Introduction to black holes :

- Black holes in 4 dimensions (monday)
  - Uniqueness
  - BH mechanics
  - BH thermodynamics
  - Information paradox

- Black holes & branes in higher dimensions
  - tuesday:
    - Introduction : Why $D > 4$?
    - Black branes
    - Stress tensor for black branes
    - Dynamics of infinitely thin branes
  - wednesday:
    - Blackfolds
    - Stationary blackfolds
  - thursday:
    - Examples of stat. blackfolds
    - Perturbations + Gregory-Laflamme instability
    - DBI action, BIon, open/closed string duality ...

D=4 Black Holes

This part will be on D=4 Black Holes. I will introduce some of the main ideas, results and concepts that modern theoretical research builds on. The intention is to do this in a basic way.

Subjects:

- BH physical properties
- BH mechanics
- BH thermodynamics and Hawking radiation
- BH uniqueness
- Information paradox

Table of contents

BH metrics:

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Einstein eq's: (cosm. const. $\Lambda = 0$)

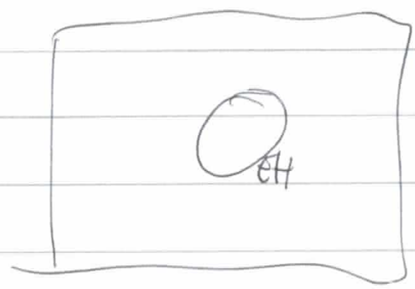
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

space-time geometry — matter

Matter curves space-time, e.g. light rays ~~[illegible]~~ bend due to gravitational field

BH's ~~[illegible]~~ arise when matter is so condensed that an event horizon appears
EH: Light cannot escape

Consider having $T_{\mu\nu} = 0$ outside the EH



Einstein eq's: $R_{\mu\nu} = 0$

Two metrics:

1) Schwarzschild metric

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

EH at $r = r_0$

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2) Kerr metric:

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}dtd\phi$$
$$+\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta \quad , \quad \Delta = r^2 - r_0 r + a^2 = (r-r_-)(r-r_+)$$

$$r_\pm = \frac{r_0}{2} \pm \sqrt{\frac{r_0^2}{4} - a^2}$$

EH at $r = r_+$

Notice: For $a=0$ Kerr $\rightarrow$ Schw.

We begin these lectures by analyzing ~~&~~ certain properties of these metrics

Mass and ang. mom. of BH: 1

Measurement of mass for Newtonian ~~...~~ (or special relativistic) matter:

$$\int d^3x\, \rho = \int d^3x\, T_{00}$$

But, we can't do this for BH's

Schw BH → purely geometric, no $T_{\mu\nu}$,
e.g. $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$

In what sense does a Schw BH have a mass?

Geometry → Physical object

The ~~...~~ properties as physical object can be defined by seeing the influence of BH on Newtonian matter

Same gravitational pull as a Newt. point mass M ⇒ BH has mass M

2

Consider a Newtonian localized matter distribution:

Gravitational field weak everywhere

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \quad |h_{\mu\nu}| << 1$$

Mink. metric $\eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$
$= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

Using Einstein eqs one can derive

$$g_{00} = -1 + \frac{2GM}{r} + O(r^{-2}) \qquad M = \int d^3\vec{r}\, T_{00}(\vec{r})$$

for large r, as well as

$$g_{0\phi} = -2GJ \frac{\sin^2\theta}{r} + O(r^{-2})$$

with $J = \int d^3\vec{r} \left[x T^0{}_y - y T^0{}_x \right]$

This is derived for a localized Newt. matter distribution as seen by an observer far away, hence $r \to \infty$

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Compare with Schw. metric

$$g_{00} = -1 + \frac{r_0}{r}$$

Same grav. pull as Newt. matter distr. with mass M if we identify

$r_0 = 2GM$

In this way we define the mass of a BH → called the ADM mass (Arnowitt – Deser – Misner)

Similarly one can take

$g_{0\phi} = \dots$

to define the ~~...~~ ang. mom. J of a BH

For the Kerr-metric

$$M = \frac{r_0}{2G}\ , \quad J = \frac{a r_0}{2G}$$

1

Killing vector fields:

Important for

→ Symmetries of metrics/space-times

→ Event horizon definition

Consider a 4 dim manifold/space-time M

Local coord's x^μ, $\mu = 0, 1, 2, 3$

Vector field on M: For each point $p \in M$ we have a vector

$V = V^\mu \frac{\partial}{\partial x^\mu} \in T_p(M)$: tangent space of M at p

Flow of a vect. field:

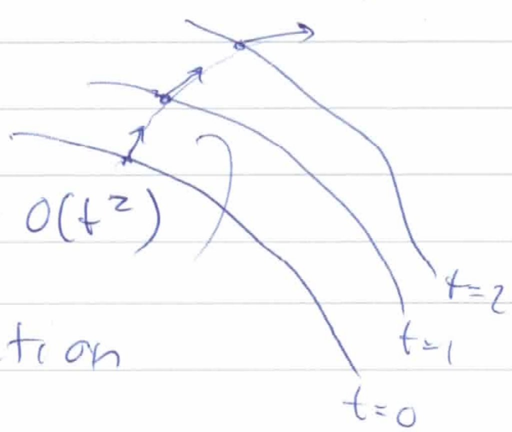
Flow of V is a (smooth) map $\sigma_t : M \to M$ for each $t \in \mathbb{R}$ such that

$$\frac{d}{dt}\sigma_t = V(\sigma_t(p))$$

and $\sigma_0(p) = p$, $\sigma_{t+s}(p) = \sigma_t(\sigma_s(p))$.

In local coord's

$\sigma_t^\mu(p) = x^\mu(p) + t V^\mu(p) + O(t^2)$



⇒ The flow goes in direction of V

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Can we drag things along the flow?

