar\partial^+ + \partial^+\partial + \cancel{\partial^+\bar\partial} + \bar\partial^+\bar\partial + \cancel{\bar\partial^+\partial} =$$

$$= \Delta_\partial + \Delta_{\bar\partial} = 2\Delta_\partial = 2\Delta_{\bar\partial}.$$

of course $[\Delta,\partial] = 0.$ ....

Definitions

CY

Consider $\mathbb{C}^m$:

$$g = \sum_{I=1}^{m} dz_I \, d\bar{z}_I \quad , \quad \omega = \frac{i}{2} \sum_{I=1}^{m} dz_I \wedge d\bar{z}_I \qquad (h_{\bar{I}J} = \delta_{\bar{I}J})$$

$$\underset{(m)}{\Omega^{(m,0)}} = dz_1 \wedge \dots \wedge dz_m \quad , \quad \Omega^{(0,m)} = d\bar{z}_1 \wedge \dots \wedge d\bar{z}_m$$

top form

$\mathbb{C}^m$ has a group structure $GL(m, \mathbb{C})$.
(namely we can redefine the coords
$z_I \to A_I{}^J z_J \quad A \in GL(m, \mathbb{C})$)

$\downarrow$ Introduce the metric $\qquad\qquad \downarrow$

$(\mathbb{C}^m, h_{\bar{I}J} = \delta_{\bar{I}J}) \qquad\qquad U(m)$
(group which preserves the metric h)

$\downarrow$ Add the holomorphic top form Ω $\qquad\qquad \downarrow$

$(\mathbb{C}^m, h_{\bar{I}J}, \Omega) \qquad\qquad SU(m)$
(constant)

There is a unique J on M, such that $\omega_{\bar{I}J} = J_J{}^k h_{\bar{I}k}$

$\Rightarrow$ (ω, g, J) is a kähler structure on M,
Ω is a holomorphic top form (Volume form) (w.r.t. J)

$\Longrightarrow$ **Every Riemannian manifold w. holonomy $SU(m)$ is kähler manifold w. a constant hol. volume form Ω.**

$\Leftarrow$ if (M, J, g) is kähler ($d\omega = 0$) and

Ω is a holomorphic volume form on M

with $\nabla\Omega = 0$ $\Rightarrow$ $Hol(\nabla) \subseteq SU(m)$.

$\Lambda^{m,0}M$ = canonical bundle k_M on M.

hol. line bundle and

$\Omega \in \Gamma(M, \Lambda^{m,0}M) = \Gamma(M, k_M)$

$\Omega \exists$ iff k_M is trivial namely if

(non-where vanishing) on M

$k_M \cong M \times \mathbb{C} \to M$

$\Rightarrow$ if (M, J, g) is kähler and $Hol(\nabla) \subseteq SU(m)$

$\Rightarrow$ k_M is trivial.

Since $c_1(M)$ of M is a char. class of k_M (no $H^2 = 0$)

$(c_1(k_M) \in H^2(M, \mathbb{Z}))$, the triviality of $k_M \Rightarrow$

$\boxed{c_1(M) = 0 \text{ in } H^2(M, \mathbb{Z})}$

$c_1(M) = [\omega^{(1,1)}]$ $\quad \int_\alpha \omega^{(1,1)} = n \in \mathbb{Z}$.

$\Rightarrow$ (M, J, g) kähler and $Hol(\nabla) \subseteq SU(m)$

$\Rightarrow c_1(M) = 0$ (namely Ricci flat)

$\boxed{(M, J, g) \text{ a kähler. } Hol(g) \subseteq SU(m) \text{ iff } g \text{ is Ricci flat.}}$

<u>Proof</u> g is kähler, Levi-Civita connection ~~LC~~ can $\nabla \to \nabla_k$ on $K_M = \Lambda^{m,0} M$

w. $Hol(\nabla^k) \subseteq U(1)$ (line bundle).

If $A \in U(m)$ acts on $\mathbb{C}^m$,

A act on $\Lambda^{(m,0)} M$ by multiplying by $\det A$.

$$A^I{}_J z^J = z'^I,$$

$$\Omega \xrightarrow[A]{} (\det A) \Omega.$$

$\Rightarrow$ $Hol(\nabla) \to Hol(\nabla^k) = \det(Hol(\nabla))$

where $\det : U(m) \to U(1)$
(determinant map).

$\Rightarrow$ if $Hol(\nabla) = SU(m)$, $\boxed{Hol(\nabla^k) = \{1\}}$ if and ~~only~~ only if $Hol(\nabla) \subseteq SU(m)$

By Frobenius thm.

$$Hol(\nabla^k) = \{1\} \text{ iff } \boxed{Ric(\nabla^k) = 0}.$$

$\Rightarrow R(\nabla^k) = 0$. Since the gauge group of ∇^k is $U(1)$ so $R(\nabla^k)$ is a closed 2-form, $dR(\nabla^k) = 0$ (in general $\boxed{dR - \omega \wedge R = 0}$)

It can be shown ~~that~~ this

$\Rightarrow R(\nabla^k) = \partial\bar{\partial} \ln h = \underline{\text{Ricci form}} \Rightarrow$ if $Hol(\nabla) \subseteq SU(m)$ $\Updownarrow$ g is Ricci flat

(Using the dd^c-lemma)

A CY is a compact kähler manifold $\boxed{\text{DEF}}$
(M, J, g) of $\dim_{\mathbb{C}} m \geq 2$ with $Hol(g) = SU(m)$.

(the torus in this context is not a CY).

$m = 2$ $K3$ $Hol(K3) = SU(2)$

$\longrightarrow$

YAU Thm.

$\left\{ \begin{array}{l} M \text{ is kähler}, \quad c_1[M] = 0 \text{ in } H^2(M, \mathbb{R}). \\ \Rightarrow \exists\, g' \text{ on } M, \text{ with } R(g') = 0 \quad \Rightarrow \quad Ric(g') = 0. \end{array} \right.$

Thm:

(M, J) compact complex manifold w. g kähler and $c_1(M) = 0$. $\exists!\, g'$ Ricci flat $\uparrow$ + kähler class on M. The Ricci-flat kähler metric metrics on M form a smooth family of dim $h^{(1,1)}(M)$.

CALABI conj (Before Yau thm).

(M, J) compact complex manifold, g kähler on M, ω its kähler form. If $R'^{(1,1)}$ closed, $dR'^{(1,1)} = 0$

w. $\boxed{[R^{(1,1)}] = 2\pi c_1(M)}$ (Hodge manifold).

$\Rightarrow \exists\, g'$ on M, w. ω' such that $[\omega] = [\omega'] \in H^2(M, \mathbb{R})$

$\boxed{R(g') = R'}$

Properties of CY manifolds

$c_1(M) = 0$ (compact Kähler)

$$c_1(M) = \left[\frac{i}{2\pi} \operatorname{tr} R^i{}_j\right] = \frac{i}{2\pi} R^k{}_{k i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$$

$$R^i{}_{j k \bar{e}}\, dz^k \wedge d\bar{z}^{\bar{e}} = R^i{}_j \quad \text{(not with values in the group } U(M)\text{)}$$

This is the condition for an Hodge manifold $\boxed{[R] \propto [k]}$

and $c_1(M) = 0 \Rightarrow$ (the Ricci ~~class~~ form)

$$\begin{cases} R_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}} = dA = \\ \qquad = \left(\partial_i A_{\bar{j}} - \bar{\partial}_{\bar{j}} A_i\right) dz^i \wedge d\bar{z}^{\bar{j}} \\ A = A_i dz^i + \bar{A}_{\bar{i}}\, d\bar{z}^{\bar{i}} \end{cases}$$

Theorem: (YAU)

M is CY complex n-fold, $c_1(M) = 0$

$\forall\, \omega^{(1,1)} \in H^{(1,1)}\ \exists\, K = g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$ such that

i) $[K] = [\omega^{(1,1)}]$

ii) $R_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}} = 0$

SU(n) - holonomy - Ricci - flatness

$$\Gamma^{I}{}_{JK}\, dz^{K} = (\Gamma)^{I}{}_{J}$$

is $O(n)$ valued algebra.

$$e^{a} = e^{a}_{I}\, dz^{I} \qquad \overline{e^{a}} = e^{\bar a}_{\bar I}\, d\bar z^{\bar I}$$

$(e^{a}, \bar e^{a})$ Vielbeins of M. (Complex manifold in anadolta basis)

this spin connection satisfies.

$$\begin{cases} T^{a} = de^{a} - \omega^{a}{}_{b} \wedge e^{b} = 0 \\ \omega^{a}{}_{b} = - \overline{\omega}^{\bar b}{}_{\bar a} \in T[U(n)]. \end{cases}$$

$$g_{I\bar J} = e^{a}_{I}\, \overline{e}^{\bar b}_{\bar J}\, \eta_{a\bar b} \qquad \eta_{a\bar b} = (\underbrace{+, \dots, +}_{n})$$

which is left invariant under $U(n)$ rotation.

$$e^{a}_{I} \to U^{a}{}_{b}\, e^{b}_{I}$$

$$\boxed{g_{I\bar J} = e_{I}^{+} e_{\bar J}}$$

$$R^{a}{}_{b} = d\omega^{a}{}_{b} - \omega^{a}{}_{c} \wedge \omega^{c}{}_{b} = e^{a}_{I}\, \bar e^{J}_{b}\, dz^{m} \wedge d\bar z^{n}\, R^{I}{}_{Jm\bar n}$$

since $e^{a}_{I} e^{J}_{b} \delta^{b}_{a} = \delta^{J}_{I}$

$\boxed{\text{Ricci form}} \longrightarrow \mathcal{R} = \mathrm{tr}(R) = R^{a}{}_{b}\delta^{b}_{a} \Rightarrow U(1)$ part

$$\boxed{\omega^{a}{}_{b} = \underbrace{\hat\omega^{a}{}_{b}}_{\text{traceless}} + \frac{1}{n}\delta^{a}_{b} A_{U(1)}}$$

$c_1(M) = 0 \Rightarrow$

$$\boxed{\mathcal{R} = \mathrm{tr}(\mathcal{R}^a{}_b) = d A_{U(1)}.}$$

$A_{U(1)}$ is <u>globally defined</u> 1-form

If we choose the Ricci-flat metric $\Rightarrow$

$$\begin{cases} \mathcal{R} = 0 & \Rightarrow \text{ the holonomy group is } SU(n) \\ \text{and } \; d A_{U(1)} = 0 & \end{cases}$$

<u>Relation between Harmonic forms and Spinors</u>

<u>Complex manifold</u>: $M_{m\text{-dim}}$

$$\{\Gamma_a, \Gamma_b\} = 0 \qquad \{\Gamma_{\bar a}, \Gamma_{\bar b}\} = 0$$

$$\{\Gamma_a, \Gamma_{\bar b}\} = 2\eta_{a\bar b} \qquad a, b = 1 \dots m.$$

$$\Gamma_I = e^a_I \Gamma_a \qquad \Gamma^I = g^{I\bar J}\Gamma_{\bar J}$$

$$\Gamma_{\bar I} = e^{\bar a}_{\bar I}\Gamma_{\bar a} \qquad \Gamma^{\bar I} = g^{\bar I J}\Gamma_J$$

and

$$\{\Gamma_I, \Gamma_J\} = \{\Gamma_{\bar I}, \Gamma_{\bar J}\} = 0$$

$$\{\Gamma_I, \Gamma_{\bar J}\} = 2 g_{I\bar J}$$

Introduce ξ a spinor.

$$\boxed{\Gamma_{\bar I}\,\xi = 0} \quad \bar I = 1 \dots m$$

$$\begin{cases} \psi(z,\bar z) = \omega^{(0,0)}(z,\bar z)\,\xi + \\ + \omega^{(0,1)}_{\bar I}(z,\bar z)\,\Gamma^{\bar I}\xi + \\ + \omega^{(0,2)}_{\bar I\bar J}\,\Gamma^{\bar I\bar J}\xi + \\ \vdots \\ + \omega^{(0,n)}_{\bar I_1\ldots\bar I_n}\,\Gamma^{\bar I_1\ldots\bar I_n}\xi \end{cases}$$

$\omega^{(0,k)}_{\bar I_1\ldots\bar I_k}(z,\bar z)$ transforms under coords changes as a $(0,k)$-differential form.

$$\Gamma^{\bar I_1\ldots\bar I_r} = \Gamma^{[\bar I_1}\ldots\Gamma^{\bar I_r]}$$

1 <u>Clifford algebra</u> $\longleftrightarrow$ k-diff.

ξ = Clifford vacuum.