→ <u>Lie dragging</u>

Consider a fct $f: M \to \mathbb{R}$

For each $t \in \mathbb{R}$ we can define new fct $f(\sigma_t(p))$ → dragged along the flow of V

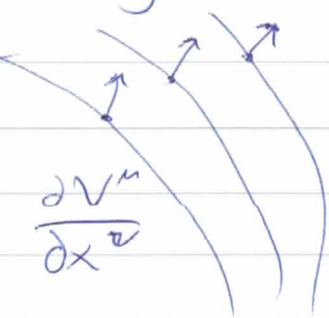
Change of f along flow:
<u>Lie derivative</u>

$$\mathcal{L}_V f(p) = \lim_{t \to 0} \frac{1}{t}\left(f(\sigma_t(p)) - f(p) \right)$$

Local coord's: $\mathcal{L}_V f = V^\mu \frac{\partial f}{\partial x^\mu}$

If $\mathcal{L}_V f = 0$: No change of f along V
⇒ Means that f is invariant along the flow of V!

We can also consider the dragging of another vec. field W along V
→ Can be used to define Lie derivative of W along V:



$$(\mathcal{L}_V W)^\mu = V^\nu \frac{\partial W^\mu}{\partial x^\nu} - W^\nu \frac{\partial V^\mu}{\partial x^\nu}$$

$$= ([V, W])^\mu$$

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2nd term: Extra term due to transf. of W, i.e. we cannot directly compare W at p and at $\sigma_t(p)$ since $T_p(M) \neq T_{\sigma_t(p)}(M)$

Note that $\mathcal{L}_V V = [V, V] = 0$
→ makes sense since V should be invariant when dragging it along V
→ 2nd term precisely ensures this

One can define $\mathcal{L}_V T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m}$ ((n,m) tensor)
in particular for a metric ((0,2) tensor)

$$(\mathcal{L}_V g)_{\mu\nu} = V^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{\partial V^\rho}{\partial x^\mu} g_{\rho\nu} + \frac{\partial V^\rho}{\partial x^\nu} g_{\mu\rho}$$

A way to understand the 2 extra terms in comparison with a scalar fct is to define a scalar

$\phi = g_{\mu\nu} W^\mu W^\nu$

Then we know

$\mathcal{L}_V \phi = V^\rho \frac{\partial \phi}{\partial x^\rho}$

But since $\mathcal{L}_V$ is a derivative we can use the chain rule

$$\mathcal{L}_V \phi = (\mathcal{L}_V g)_{\mu\nu} W^\mu W^\nu + g_{\mu\nu} (\mathcal{L}_V W)^\mu W^\nu + g_{\mu\nu} W^\mu (\mathcal{L}_V W)^\nu$$

using $(\mathcal{L}_V W)^\mu = V^\rho \frac{\partial W^\mu}{\partial x^\rho} - W^\rho \frac{\partial V^\mu}{\partial x^\rho}$
one finds the above formula for $(\mathcal{L}_V g)_{\mu\nu}$

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What does $(\mathcal{L}_V g)_{\mu\nu} = 0$ mean?
$g_{\mu\nu}$

→ That the metric does not change (is invariant) when dragging it along the flow of the vector field V

→ Hence the metric has a symmetry along V

For a given metric $g_{\mu\nu}$ we say that it has a Killing Vector Field (KVF) if we can find a vec. field V such that $(\mathcal{L}_V g)_{\mu\nu} = 0$

A KVF corresponds to a symmetry of the metric

Consider the Schw. BH metric:

$$ds^2 = -\left(1-\frac{r_0}{r}\right)dt^2 + \frac{dr^2}{1-\frac{r_0}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

We find the following KVF's:

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$$T = \frac{\partial}{\partial t}$$

$$X = \frac{\partial}{\partial \phi}, \quad Y = \sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi},$$
$$Z = \cos\phi \frac{\partial}{\partial \theta} - \sin\phi \cot\theta \frac{\partial}{\partial \phi}$$

T : time translation symmetry
X, Y, Z : spherical symmetry

Exercise:

1) Check that T, X, Y, Z are KVF's
2) Check $[X, Y] = Z$, $[Z, X] = Y$, $[Y, Z] = X$
   → commutator relations for $SO(3)$, the symmetry group of S^2
3) Check that there are no other linearly independent KVF's

Note: We have maximally two linearly independent commuting KVF's (f.eks. T & X) means at most two KVF's can be manifest in coord system of metric, i.e. that $V = \frac{\partial}{\partial x^4}$, since in general we have $[\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^2}] = 0$

Will be important later on...

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Compute example:

$$(\delta_V g)_{\mu\nu} = V^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{\partial V^\rho}{\partial x^\mu} g_{\rho\nu} + \frac{\partial V^\rho}{\partial x^\nu} g_{\mu\rho}$$

$$V = \frac{\partial}{\partial t} \Rightarrow V = \underset{t}{(1,}\underset{r}{0,}\underset{\theta}{0,}\underset{\varphi}{0)}$$

$$(\delta_V g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial t} = 0$$

Kerr-metric

$ds^2 =$

Two lin. independent KVF's

$$T = \frac{\partial}{\partial t}, \quad X = \frac{\partial}{\partial \phi}$$

→ Spherical symmetry is broken

Static & stationary: 7

Def. A stationary asympt. flat space-time is such that

1) $\exists$ coordinate r such that for $r \to \infty$ the space-time asymptotes to Minkowski space-time

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

2) $\exists$ a KVF T such that for $r \to \infty$ we have $T = \frac{\partial}{\partial t}$, where $\frac{\partial}{\partial t}$ is the generator of time-translation in Mink. space

---

We can find coord's ~~for an asympt flat stationary space-time~~ ~~such that~~ ~~(t,r,θ,ϕ)~~ ~~such that~~ (t, r, θ, ϕ) for a stationary, asympt. flat space-time such that

$$T = \frac{\partial}{\partial t}$$

outside a possible EH.