2 <u>Parity</u> $\Gamma_{2n+1}\xi = \xi \qquad \Gamma_{2n+1} = \Gamma_1\ldots\Gamma_{2n}$.

$$\omega_R = \{\quad,\ \omega^{(0,1)}\Gamma\xi,\ \omega^{(0,3)}\Gamma^3\xi,+\ldots\}$$
$$\omega_L = \{\omega^{(0,0)}\xi,\ \omega^{(0,2)}\Gamma^2\xi,\ \ldots\quad\}$$

3 <u>Dirac operator</u>

$$\not{D} = \Gamma^I\partial_I + \Gamma^{\bar I}\partial_{\bar I} = \not{D}_+ + \not{D}_-$$

$$\Gamma^I\partial_I\left(\omega_{\bar J}\Gamma^{\bar J}\xi\right) = \left(\partial_I\omega_{\bar J}\right)\Gamma^I\Gamma^{\bar J}\xi =$$
$$= \left(\partial_I\omega_{\bar J}\right) 2g^{I\bar J}\xi$$

By assuming that the spinor ξ is covariant constant. $\Rightarrow \boxed{\not{D}\xi = 0}$

$$\begin{cases} \not{D}_- : \omega^{(0,k)} \longrightarrow \bar\partial\omega^{(0,k)} = \partial_{[\bar I}\omega_{\bar J_1\ldots\bar J_k]}\,d\bar z^{\bar I}\wedge\ldots\wedge d\bar z^{\bar J_{k-1}} \\ \not{D}_+ : \omega^{(0,k)} \longrightarrow \bar\partial^\dagger\omega^{(0,k)} = \partial_I\,\omega_{\bar J\bar J_1\ldots\bar J_{k-1}}\,g^{I\bar J}\,d\bar z^{\bar J_1}\wedge\ldots\wedge d\bar z^{\bar J_k} \end{cases}$$

Al contat spinor $\boxed{\nabla \xi = 0}$

$) = [\nabla_A, \nabla_B] \Gamma^A \Gamma^B \xi =$

$\Gamma^{AB} \xi = 0 \quad \Rightarrow$

$\Rightarrow$ multiplying by Γ^C and using the Bianchi identity $R_{C[DAB]} = 0$.

Non fare

$\xi = 0$

tion only if $\boxed{R^c{}_{eAB} = 0}$

e is Ricci flat)

An important theorem:

A compact Kähler mfld M_m with $c_1(M) = 0$ iff admits a unique holomorphic m- form

$$\Omega = \frac{1}{m!} \Omega_{I_1 \dots I_m}(z)\, dz^{I_1} \wedge \dots \wedge dz^{I_m}$$

with properties

i) Ω is harmonic

ii) the component of Ω are cov. contant in the Ricci-flat metric.

$$\nabla_I \Omega_{J_1 \dots J_m} = 0 \, , \quad \Omega \in H^{(m,0)}(M).$$

and it follows:

$$\boxed{\text{A CY } n\text{-fold has a } \underline{\text{one-dim.}} \text{ Dolbeault cohomology group } H^{(n,0)} : \boxed{h^{(n,0)} = 1}}$$

I). suppose that $\exists\, \Omega$ i) harmonic
ii) $\nabla_g \Omega = 0$.

and we would like to prove that
$\Rightarrow R(g) = 0$.

$$\|\Omega\|^2 = \frac{1}{n!} \Omega_{I_1 \dots I_n} g^{I_1 \bar{J}_1} \dots g^{I_n \bar{J}_n} \overline{\Omega}_{\bar{J}_1 \dots \bar{J}_n}.$$

$$= \frac{1}{n!} \Omega_{I_1 \dots I_n} \overline{\Omega}^{I_1 \dots I_n}$$

in each coord. patch: U

$$\Omega_{I_1 \dots I_n}(z) = f(z)\, \epsilon_{I_1 \dots I_n}$$

$$\begin{cases} f(z) \in C^\infty(M) \text{ holomorphic.} \\ \neq 0. \text{ on the patch. } U. \end{cases}$$

$$\overline{\Omega}^{I_1 \dots I_n} = \bar{f}(\bar{z})\, g^{I_1 \bar{J}_1} \dots g^{I_n \bar{J}_n} \epsilon_{\bar{J}_1 \dots \bar{J}_n} =$$

$$= \bar{f}(\bar{z}) \frac{1}{\sqrt{g}} \epsilon^{I_1 \dots I_n}.$$

the square root is coming from the fact that we consider only the holomorphic sector.

$$\Rightarrow \boxed{\|\Omega\|^2 = \frac{1}{\sqrt{g}} |f|^2}$$

$$\rightsquigarrow \sqrt{g} = \frac{|f|^2}{\|\Omega\|^2}$$

$$\Rightarrow \frac{i}{2\pi}\mathcal{R} = \frac{i}{2\pi}\operatorname{tr} R = \frac{i}{2\pi} R_{I\bar{J}}\, dz^I \wedge d\bar{z}^{\bar{J}} =$$

$$= \frac{i}{2\pi}\partial\bar{\partial}\ln\sqrt{\det g} =$$

$$= \frac{i}{2\pi}\partial\bar{\partial}\ln\frac{|f|^2}{\|\Omega\|^2} = -\frac{i}{2\pi}\partial\bar{\partial}\ln\|\Omega\|^2.$$

This is globally defined since by def. $\|\Omega\|^2$ is a scalar.

$$\Rightarrow \boxed{[\mathcal{R}] = 0. \text{ (since it is exact)}}$$

[Notice that in general $\partial\bar{\partial}\sqrt{\det g}$ is not globally define.]

II) if $c_1(M) = 0$ $\Rightarrow$ $\boxed{\mathcal{R} = R_{I\bar{J}}\, dz^I \wedge d\bar{z}^{\bar{J}} = dA}$

$$A = A^{(1,0)} + A^{(0,1)}$$

$$dA = (\partial + \bar{\partial})A^{(1,0)} + (\partial + \bar{\partial})A^{(0,1)} =$$

$$= \partial A^{(1,0)} + \left[\bar{\partial}A^{(1,0)} + \partial A^{(0,1)}\right] + \bar{\partial}A^{(0,1)}$$

$$\mathcal{R} \in H^{(1,1)} \rightarrow \begin{cases} i\mathcal{R} = \bar{\partial}A^{(1,0)} + \partial A^{(0,1)} \\ \partial A^{(1,0)} = \bar{\partial}A^{(0,1)} = 0 \end{cases}$$

If $\chi_{EULER} \neq 0$ $\quad h^{(1,0)} = h^{(0,1)} = 0$

(to see below).

$\Rightarrow \bar{\partial} A^{(1,0)} = 0 \qquad \partial A^{(0,1)} = 0 \Rightarrow$

$A^{(0,1)} = \bar{\partial} \bar{\alpha}(z, \bar{z})$

$A^{(1,0)} = \partial \alpha$ $\qquad \alpha$, is globally defined $(0,0)$-form on M_n.

So, the solution of $iR = \bar{\partial} A + \partial \bar{A}$ is given by

$\Rightarrow$ so we can define:

$$\boxed{\begin{aligned} \Omega &= \Omega_{I_1 \cdots I_n} dz^{I_1} \wedge \cdots \wedge dz^{I_n} \\ \Omega_{I_1 \cdots I_n} &= e^{-i\bar{\alpha}} \left(\xi^T \Gamma_{I_1 \cdots I_n} \xi \right) \end{aligned}}$$

where ξ satisfies

$$\boxed{\nabla \xi = \frac{i}{2} A \xi} \Rightarrow \begin{cases} \nabla_{\bar{k}} \xi = \frac{i}{2} (\partial_{\bar{k}} \bar{\alpha}) \xi \\ \nabla_k \xi = \frac{i}{2} (\partial_k \alpha) \xi \end{cases}$$

but this equation needs an auxiliary condition:

$$\nabla^2 \xi = \left(-\frac{1}{4} R^{AB}{}_{CD} \Gamma_{AB} + F_{CD} \right.$$

$$\nabla^2 \xi = \frac{i}{2} (\nabla A) \xi - \frac{i}{2} A \nabla \xi =$$

$$= \frac{i}{2} (\nabla A) \xi - \frac{i}{2} \underbrace{A \wedge \left(\frac{i}{2} A \xi \right)}_{0}$$

$$\boxed{\left(-\frac{1}{2} R^{AB}{}_{CD} \Gamma_{AB} + \frac{i}{2} F_{CD} \right) \xi = 0}$$

if we choose also $\Gamma_{a} \xi = 0$

$$\rightarrow \boxed{R^{\bar{b}}{}_{a} \Gamma^{b}{}_{b} + i\, \Gamma^{\bar{b}} F_{\bar{b} a} = 0} \Leftrightarrow \boxed{\bar{\partial} A^{(1,0)} + \partial A^{(0,1)} = iR}$$

Since ξ $\exists$. we have:

$$\nabla_{\bar{k}}\left(e^{-i\bar{\alpha}}\ \xi^{T}\Gamma_{I_1\ldots I_n}\xi\right) =$$

$$= -i\ \nabla_{\bar{k}}\bar{\alpha}\ \Omega_{I_1\ldots I_n} + e^{-i\bar{\alpha}}\ (\nabla_{\bar{k}}\xi)^{T}\Gamma_{I\ldots I_n}\xi +$$

$$+ e^{-i\bar{\alpha}}\ \xi^{T}\Gamma_{I_1\ldots I_n}\nabla_{\bar{k}}\xi =$$

$$= -i\ \nabla_{\bar{k}}\bar{\alpha}\ \Omega_{I_1\ldots I_n} + \frac{1}{2}(\nabla_{\bar{k}}\bar{\alpha})\ \Omega_{I_1\ldots I_n} + \frac{1}{2}\nabla_{\bar{k}}\bar{\alpha}\ \Omega_{I_1\ldots I_n}$$

$$= 0$$

$\Rightarrow$ $\boxed{\bar{\partial}\Omega = 0}$ $\boxed{\partial\Omega = 0}$ this is because Ω is a top form.

Since Ω is a n-form. top-form $\Rightarrow$ $\boxed{d\Omega = 0}$

but

$$\nabla^{I}\Omega_{IJ_2\ldots J_n} = g^{I\bar{k}}\nabla_{\bar{k}}\Omega_{I_1\ldots I_n} = 0$$

$\Rightarrow$ <u>Ω is Harmonic</u>.

$\Omega_{I_1\ldots I_n}$ are covariantly constant. if

$$\nabla_{k}\Omega_{I_1\ldots I_n} = 0.$$

this in general is not true. since I would need an $e^{i\alpha}$ in the exponential. this means that $\nabla_k\Omega_{I_1\ldots I_n} = 0$ $\boxed{\text{only if } \alpha = 0 \text{ which is true if } c_1(M) = 0}$

In addition $\|\Omega\|^2 = e^{i(\alpha-\bar{\alpha})}\xi^{+}\xi > 0.$

(by using Fierz rearrangements).

$(4,0)$- form $\tilde{\Omega}$

$$\tilde{\Omega} = h(z,\bar{z})\, dz^{I_1} \wedge \dots \wedge dz^{I_4}\, \epsilon_{I_1 \dots I_4}$$

h is non-singular local function.

However h is not a function $\rightarrow$ (h : is $\Gamma(M, \Omega^{(4,0)}(M))$

canonical bundle over M.
(determinant)

$$h(z,\bar{z}) \xrightarrow[\text{another patch}]{} h'(z,\bar{z}) = \underbrace{\det\left(\frac{\partial z^{I'}}{\partial z^{I}}\right)}_{\substack{\text{Jacobian} \\ \text{(holomorphic)}}} h(z,\bar{z})$$

$R(\Omega^{4}$

$R(\nabla^{4}) = \operatorname{tr} R(\nabla)$

$\boxed{R(}$ $\Rightarrow$ $\cancel{c_1(\nabla^{4})}$ $c_1(k_M) = c_1(M)$

$\Rightarrow c_1(M) = 0 \Rightarrow \exists$ global hol. section.

$\Rightarrow c_1(\Omega$

(the canonical bundle is trivial) for section.

The Kähler form can be also written as follow

$$k = (\xi^{+} \Gamma_{I\bar{J}}\, \xi)\, dz^{I} \wedge dz^{J}$$

Using again the killing spinors.

Some other remarks.

CY_1 $(\dim = 1)$: $h^{(0,0)}, h^{(1,0)} = h^{(0,1)}, h^{(1,1)}$.

Since we know that $h^{(0,0)} = h^{(1,1)} = 1$.

$$h^{(p,0)} = h^{(0,n-p)} \Rightarrow h^{(1,0)} = h^{(0,0)}$$
$$h^{(0,0)} = h^{(0,1)}$$

$$h^{(r,s)} = h^{(s,r)} \Rightarrow h^{(1,0)} = h^{(0,1)}$$

$$\Rightarrow \boxed{h^{(0,0)} = h^{(1,0)} = h^{(0,1)} = h^{(1,1)} = 1}$$

Notice that

$$\chi_E = \int c_1(CY_1) = 0$$

$$\chi_E = h^{(0,0)} - [h^{(1,0)} + h^{(0,1)}] + h^{(1,1)} = 2(h^{(0,0)} - h^{(1,0)})$$

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \Rightarrow CY_1 = T_2$$

(torus)

CY_2 $(n=2)$

$$\begin{cases} h^{(0,0)} = h^{(2,2)} = 1 \\ h^{(1,0)} = h^{(0,1)} \\ h^{(2,0)} = h^{(0,0)} = h^{(0,2)} = 1 \\ h^{(1,2)} = h^{(2,1)} = h^{(2-1,2-2)} = h^{(1,0)} \end{cases}$$

$$\begin{array}{ccccc} & & 1 & & \\ & h^{(1,0)} & & h^{(1,0)} & \\ 1 & & h^{(1,1)} & & 1 \\ & h^{(1,0)} & & h^{(1,0)} & \\ & & 1 & & \end{array}$$

$$\chi_E = h^{(0,0)} - [h^{(1,0)} + h^{(0,1)}] + [h^{(2,0)} + h^{(1,1)} + h^{(0,2)}]$$
$$- [h^{(2,1)} + h^{(1,2)}] + h^{(2,2)} =$$
$$4h^{(0,0)} - 4h^{(1,0)} + h^{(1,1)} = 4 - 4h^{(1,0)} + h^{(1,1)} =$$

$$\boxed{\chi_E = 4\left(1 - h^{(1,0)} + \frac{1}{4} h^{(1,1)}\right)}$$

if $\chi_E \neq 0 \Rightarrow h^{(1,0)} = 0 \Rightarrow \chi_E = 4\left(1 + \frac{1}{4} h^{(1,1)}\right)$