A stationary AF space-time is static if the space-time is invariant under $t \to -t$ (time-reversal)

Means that $g_{tr} = g_{t\theta} = g_{t\phi} = 0$

Schwarzschild metric $\to$ Static

Kerr metric $\to$ Stationary

BH Basics D=4

Side remark

All stationary (incl. static) black hole space-times have a particular KVF which is null at the event horizon

So, there exists a vector field k^μ such that

- It is a KVF: $(\mathcal{L}_k g)_{\mu\nu} = 0$
- $g_{\mu\nu} k^\mu k^\nu \big|_{EH} = 0$

This can also be turned around since such a ~~vector field~~ KVF defines a so-called Killing horizon and one can show: EH $\Longleftrightarrow$ Killing horizon

Examples:

Schwarzschild: $k = \frac{\partial}{\partial t}$

Kerr: $k = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$

$$\Omega = \frac{a}{a^2 + r_+^2}$$

Uniqueness of BH's: 1

Schwarzschild & Kerr metrics

i) Stationary, asympt. flat space-times

ii) $\exists$ an EH

iii) outside the EH: $R_{\mu\nu} = 0$

→ Obey the vacuum Einstein eqs

But are there other metrics with same properties i) + ii) + iii) ?

→ i.e. other types of BH's ?

Answer: No!

i) + ii) + iii) $\Rightarrow$ Schwarzschild or Kerr metric

Formulated by the Uniqueness theorems for BH's

Case I: (Israel's theorem + Hawking)

Assume i) + ii) + iii) + static

$\Rightarrow$ Metric of space-time is Schwarzschild metric

In particular, notice that space-time is spherically symmetric

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Thus any static BH $\Rightarrow$ spherically sym.

Compare to matter-distributions
$\exists$ infinitely many static matter ~~distri~~ ~~~~ distributions, can take all possible multipole moments

For ~~~~ static BH's we only need to know the mass M (assuming $R_{\mu\nu}=0$)

<u>Case II</u> (Hawking, Carter, Robinson, ...)

Assume i) + ii) + iii) + not static

$\Rightarrow$ Metric of space-time is Kerr-metric

In particular, we get that there $\exists$ an axial KVF $\chi = \frac{\partial}{\partial \phi}$ of the space-time

Thus, in this case a unique BH space-time for given M & J

Generalization to charged BH's
$\rightarrow$ specify in addition magn + electric charges

Classical BH's

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- Nothing (e.g. no γ's) escapes from behind the EH
  → The BH can only absorb, not emit

- At equilibrium, BH is characterized by very few parameters: M, J, Q_e, Q_m

⇒ We loose information into the BH
   i) star → BH
   ii) BH + astronaut → BH

⇒ In particular, entropy outside a BH can decrease by having the BH absorb matter

Possible resolution: BH ~~has~~ an entropy
But how is this in accordance with being characterized by few parameters?

We now turn to further properties of BH's

Including exploring geometry near the horizon of a BH

~~As we shall see this will ultimately lead to~~

All this will be for classical BH's but the findings will be a precursor to an understanding of certain quantum properties of BH's

Rindler space-time & surface gravity: 1

Consider schw. metric

$$ds^2 = -\left(1-\frac{r_0}{r}\right)dt^2 + \frac{dr^2}{1-\frac{r_0}{r}} + r^2(\qquad)$$

How does this metric look close to EH at $r = r_0$?

Write $r = r_0\left(1+\frac{x^2}{4r_0^2}\right)$

We consider limit $x \ll r_0$
→ close to EH

$$1-\frac{r_0}{r} = 1 - \frac{1}{1+\frac{x^2}{4r_0^2}} \simeq \frac{x^2}{4r_0^2}$$

$$dr^2 = \left(\frac{1}{4r_0} 2x\,dx\right)^2 = \frac{x^2}{4r_0^2}dx^2$$

$$ds^2 \simeq \underbrace{-\frac{x^2}{4r_0^2}dt^2 + dx^2}_{\text{Rindler space-time}} + \underbrace{r_0^2(d\theta^2+\sin^2\theta\, d\varphi^2)}_{\text{\& } S^2 \text{ with constant radius}}$$

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Consider a static particle
→ lying at a constant value of $x > 0$
outside the EH

Thus $u^\mu = (u^t, 0, 0, 0)$, i.e. $u \propto \frac{\partial}{\partial t}$

$u^2 = -1 \Rightarrow g_{tt}(u^t)^2 = -1 \Rightarrow u^t = \frac{2r_0}{x}$

The acceleration wrt. the proper time is

$$a^\mu = D_{(\tau)} u^\mu = \frac{dx^\rho}{d\tau} D_\rho u^\mu$$ ← cov. deriv.

$$= u^\rho D_\rho u^\mu = u^t D_t u^\mu$$

$$= u^t(\partial_t u^\mu + \Gamma^\mu_{t\sigma} u^\sigma)$$

$$= (u^t)^2 \Gamma^\mu_{tt}$$

$$\Gamma^\mu_{tt} = \frac{1}{2} g^{\mu\nu}(2\partial_t g_{t\nu} - \partial_\nu g_{tt}) = -\frac{1}{2} g^{\mu\nu}\partial_\nu g_{tt}$$

$$\Gamma^t_{tt} = 0, \quad \Gamma^x_{tt} = -\frac{1}{2}\partial_x g_{tt} = \frac{1}{2}\partial_x\left(\frac{x^2}{4r_0^2}\right) = \frac{x}{4r_0^2}$$

So, $a^t = 0$, $a^x = \left(\frac{2r_0}{x}\right)^2 \frac{x}{4r_0^2} = \frac{1}{x}$

Particle accelerating in direction away from EH

$F^\mu = m a^\mu$ → External Force outwards to keep particle static near the EH

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For $x \to 0$ (the EH) the acc. $a^x \to \infty$ $\Rightarrow$ it is harder and harder to keep particle static near the EH.

Consider instead the acc. of the particle as seen by an observer with proper time t. She will observe

$$a^{\mu}_{(t)} = \frac{d\varrho}{dt} a^{\mu} = \frac{x}{2r_0} a^{\mu} \quad \text{since } d\varrho^2 = \frac{x^2}{4r_0^2} dt^2$$

so, $a^{\mu}_{(t)} = \left(0, \frac{1}{2r_0}\right)$

$\Rightarrow$ The observer sees the particle as having acceleration $\frac{1}{2r_0}$

Go back to full Schw metric

Observer in asympt. region $r \to \infty$ has proper time t
$\Rightarrow$ observes a static particle near horizon having acceleration $\frac{1}{2r_0}$

Accel $\Leftrightarrow$ Force of gravity by equivalence principle

~~a acceleration corresponds to~~

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We can think of the BH geom near the EH as making a gravitational pull of magnitude $\frac{1}{2r_0}$

This is called the "surface gravity" of the BH

$$\kappa = \frac{1}{2r_0} \quad \leftarrow \text{For Schw. BH}$$

5

Take another look at the 2D Rindler space-time:

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2$$

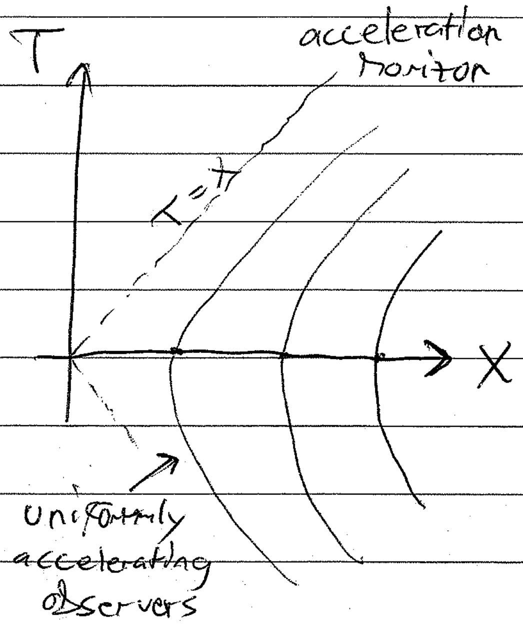
Define new coordinates:

$$T = x\sinh(\kappa t)\ , \quad X = x\cosh(\kappa t)$$

Then $ds^2 = -dT^2 + dX^2$

→ 2D Minkowski space!

<u>Interpretation:</u>



Rindler space-time is the space-time a family of uniformly accelerating observers sees.

They all accelerate towards positive X, and are at rest at $T=0$.

~~[illegible]~~

Proper acceleration $\to \infty$

Towards $X \to 0$

⇒ Acceleration horizon

acc. $= \frac{1}{x}$ (proper time)

Event horizon ⇔ Acceleration horizon

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What about for Kerr BH?

We introduce co-moving coordinates

$$\tilde{t} = t \,, \quad \tilde{\phi} = \phi - \Omega t \,, \quad \Omega = \frac{a}{a^2 + r_+^2}$$

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} d\tilde{t}^2$$

$$- 2a\sin^2\theta \frac{r^2 + a^2 - \Delta}{\Sigma} d\tilde{t}\,(d\tilde{\phi} + \Omega d\tilde{t})$$

$$+ \sin^2\theta \cdot \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} (d\tilde{\phi} + \Omega d\tilde{t})^2$$

$$g_{\tilde{t}\tilde{\phi}} = -a\sin^2\theta \frac{r^2 + a^2 - \Delta}{\Sigma} + \Omega \sin^2\theta \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}$$

$$g_{\tilde{t}\tilde{\phi}}\big|_{r=r_+} = -a\sin^2\theta \frac{r_+^2 + a^2}{\Sigma} + \Omega \sin^2\theta \frac{(r_+^2 + a^2)^2}{\Sigma}$$

$$= 0$$

shows angular velocity of EH is Ω in ϕ-direction!

$$g_{\tilde{t}\tilde{t}} = -\frac{\Delta - a^2\sin^2\theta}{\Sigma} - 2a\sin^2\theta \frac{r^2 + a^2 - \Delta}{\Sigma} \cdot \Omega$$

$$+ \sin^2\theta \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \Omega^2$$

$$\Delta = (r - r_-)(r - r_+) \simeq (r_+ - r_-)(r - r_+)$$

$$r = r_+\left(1 + \frac{r_+ - r_-}{4\Sigma_+ r_+} x^2\right)$$

$$\Sigma_+ = r_+^2 + a^2\cos^2\theta$$

We consider $\theta = \text{const}$ (we look at θ near a fixed θ value)

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$$\frac{\Sigma}{\Delta} dr^2 \simeq \frac{\Sigma_+}{(r_+ - r_-)(r - r_+)} dr^2 .$$

$$\simeq \frac{\Sigma_+}{(r_+ - r_-) r_+} \cdot \frac{4 \Sigma_+ r_+}{r_+ - r_-} \cdot \frac{1}{x^2} \cdot \left(\frac{r_+ (r_+ - r_-)}{4 \Sigma_+ r_+} 2 x dx \right)^2 = dx^2$$

$$g_{\tilde{t}\tilde{t}} = - \frac{\Delta}{\Sigma} \left(1 - \Omega a \sin^2\theta \right)^2 + \frac{1}{\Sigma} \left(a - \Omega (r^2 + a^2) \right)^2 \sin^2\theta$$

$$\simeq - \frac{(r_+ - r_-)(r - r_+)}{\Sigma_+} \left(1 - \Omega a \sin^2\theta \right)^2$$

gives $(r - r_+)^2$ term!