We define the harmonic forms:

$$\mathrm{Harm}_{\partial}^{(r,s)} = \{ \omega \in \Lambda^{(r,s)}(M) \mid \Delta_{\partial}\,\omega = 0 \}$$

$$\mathrm{Harm}_{\bar\partial}^{(r,s)} = \{ \omega \in \Lambda^{(r,s)}(M) \mid \Delta_{\bar\partial}\,\omega = 0 \}$$

Hodge theorem

$$\Lambda^{(r,s)}(M) = \mathrm{Harm}_{\partial}^{(r,s)} \oplus \partial\Lambda^{(r-1,s)}(M) \oplus \partial^{\dagger}\Lambda^{(r+1,s)}(M)$$

$$= \mathrm{Harm}_{\bar\partial}^{(r,s)} \oplus \bar\partial\Lambda^{(r,s-1)}(M) \oplus \bar\partial^{\dagger}\Lambda^{(r,s+1)}(M).$$

means:
$$\omega = \bar\partial\alpha + \bar\partial^{\dagger}\beta + \gamma =$$
$$= \partial\alpha' + \partial^{\dagger}\beta' + \gamma'.$$

Corollary:

$$\boxed{\begin{aligned} \mathrm{Harm}_{\partial}^{(r,s)} &\simeq H_{\partial}^{(r,s)} \\ \mathrm{Harm}_{\bar\partial}^{(r,s)} &\simeq H_{\bar\partial}^{(r,s)} \end{aligned}}$$

Given M a Kähler mfld of cmplx dim $\dim_{\mathbb{C}} M = n$

$$\Rightarrow \quad h^{(r,s)} = \dim_{\mathbb{R}} H^{(r,s)}$$

$$\boxed{\begin{array}{l} 1)\ h^{(r,s)} = h^{(s,r)} \\ 2)\ h^{(r,s)} = h^{(n-r,\, n-s)} \end{array}}$$

Proof: $\omega \in \Lambda^{(r,s)} M$. and $\mathrm{Harm}^{(r,s)}_{\Delta_\partial} \approx \mathrm{Harm}^{(r,s)}_{\Delta_{\bar\partial}}$

1)

$$\to \Delta_\partial \omega = 0 \qquad \Delta_{\bar\partial} \omega = 0$$

$$\Rightarrow \boxed{\overline{\Delta_\partial \omega} = \Delta_{\bar\partial} \bar\omega = \Delta_\partial \bar\omega}$$

So $\omega \in H^{(r,s)}_{\Delta_\partial} \to \bar\omega \in H^{(s,r)}_{\Delta_\partial} \Rightarrow \boxed{h^{(r,s)} = h^{(s,r)}}$

2) $\omega \in H^{(r,s)}_{\bar\partial}$, $\tau \in H^{(n-r,\, n-s)}_{\bar\partial}$

$$\int : \ H^{(r,s)}_{\bar\partial} \otimes H^{(n-r,\, n-s)}_{\bar\partial} \longrightarrow \mathbb{C}$$

$$\Rightarrow H^{(r,s)}_{\bar\partial} \approx H^{(n-r,\, n-s)}_{\bar\partial} \Rightarrow \boxed{h^{(r,s)} = h^{(n-r,\, n-s)}}$$

<u>Hodge diamond</u> $\quad n = 3$

$h^{(0,0)}$

$h^{(1,0)} \quad h^{(0,1)}$

$h^{(2,0)} \quad h^{(1,1)} \quad h^{(0,2)}$

$h^{(3,0)} \quad h^{(2,1)} \quad h^{(1,2)} \quad h^{(3,0)}$

$h^{(2,0)} \quad h^{(1,1)} \quad h^{(2,0)}$

$h^{(1,0)} \quad h^{(1,0)}$

$h^{(0,0)}$

Now we have:

if $c_1(M) = 0 \Rightarrow$

$$\boxed{h^{(p,0)} = h^{(0,n-p)}}$$

<u>Proof</u> (for a CY $\exists\, \Omega$).

Given an holomorphic $(p,0)$-form $u_{I_1 \cdots I_p}$

$$\Rightarrow \quad u_{\bar{I}_1 \cdots \bar{I}_{n-p}} = \bar{\Omega}_{\bar{I}_1 \cdots \bar{I}_{n-p} \bar{I}_{n-p+1} \cdots \bar{I}_n}\, u_{I_1 \cdots I_p}\, g^{\bar{I}_1 I_1} \cdots g^{\bar{I}_p I_p}$$

notice that this expression is invertible.

$$\Rightarrow \forall\, \omega \in H^{(p,0)} \Rightarrow \exists\, u \in H^{(0,n-p)}$$

$$\boxed{\begin{aligned} h^{(0,0)} &= h^{(0,3)} = h^{(3,0)} \\ h^{(1,0)} &= h^{(0,2)} = h^{(2,0)} \\ h^{(2,0)} &= h^{(0,1)} = h^{(1,0)} \end{aligned}}$$

Since there is a <u>unique</u> $\Omega \in H^{(3,0)}_{\bar\partial}$, $\bar{\Omega} \in H^{(0,3)}_{\partial}$

$$\Rightarrow \quad \boxed{h^{(0,0)} = h^{(3,0)} = 1}$$

globally define

$h^{(1,0)}$
$h^{(2,1)}$
$h^{(1,1)}$

1

$h^{(1,0)}$ $h^{(1,0)}$

$h^{(1,0)}$ $h^{(1,1)}$ $h^{(1,0)}$

1 $h^{(2,1)}$ $h^{(2,1)}$ 1

$h^{(1,0)}$ $h^{(1,1)}$ $h^{(1,0)}$

$h^{(1,0)}$ $h^{(1,0)}$

1

we see left with $\boxed{h^{(1,0)},\ h^{(2,1)},\ h^{(1,1)}}$ 7

$$\chi_{Eul} = \sum_{r=0}^{6 \leftarrow \mathbb{R}\text{-dim}} (-)^r b_r =$$

$$= \sum_{r=0}^{\cancel{6}} (-)^{r} \sum_{k=0}^{r} h^{(r-k,k)} =$$

$$= h^{(0,0)} - \left[h^{(1,0)} + h^{(0,1)} \right] + \left[h^{(2,0)} + h^{(1,1)} + h^{(0,2)} \right] +$$

$$- \left[h^{(3,0)} + h^{(2,1)} + h^{(1,2)} + h^{(0,3)} \right] +$$

$$+ \left[h^{(3,0)} + h^{(1,1)} + h^{(0,2)} \right] +$$

$$- \left(h^{(1,0)} + h^{(0,1)} \right)$$

$$+ 1 =$$

$$= \cancel{1} - \cancel{\left[2h^{(1,0)} \right]} + \left[\cancel{2h^{(1,0)}} + h^{(1,1)} \right]$$

$$- \left[\cancel{1 \cdot 2} + 2h^{(2,1)} \right] +$$

$$+ \left(\cancel{2h^{(1,0)}} + h^{(1,1)} \right)$$

$$- \cancel{2h^{(1,0)}}$$

$$+ \cancel{1}$$

$$= 2\left(h^{(1,1)} - h^{(2,1)} \right)$$

$$\boxed{\chi_{EUL} = +2\left(h^{(1,1)} - 2h^{(2,1)} \right)}$$

$$\boxed{\begin{array}{l}\underline{\text{Thm}}!\\ \text{On a mfld w. } \chi_E \text{ a non-singular 1-form}\\ \text{has at least } |\chi_E| \text{ zeros.}\end{array}}$$