$$+ \frac{1}{\Sigma_+} \Big(\underbrace{a - \Omega (r_+^2 + a^2)}_{=0} - \Omega \cdot 2 r_+ (r - r_+) \Big)^2 \sin^2\theta$$

$$\simeq - \frac{r_+ - r_-}{\Sigma_+} r_+ \frac{r_+ - r_-}{4 \Sigma_+ r_+} x^2 \left(1 - \frac{a^2}{r_+^2 + a^2} \sin^2\theta \right)^2$$

$$= - x^2 \cdot \frac{1}{4} \frac{(r_+ - r_-)^2}{\cancel{\Sigma_+^2}} \cdot \frac{1}{(r_+^2 + a^2)^2} \cdot \cancel{\Sigma_+^2}$$

$$\simeq - x^2 \cdot \frac{1}{4} \frac{(r_+ - r_-)^2}{(r_+^2 + a^2)^2}$$

$\Rightarrow$ Surface gravity is

$$\kappa = \frac{r_+ - r_-}{2 (r_+^2 + a^2)}$$

Area of event horizon:

M, J, Q: Quantities for BH's found by analogy to Newt. matter

Surface grav. κ and angular velocity Ω: Special properties only for BH's

We now consider another special property of BH's

Consider the Schw. metric $ds^2 = \cdots$

~~For ... ~~

We just found that close to the horizon at $r = r_0$, the metric is approximately

$$ds^2 \simeq \underbrace{-\kappa^2 x^2 dt^2 + dx^2}_{\text{Rindler space-time}} + r_0^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

Thus, at $x = 0$ (at BH) and $t = \text{const}$ we find $ds^2 = r_0^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$

$\Rightarrow$ This 2D slice of the space-time is a sphere of radius r_0

We can compute the area of this sphere

$$A = 4\pi r_0^2 = 16\pi G^2 M^2$$

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This quantity is known as the <u>area of the EH</u>

What about the Kerr-metric?
Take comoving coord's
$$\tilde{t} = t, \quad \tilde{\phi} = \phi - \Omega t$$

Locally near the EH Kerr-metric becomes

$$ds^2 = \underbrace{-\kappa^2 x^2 dt^2 + dx^2}_{\text{Rindler}} + g_{\tilde{\phi}\tilde{\phi}} d\tilde{\phi}^2 + g_{\theta\theta} d\theta^2$$

For $x=0$ and $t=\text{const.}$
$$ds^2 = g_{\tilde{\phi}\tilde{\phi}} d\tilde{\phi}^2 + g_{\theta\theta} d\theta^2$$

Thus we can define the area of the EH as
$$A = \int_0^{\pi} d\theta \int_0^{2\pi} d\tilde{\phi} \sqrt{g_{\theta\theta} g_{\tilde{\phi}\tilde{\phi}}}\Big|_{x=0,\, t=\text{const.}}$$
$$= \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}}\Big|_{r=r_+,\, t=\text{const.}}$$
since $g_{\tilde{\phi}\tilde{\phi}} = g_{\phi\phi}$. We compute

$$A = 4\pi r_0 r_+$$

Since $\sqrt{g_{\theta\theta} g_{\phi\phi}}\Big|_{r=r_+} = (r_+^2 + a^2)\sin\theta = r_0 r_+ \sin\theta$

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The area of the EH is an important quantity for several reasons, as we shall see

Hawkings area theorem:

The area of the (future) EH of an asympt. flat space-time is a non-decreasing function of time

Remark 1: Should be understood as a statement about dynamics in classical GR → not including quantum effects

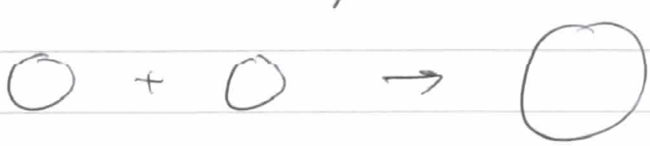
Remark 2: The EH can be disconnected

Example 1:

Compare area of two Schwarzschild BH's with one:

$$A_1 = 16\pi G^2 M^2 \quad , \quad A_2 = 16\pi G^2 (M_1^2 + M_2^2)$$

~~Ex~~ Consider processes

1) 

2) 

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Mass conservation $M = M_1 + M_2$

In process 1)

$$\mathcal{A}_{\text{before}} = \mathcal{A}_2 < \mathcal{A}_{\text{after}} = \mathcal{A}_1$$

— " — 2)

$$\mathcal{A}_{\text{before}} = \mathcal{A}_1 > \mathcal{A}_{\text{after}} = \mathcal{A}_2$$

since $M^2 = (M_1 + M_2)^2 = M_1^2 + M_2^2 + 2M_1 M_2$
$> M_1^2 + M_2^2$
$\Rightarrow \mathcal{A}_1 > \mathcal{A}_2$

Thus, two BH's merging into one is an irreversible process!

Example 2:

~~Normally, one~~

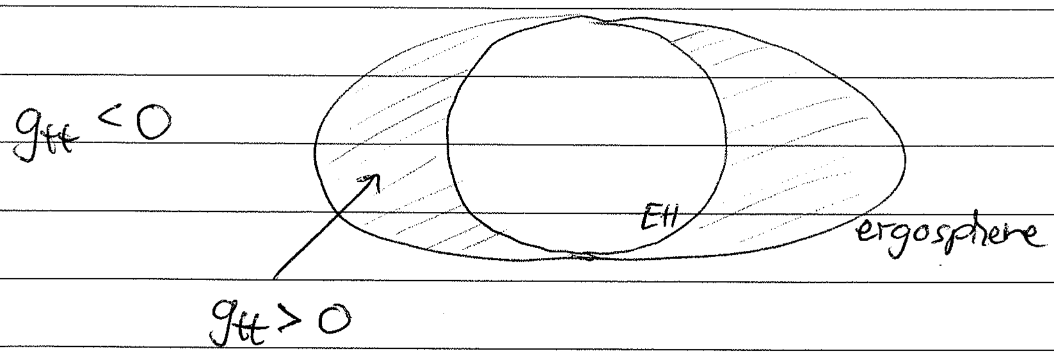
Since (classically) BH's can only absorb one might think that the mass of a BH also never decreases

However, this intuition is wrong

This is shown by the so-called Penrose process for the Kerr BH

5

Kerr BH has the ergo-region near the EH

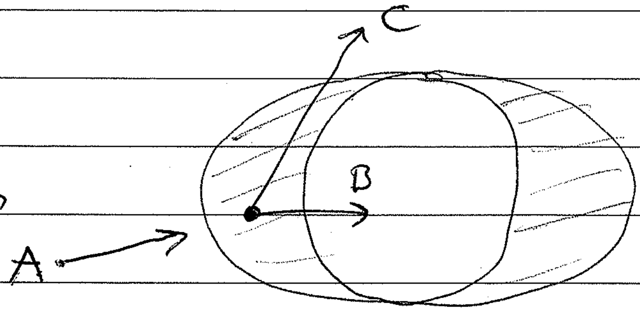


$g_{tt} > 0$ in ergoregion: Time-translation KVF $T = \frac{\partial}{\partial t}$ is space-like

Penrose process

$A \to B + C$

decay of A in ergoregion



B: Falls in to BH
C: Escapes BH

Energy conservation

$E_A = E_B + E_C$

Momentum conservation

$(p_A)_\mu = (p_B)_\mu + (p_C)_\mu$

In local frame at decay point

We have $E = -p \cdot T$ $\left(T = \frac{\partial}{\partial t}\right)$.

Outside ergoregion (and BH): $p^2 < 0$, $T^2 < 0$

This implies $p \cdot T < 0$ hence $E > 0$

6

At decay point: $T^2 > 0$ ~~[scribbled]~~ and $p^2 < 0$
$\Rightarrow$ Possible to have $\mathcal{E} = -p \cdot T < 0$

Assume $\mathcal{E}_B = -p_B \cdot T < 0$

This gives $\mathcal{E}_C = \mathcal{E}_A - \mathcal{E}_B > \mathcal{E}_A$

$\Rightarrow$ We have extracted energy from the BH

However, one can show that processes with maximal energy extraction are the ones for which $\delta A = 0$ and for any process $\delta A \geq 0$.

<u>Note</u>: $a^2 < 0$, $b^2 < 0 \Rightarrow a \cdot b < 0$

Proof: Assume $a \cdot b \geq 0$

Choose local Lorentz frame $g_{\mu\nu}(p) = \eta_{\mu\nu}$

$a^2 = -a_0^2 + \bar{a}^2 < 0 \Rightarrow a_0^2 > \bar{a}^2$

$b^2 < 0 \Rightarrow b_0^2 > \bar{b}^2$

Then

$a \cdot b \geq 0 \Rightarrow \bar{a} \cdot \bar{b} \geq a_0 b_0 \Rightarrow (\bar{a} \cdot \bar{b})^2 \geq a_0^2 b_0^2 > \bar{a}^2 \bar{b}^2$

But this contradicts $|\bar{a} \cdot \bar{b}| \leq |\bar{a}||\bar{b}|$

1

BH mechanics:

We introduced the quantities [For BH's with $T_{\mu\nu}=0$ outside EH]

M, J : In analogy with ordinary matter

$\Omega, \kappa, \mathcal{A}$: Found by analysing NH region

We considered change of $\mathcal{A}$ in physical processes. We now consider all the quantities

3 laws of BH mechanics:

0th law: Surface gravity κ is constant on the EH

1st law: If we perturb the BH such that (when it has settled down) there is a small change in the quantities $M, J, \Omega, \kappa, \mathcal{A}$, then the 1st law holds

$$\delta M = \frac{\kappa}{8\pi G}\,\delta\mathcal{A} + \Omega\,\delta J$$

2nd law: (Hawkings area theorem) The area $\mathcal{A}$ of the EH does not decrease as a function of time.

2

Exercise: Show that the Kerr BH obeys the 1st law

~~Comments~~ This looks like the 3 laws of thermodynamics!

$\kappa \Leftrightarrow$ temperature

$A \Leftrightarrow$ entropy

This is what we shall show now!

Hawking radiation: 1

So far we only considered classical physics → GR
Now we consider quantum effects for BH's.

General lesson for QFTs:
The partition fct for a QFT at finite temperature T can be computed as the Euclidean path integral of the Wick rotated QFT

$$\text{Wick rotation: } \quad t = i\tau$$

where the Euclidean time τ has period $\beta = \frac{1}{T}$.

$$Z(\beta) = \int_{\phi(\tau+\beta)=\phi(\tau)} D\phi \, e^{-S_E[\phi]}$$

We can use this as an easy way to ~~xxxx~~ see that BH's have a temperature.

Consider Schwarzschild:

$$ds^2 = -\left(1-\frac{r_0}{r}\right)dt^2 + \frac{dr^2}{1-\frac{r_0}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Do Wick rotation:

$$ds_E^2 = \left(1-\frac{r_0}{r}\right)d\tau^2 + \frac{dr^2}{1-\frac{r_0}{r}} + r^2(\qquad)$$

2

Let's do the same for the near-horizon metric:

$$ds^2 = \underbrace{-\kappa^2 x^2 dt^2 + dx^2}_{\text{Rindler}} + r_0^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

After Wick rot:

$$ds_E^2 = \underbrace{\kappa^2 x^2 d\tau^2 + dx^2} + r_0^2(\qquad)$$

This looks like the plane $\mathbb{R}^2$ in polar coordinates!

However, one gets a conical ~~[illegible]~~ singularity if τ does not have the right period.

Write $\alpha = \kappa \cdot \tau$, then the relevant part of ds_E^2 is $x^2 d\alpha^2 + dx^2$

No conical singularity $\Leftrightarrow$ α period 2π

Hence we see that τ has period $\frac{2\pi}{\kappa}$

Thus in the Wick rotated QFT interpretation

$$\beta = \frac{2\pi}{\kappa} \Rightarrow T_H = \frac{1}{\beta} = \frac{\kappa}{2\pi}$$

$\Rightarrow$ Hence we can interpret this as the BH having temperature

$$T_H = \frac{\kappa}{2\pi}$$

3

This is called the Hawking temp. of a BH

Note: same derivation for Kerr since we also find Rindler $ds_{2D}^2 = -\kappa^2 x^2 dt^2 + dx^2$ near the EH → Also gives $T_H = \frac{\kappa}{2\pi}$

Using analysis of QFT in curved space-time Hawking found that the temp. T_H corresponds to blackbody radiation of that temp. emitted from the BH

Quantum mech. a BH is not entirely black → it emits radiation (Hawking rad.)