<u>Hint</u>: <u>M = Riemann surface</u>

$$\chi_M = h^{(0,0)} - [h^{(1,0)} + h^{(0,1)}] + h^{(1,1)}$$
$$= 2\left(h^{(0,0)} - h^{(1,0)}\right) =$$
$$= 2\left(1 - h^{(1,0)}\right)$$

Canonical divisor:

$$D_K = \frac{z_1 \cdots z_N}{p_1 \cdots p_M}$$

$$\Rightarrow \deg D_K = \#\text{zero} - \#\text{poles} = 2(g-1) = -\chi_E$$

$$\chi_E = 2 - 2g$$

$g = 0$ Riemann sphere $\Rightarrow \chi_E = 2$

but $\chi_E = 2h^{(1,1)}$

$$\Rightarrow \begin{cases} h^{(1,1)} = 1 & (\text{fundament form (kähler form) } k) \\ h^{(1,0)} = 0 \end{cases}$$

$\Rightarrow$

$g = 1 \qquad \chi_E = 0 \Rightarrow h^{(1,0)} = 1$

$g \geq 2 \qquad \chi_E = -2$

$$\chi = 2\left(h^{(1,1)}\right)$$

$$\chi = \left[h^{(0,0)} - \left(h^{(1,0)} + h^{(0,1)}\right) + h^{(1,1)}\right]$$ $\boxed{\text{Euler char.}}$

$$h^{(0,0)} - 2h^{(1,0)} + h^{(1,1)} = 2\left(h^{(0,0)} - h^{(1,0)}\right) = 2 - 2g$$

$$2\left(h^{(0,0)} - h^{(1,0)}\right) = 2(1-g)$$

$$h^{(0,0)} - h^{(1,0)} = (1-g)$$

$g = 0 \qquad h^{(0,0)} - h^{(1,0)} = 1$

$$h^{(0,0)} = h^{(1,0)} + (1-g)$$

for $g=0$ $\quad \boxed{1 = h^{(1,0)} + (1-g)} \Rightarrow \boxed{h^{(1,0)} = g}$

number of abelian differentials

Hodge Diamond for Riemann surface

$$\boxed{\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}}$$

sfere : $\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$ 

torus : $\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$ 

bi torus : $\begin{array}{ccc} & 1 & \\ 2 & & 2 \\ & 1 & \end{array}$ 

$$\boxed{\chi = 2(1-g)}$$

CY $\quad g = 1 \qquad \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$

$g = 2 \qquad \begin{array}{ccc} & 1 & \\ 2 & & 2 \\ & 1 & \end{array}$

$\vdots$

Then $\quad \boxed{CY_3 \text{ with } \chi \neq 0 \Rightarrow h^{(1,1)} \neq h^{(2,1)}, \ \boxed{h^{(1,0)} = 0}}$

Thm

$$\boxed{\begin{array}{l}\text{for a CY 3-fold } (h^{(1,0)}, h^{(1,1)}, h^{(2,1)}) \neq 0. \\ \text{if } \chi_E \neq 0 \Rightarrow h^{(1,1)} \neq h^{(2,1)}. \\ \text{and } \boxed{h^{(1,0)} = 0}\end{array}}$$

Proof: $h^{(1,0)} \neq 0$ if $\exists\ \mathrm{Harm}_{\bar\partial}^{(1,0)}(M)$. $\left(\begin{array}{l}\bar\partial \omega^{(1,0)} = 0 \\ \partial \omega^{(0,1)} = 0\end{array}\right)$

we use the Weitzenbock formula.

$$\boxed{(dd^\dagger + d^\dagger d)\,\omega_A = -\nabla * \nabla \omega_A + (R\,\omega)_A .}$$

where $\omega^{(1} = \omega_A dx^A = \omega^{(1,0)}_I dz^I + \omega^{(0,1)}_{\bar I} dz^{\bar I}$

$R^A{}_B$ is the Ricci tensor.

If $\omega \in \mathrm{Harm}_d(M) \Rightarrow$

$(dd^\dagger + d^\dagger d)\,\omega_A = 0 \Rightarrow -\nabla^A \nabla_A \omega_B + R_B{}^A \omega_A = 0.$

but for a CY $R^A{}_B = 0. \rightarrow$

$$\boxed{\nabla^A \nabla_A \omega_B = 0}$$

$$\Rightarrow 0 = \int_M \sqrt{+g}\, \omega^B \nabla^A \nabla_A \omega_B = -\int_M \sqrt{g}\, (\nabla_A \omega_B)^2 = -\|\nabla_A \omega_B\|^2$$

$\rightarrow \boxed{\nabla_A \omega_B = 0}$ covariantly constant.

But if $D_A \omega_B = 0$ $\Rightarrow$ and ω_B has a zero.

$\Rightarrow$ $\omega_B = 0$. (So if $\chi_E \neq 0$ $\Rightarrow$ # zeros of $\omega_B = |\chi_E|$) $\Rightarrow$ $\boxed{h^{(1,0)} = 0}$.

$\Rightarrow$ Hodge diamond: $h^{(1,1)}, h^{(2,1)} \neq 0$ $\boxed{h^{(1,0)} = 0}$

$$\begin{cases} h^{(1,1)} = \text{Kähler class deformations} \\ h^{(2,1)} = \text{Complex-structure deformations} \end{cases}$$

if, however, $\chi_E = 0$ $\Rightarrow$ $h^{(1,1)} = h^{(2,1)}$ $\Rightarrow$ and $\boxed{h^{(1,0)} \neq 0.}$

example the T_3 (three torus).

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 9 & & 3 & \\ 1 & & 9 & & 9 & & 1 \\ & 3 & & 9 & & 3 & \\ & & 3 & & 3 & & \\ & & & 1 & & & \end{array}$$

~~Sig~~

$\cancel{\delta\delta^{\bar I}_{K} = \omega_{\bar L MN}\, \Omega^{RST}_{\bar L\bar M R}\, g^{L\bar L} g^{M\bar M}}$

$\omega_{LM\bar N}\, g^{L\bar L} g^{M\bar M} g^{N\bar N}\, \Omega_{\bar L\bar M R}\, g^{R}$

Thm

(M, J, g) a CY space of dim $m \geq 3 \Rightarrow$
M is projective. (M, J) is isomorphic to as a complex mfld to a submanifold of $\mathbb{CP}^N$ — (that is an algebraic variety).