Note that the perceived temperature depends on the observer
→ what proper time is for the observer

We assumed proper time t: Observer at $r \to \infty$ far away from BH

Observer at $r = \text{const.}$ would have proper time
$dt'^2 = (1 - \frac{r_0}{r}) dt^2$

⇒ BH temperature measured: $\frac{1}{\sqrt{1-\frac{r_0}{r}}} \cdot \frac{\kappa}{2\pi}$

Called the Tolman law

4

Notice the temp. $\rightarrow \infty$ as we approach the BH

Near the BH in Rindler coord's:

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + r_0^2 (\qquad)$$

Temp. at $x = \text{const}$: $T_{\text{local}} = \frac{1}{\kappa x} \cdot \frac{\kappa}{2\pi} = \frac{1}{2\pi x}$

Compare with local acceleration: $a = \frac{1}{x}$

We see that

$$T_{\text{Local}} = \frac{a}{2\pi}$$

→ This is called the Unruh temp.

A general relation between locally accelerated inertial frames and the temperature of the perceived horizon

BH thermodynamics 1

Hawking temp. $T_H = \frac{\kappa}{2\pi}$

Postulate: (Bekenstein & Hawking)
BH's have an entropy

$$\boxed{S_{BH} = \frac{A}{4G}}$$

Why? Since then the 3 laws of BH mechanics $\Rightarrow$ 3 laws of thermodynamics:

0th law: Hawking temp. T_H is constant on the EH

1st law:
$$\delta M = T_H \delta S_{BH} + \Omega \delta J$$

2nd law:
S_{BH} never decreases as fct. of time (classical GR)

2nd law $\Rightarrow$ $S \propto A$

1st law $\Rightarrow$ $\frac{\kappa}{8\pi G} \delta A = T_H \delta S = \frac{\kappa}{2\pi} \delta S$

2

Bekenstein further introduced the generalized entropy (which never decreases):

$$S_{gen} = S_{universe} + S_{BH}$$

for a Universe with a BH in where $S_{universe}$ is entropy of everything outside the BH

⇒ Resolves the missing entropy problem that $S_{universe}$ can decrease when things are absorbed in the BH.

⇒ S_{BH} should thus be thought of as the entropy of everything which is hidden behind the horizon

Problem for this interpretation:
$S_{BH} \propto A$, normally entropy would go like the volume
We get back to this in a moment...

Hawking realized another problem with S_{BH} and S_{gen}

Hawkings Information Paradox:

1

Consider a star → supernova → BH

→ For a star we have in principle precise information about all particles in star available

→ When it became a BH, some of this info cannot be retrieved since it is hidden behind the EH

However Hawking radiation means that the BH can fully evaporate by emitting Hawk. radiation
→ No EH after evaporation

However the info that was hidden behind the EH seems lost because black body radiation contains almost no info

Where did the info about the star go? → Info paradox

2

Problem in unifying QM with GR

In QM: A state $|\alpha;t\rangle$ of a closed system at time t can be evolved to another time t' using a Unitary operator $U(t,t')$:

$$|\alpha;t'\rangle = U(t,t')|\alpha;t\rangle$$

(relation to Ham. H: $i\hbar \frac{d}{dt} U = HU$)

Since $U^\dagger U = I$ $U(t,t')$ is invertible
→ we can evolve from t' back to t
In this sense QM is fully deterministic

→ Seems not possible to do this after evaporation

More precisely: A star → Pure state
Hawking radiation → Thermal state

Thus we go from a pure state (the star) to a mixed state (Hawk. rad. + remnants always outside the EH)

Not possible in QM!

3

Two possibilities:

1) Info lost, QM is wrong
- Bad news since if QM is wrong we lack fundamental principles to guide us in finding a theory of quantum gravity
- And it conflicts with all experiments that information is not preserved

2) Info not lost, QM is right
We should resolve paradox by finding where the info went
→ Most likely means that Hawking radiation is only approximately thermal

Black holes for $D > 4$

Why?

- String theory
- Phenomenological scenario (incl. low Planck scale)
- Learn more about gravity (like considering $N_c > 3$ for Yang-Mills)
- $D > 4$ gravity can be used as effective description for strongly coupled non-gravitational theories with $D \leq 4$

(large N_c $\mathcal{N}=4$ SYM $\leftrightarrow$ AdS/CFT, QCD, YM, CMT, ...)

Similarities between D=4 and D>4 gravity:

- BH mechanics
- Hawking radiation, BH entropy, BH thermodynamics
- Information paradox
- D=4 Schwarzschild ~~metric~~ generalizes to Schwarzschild-Tangherlini metric:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{D-2}^2$$

$$f = 1 - \frac{r_0^{D-3}}{r^{D-3}}$$

- Similar generalization for charged BH + cosmological constant
- Kerr BH $\longrightarrow$ Myers-Perry BH

Kerr BH: rotation in one plane

Myers-Perry BH: $\left[\frac{D-1}{2}\right]$ rotation planes

Differences between $D=4$ and $D>4$ gravity:

- No general uniqueness theorems
  (specific ones for static BH's + $D=5$ BH's)

  $\Updownarrow$

- Many more kinds of BH's

  $\Updownarrow$

- "Less unique"
  $\rightarrow$ New BH invariants
  (rod/domain structure)

  $\rightarrow$ Harder to characterize a BH
  More things to measure

- Instabilities of BH's
  (unlike in 4D: Schwarzschild and Kerr BH's are stable under perturbations)