Examples

Complete intersections

n-dim hypersurface of $\mathbb{CP}^N$.

$$M_n = \{ W_\alpha(X) = 0, \quad \alpha = 1 \dots r \}$$

dim $n = N - r$

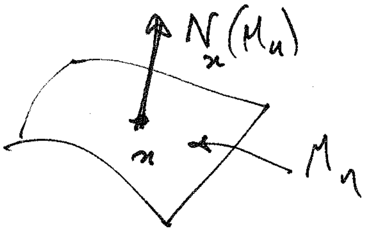
$$W_\alpha(x) \approx X^{\nu_\alpha} + \dots$$

this is a smooth compact complete if the intersection is complete.

this i.e.

$$\boxed{\Theta = dW_1 \wedge dW_2 \wedge \dots \wedge dW_r \neq 0} \text{ on } M_n.$$

$\forall x \in M_n$ the gradient to M_n identify a complete non-deg normal space to M_N



$$W_\alpha(X^\Lambda, \psi^{A_\alpha})$$

coords on $\mathbb{CP}^N$

parameters of the polynomials.

- $\forall$ choice of $\{\psi^{A_\alpha}\}_{A_\alpha = 1 \dots m_\alpha} \in \mathbb{C}^{m_\alpha}$ identifies a new $M_n(\psi)$ with varying complex structure.

- $\forall$ $M_n(\psi)$ are topologically equivalent (indeed $ch(M_n)$ depends on ν_α and not on the parameters ψ^{A_α}).

$$\mathbb{CP}^N(\nu_1, \dots \nu_r) \Rightarrow M_N(\psi) \subset \mathbb{CP}^N$$

with all possible choices of $\boxed{W_\alpha(x, \psi)}$

$$c_1\left(\mathbb{CP}^N[\nu_1 \dots \nu_r]\right) = \left(n + r + 1 - \sum_{\alpha=1}^{r} \nu_\alpha\right) k$$

$$\Rightarrow c_1 = 0 \Rightarrow n + r + 1 = \sum_{\alpha=1}^{r} \nu_\alpha$$

If we require that $\nu_\alpha \geq 2 \quad \forall \alpha. \Rightarrow$

$$1 + n + r = \sum_{\alpha=1}^{r} \nu_\alpha \geq 2r \rightarrow (1+n) \geq r$$

3-folds $\boxed{n=3}$

$r = 1 \qquad \nu_1 = 5 \qquad \alpha = 1 \qquad \mathbb{CP}_{n+r}[\nu_1] = \mathbb{CP}_4[5]$

$r = 2 \qquad \nu_1 + \nu_2 = 6 \qquad \alpha = 1, 2.$

$\mathbb{CP}_5[3,3], \quad \mathbb{CP}_5[4,2]$

$r = 3 \qquad \nu_1 + \nu_2 + \nu_3 = 7 \qquad \mathbb{CP}_6[2,2,3]$

$r=4 \qquad \nu_1+\nu_2+\nu_3+\nu_4=8 \qquad \mathbb{CP}_7[2,2,2,2].$

$$\boxed{\chi_E = \int_{M_3} c_3(M_3)}$$

(recall that: $p_3(M) = c_6(M^{\mathbb{C}}) = c_3(M) \wedge c_3(\bar{M})$.

and $e^2(M) = p_3 \Rightarrow \boxed{e(M) = c_3(M)}$

$$c_3(\mathbb{CP}_{r+4}[\nu_1,\dots,\nu_r]) = \underbrace{f_{(r+4)}(\nu_1,\dots\nu_r)}_{\Downarrow} \; k\wedge k\wedge k$$

obtained from:

$$\left[\frac{(1+k)^{r+4}}{\bigwedge_{i=1}^{r}(1+\nu_i k)}\right]_{3\text{-form}} = \sum_{\substack{l_0+\dots+l_r=3 \\ l_0,\dots l_r\in\mathbb{Z}_+}} \binom{4+r}{l_0}\prod_{\alpha=1}^{r}(-\nu_\alpha)^{l_\alpha} k\wedge k\wedge k$$

$k=$ pull back on $\mathbb{CP}_{r+4}[\nu_1,\dots\nu_r]$ from $\mathbb{CP}_{r+4}$.

$$\Rightarrow \int_{\mathbb{CP}_{r+4}[\nu_1\dots\nu_r]} k\wedge k\wedge k = \prod_{\alpha=1}^{r}\nu_\alpha .$$

$$\Rightarrow \boxed{\chi(\mathbb{CP}_{4+r}[\nu_1\dots\nu_r]) = f_{(r+4)}(\nu_1,\dots\nu_r)\prod_{\alpha=1}^{r}\nu_\alpha .}$$

$\chi(\mathbb{CP}_4[5]) = -200 \qquad \chi(\mathbb{CP}_5[3,3]) = -144$

$\chi(\mathbb{CP}_5[2,4]) = -176 \qquad \chi(\mathbb{CP}_6[2,2,3]) = -144$

$\chi(\mathbb{CP}_7[2,2,2,2]) = -128.$

Since all these manifolds are embedded in $\mathbb{CP}^N$.

(there is a single harmonic $(1,1)$-form) $\Rightarrow k$

$$\Rightarrow H^{(1,1)} \approx [k] \qquad h^{(1,1)} = 1 \quad , \; h^{(1,0)} = 0.$$

$$\chi = 2\left(h^{(1,1)} - h^{(2,1)}\right)$$

$$\chi = 2\left(1 - h^{(2,1)}\right) \qquad \frac{\chi}{2} = \left(1 - h^{(2,1)}\right)$$

$$h^{(2,1)} = 1 - \frac{\chi}{2}.$$

1) $h^{(2,1)} = 101$ 3) $h^{(2,1)} = 73$ 4) $h^{(2,1)} = 73$

2) $h^{(2,1)} = 89$ 5) $h^{(2,1)} = 65$

1)
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

2)
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 89 & & 89 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

3)
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 73 & & 73 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

4)
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 73 & & 73 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Same Hodge diamonds. [the invariant of χ, betti numbers, hodge numbers are not enough to parametrize]

5)
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 65 & & 65 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

How to count this number:

$$\mathbb{C}_{N, \nu_1 \dots \nu_r}[X] = \{ W_\alpha(x;\psi) \quad \alpha = 1 \dots r \mid \mathrm{deg}\, W_\alpha = \nu_\alpha \}$$

(homogeneous polynomials).

$$\dim \mathbb{C}_{4;5}[X] = \underset{\sum \nu_i = 5}{X_1^{\nu_1} X_2^{\nu_2} X_3^{\nu_3} X_4^{\nu_4} X_5^{\nu_5}} \quad \psi(\nu_1 \dots \nu_5) =$$

$$= \binom{N+\nu_1}{\nu_1} = \frac{(N+\nu_1-1)\cdots N}{\nu_1 !}$$

$$\mathbb{C}_{5;2,4}[X] = \binom{N+\nu_1}{\nu_1} + \binom{N+\nu_2}{\nu_2} =$$

$$= \binom{5+2}{2} + \binom{5+4}{4} =$$

$$= \binom{6}{2} + \binom{8}{4} = 147.$$

$$\boxed{\dim \mathbb{C}_{N;\nu_1 \dots \nu_r}[X] = \sum_{i=1}^{r} \binom{N+\nu_i}{\nu_i}}$$

Now consider. $\quad Q_{N;\nu_1 \dots \nu_r}[X] \subset \mathbb{C}_{N,\nu_1 \dots \nu_r}[X]$.

$$W_\alpha(x;\psi) = G^\Lambda(x,\psi) \frac{\partial}{\partial X^\Lambda} W_\alpha(x,\psi_0) + f_\alpha^\beta W_\beta(x,\psi_0).$$

$$\Lambda = 1, \dots, N+1$$

where c_g^{A} are fixed parameters.

$c^{\Lambda}(x,y)$ are $N+1$ polynomials in x,y.

$f_\alpha{}^\beta(x,y)$ is r^2 matrix of ~~degree~~ polynomials of degree $\nu_\alpha - \nu_\beta$.

(if $\nu_\alpha - \nu_\beta < 0 \Rightarrow f_\alpha^\beta = 0$).

and $f_\alpha{}^\beta \neq \delta_\alpha{}^\beta$.

$\dim \mathcal{Q}_{N;\nu_1 \dots \nu_r}(x,y) =$

$$= \left(\sum_{\alpha=1}^{r} \nu_\alpha\right)^2 + \sum_{\alpha,\beta=1}^{r} \dim f_\alpha{}^\beta - 1$$

$\left(\begin{array}{l}\text{if } \nu_\alpha - \nu_\beta < 0 \\ \dim f = 0\end{array}\right)$

$$\Rightarrow \qquad \mathbb{P}_{N;\nu_1 \dots \nu_r}[X] = \frac{\mathbb{C}_{N;\nu_1 \dots \nu_r}}{\mathcal{Q}_{N;\nu_1 \dots \nu_r}}$$

$$\dim \mathbb{P}_{N;\nu_1 \dots \nu_r} = \dim \mathbb{C}_{N;\nu_1 \dots \nu_r} - \dim \mathcal{Q}_{N;\nu_1 \dots \nu_r}$$

$$= \sum_{\alpha=1}^{r} \binom{N+\nu_\alpha}{\nu_\alpha} - \sum_{\substack{\alpha=1 \\ \beta}}^{r} \dim f_\alpha{}^\beta - \left(\sum_{\alpha=1}^{r} \nu_\alpha\right)^2 + 1.$$

$\Rightarrow$ coincides with $\boxed{h^{(2,1)}}$.

Def. of complex structure:

$$W_\alpha(x,\psi_0) + \underbrace{\delta W_\alpha}_{\in \mathbb{P}_{N;\nu_1\ldots\nu_r}[x]}$$

but we have to identify w.r.t. those polynomials which are related to $Q_{N;\nu_1\ldots\nu_r}$..

- linear transformations of the hom. coords. in $\mathbb{CP}^N$: $C^\lambda \frac{\delta}{\delta x^\lambda} W_\alpha(x,\psi_0)$.

- hypersurface defined by polynomial constraints : $f_\alpha^\beta(x,\psi)\, W_\beta(x,\psi_0)$.

$\Rightarrow$ genuine deformations $\mathbb{P}_{N;\nu_1\ldots\nu_r}[x]$.

Not only the numerology $\rightarrow$ it explains also the sections : polynomial rings $\Leftrightarrow$ Dolbeault cohomology.

Now we can compute

$$\chi_E = \int_M c_2(M)$$

$M = \mathbb{CP}^3[4]$
$\{ x_1^4 + x_2^4 + \dots = 0 \text{ in } \mathbb{P}^3 \}$.

$$c_2(M) = \left[\frac{(1+k)^4}{(1+4k)} \right]_{2\text{-form}} = 6 k \wedge k$$

$$\Rightarrow \chi_E = \int_{\mathbb{CP}^3[4]} 6 k \wedge k = 6 \int_{\mathbb{CP}^3(4)} k \wedge k = 6 \cdot 4$$

$$\Rightarrow 24 = 4\left(1 + \frac{1}{4} h^{(1,1)}\right) \Rightarrow 6 = 1 + \frac{1}{4} h^{(1,1)}$$

$$h^{(1,1)} = 20$$

Hodge diamond:

Since $\chi_E = 24 \neq 0 \Rightarrow \boxed{h^{(1,0)} = 0}$

Then we have:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

This is the Unique CY_2 - if we take the Radius $\to \infty$ we can expect. $K3 \approx T^4/\mathbb{Z}_2$

Computation by polynomials.

$$\dim \mathbb{C}_{3;4}[X] = \binom{3+4}{4} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4 \cdot 3 \cdot 2}} = 35$$

$$\dim Q_{3;4}[X] = 4^2 + \dim f_1^1 - 1 = 16$$

$\Rightarrow$ This gives 19 (moduli) + we have to add the Kähler form $\Rightarrow$ 20

Example for a Riemann surface.

Consider the algebraic curve

$$\boxed{x^n + y^n + z^n = 0 \quad \text{in} \quad \mathbb{CP}^2.}$$

This is denoted by $\mathbb{CP}_{2;n}$

Let us compute the total Chern class:

$$c(\mathbb{CP}_{2;n}) = \frac{(1+k)^{1+1+1}}{(1+nk)} = \frac{(1+k)^3}{(1+nk)}$$

$$= (1 + 3k + 3k\wedge k + k\wedge k\wedge k) + (1 - nk + n^2 k\wedge k + - n^3 k\wedge k\wedge k) =$$

$$= (3-n)k + (3+n^2) k\wedge k + (1-n^3) k\wedge k\wedge k$$

$$\boxed{c_1(\mathbb{CP}_{2;n}) = (3-n)k}$$

$$c_2(\mathbb{CP}_{2;n}) = 0 \text{ at } k\Big|_{\mathbb{CP}_{2;n}}$$

$$\chi(\mathbb{CP}_{2;n}) = \int c_1(\mathbb{CP}_{2;n}) = (3-n)\int_{\mathbb{CP}_{2;n}} k = (3-n)n$$

$$\Rightarrow 2(1-g) = (3-n)n \quad \Rightarrow \boxed{g = \frac{(n-2)(n-1)}{2}}$$

Where we used the eq:

$$\chi = 2(1-g) \quad \text{for a Riemann surface.}$$