- Rich phase structure & dynamics

Reasons for this

- Extended objects with event horizons
  $\rightarrow$ Basic ones: Black p-branes

- Largely seperated scales at event horizons possible

Black p-branes:

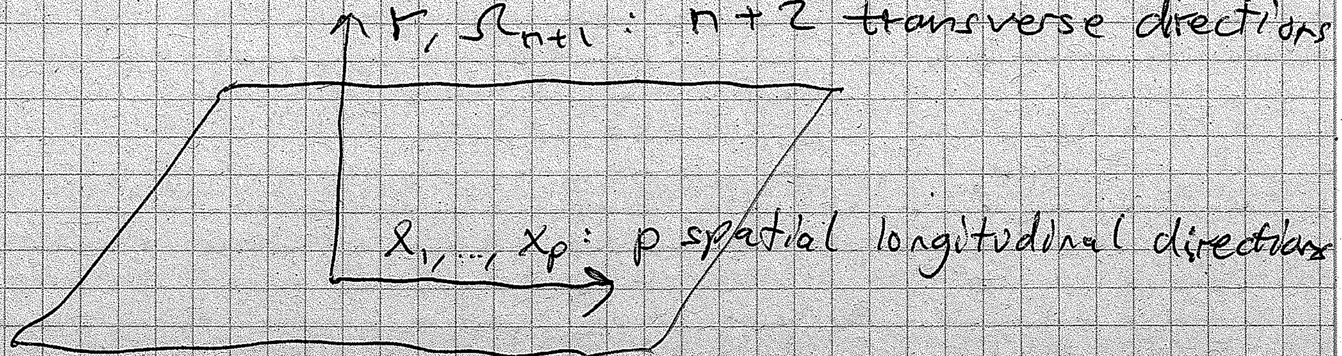
- Neutral and static black p-brane
  Metric = Schwarzschild BH in D-p dim × $\mathbb{R}^p$

$$ds^2 = -f dt^2 + \sum_{i=1}^{p} dx_i^2 + \frac{dr^2}{f} + r^2 d\Omega_{n+1}^2$$

$$f = 1 - \frac{r_0^n}{r^n} \quad \text{with} \quad n = D-p-3$$

- Solution of pure gravity Einstein eqs:
  $R_{\mu\nu} = 0$

- Describes infinitely extended black p-brane with flat embedding



Event horizon with topology $\mathbb{R}^p \times S^{n+1}$

NOTE: Metric is not asymptotically flat

Stress tensor for Black p-brane:

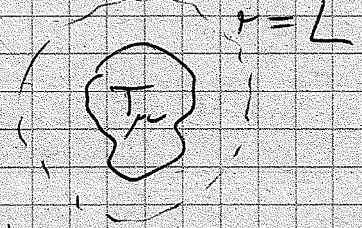
Go back to BH case $(p=0)$:

Consider localized matter distribution in D dim.
Use coordinates:

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} dy_i^2$$

$$r \equiv \sqrt{\sum_{i=1}^{D-1} y_i^2} \quad \text{distance from } y_1 = \dots = y_{D-1} = 0$$

Matter localized within $r < L$:



- For $r \gg L$ the leading contribution to the gravitation field comes from the monopole moment given by the total mass

$$M = \int d^{D-1}y \, T_{00}$$

- Then there are higher multipole moments that gives increasingly refined characterization of of $T_{\mu\nu}$, e.g.

$$J = \int d^{D-1}y \left(-y_1 T_{02} + y_2 T_{01} \right)$$

Angular momentum in 12-plane
(dipole moment)

Consider only leading order (monopole moment)
We can "integrate out" length scale L by replacing $T_{\mu\nu}$ with

$$T_{00} = M\,\delta^{D-1}(y_i) \qquad \begin{pmatrix}\text{other components}\\ \text{of } T_{\mu\nu} = 0\end{pmatrix}$$

Gravitational field from this?

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \quad |h_{\mu\nu}| \ll 1$$

In harmonic gauge: $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu(\eta^{\alpha\beta}h_{\alpha\beta})$

To first order in h: $R_{\mu\nu} = -\frac{1}{2}\partial^\alpha\partial_\alpha h_{\mu\nu}$

Einstein eqs: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\,T_{\mu\nu}$

$$\Rightarrow R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{D-2}g_{\mu\nu}T^\alpha{}_\alpha\right)$$

$$\Rightarrow \partial^\alpha\partial_\alpha h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{D-2}g_{\mu\nu}T^\alpha{}_\alpha\right)$$

Assume $\partial_t h_{\mu\nu} = 0$

$$\vec{\nabla}^2 h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{D-2} \eta_{\mu\nu} T^\alpha{}_\alpha \right)$$

$$\Rightarrow h_{\mu\nu}(y) = \frac{16\pi G}{(D-3)\Omega_{D-2}} \int d^{D-1}y' \frac{T_{\mu\nu}(y') - \frac{1}{D-2}\eta_{\mu\nu} T^\alpha{}_\alpha(y')}{\left(\sum_{i=1}^{D-1} (y_i - y_i')^2 \right)^{\frac{D-3}{2}}}$$

So ~~in~~ inserting the leading order $T_{\mu\nu}$ we have

$$h_{00} = \frac{16\pi G}{(D-3)\Omega_{D-2}} \frac{\int d^{D-1}y' \frac{D-3}{D-2} M \delta^{D-1}(y')}{r^{D-3}}$$

$$= \frac{16\pi G M}{(D-2)\Omega_{D-2} r^{D-3}}$$

$$D=4: \quad h_{00} = \frac{16\pi G M}{2 \cdot 4\pi r} = \frac{2GM}{r}$$

So for Schwarzschild metric $g_{00} = -1 + \frac{r_0}{r}$

$$\Rightarrow M = \frac{r_0}{2G}$$

We now generalize this to p-branes:

- Consider D-dim. flat space in coord's

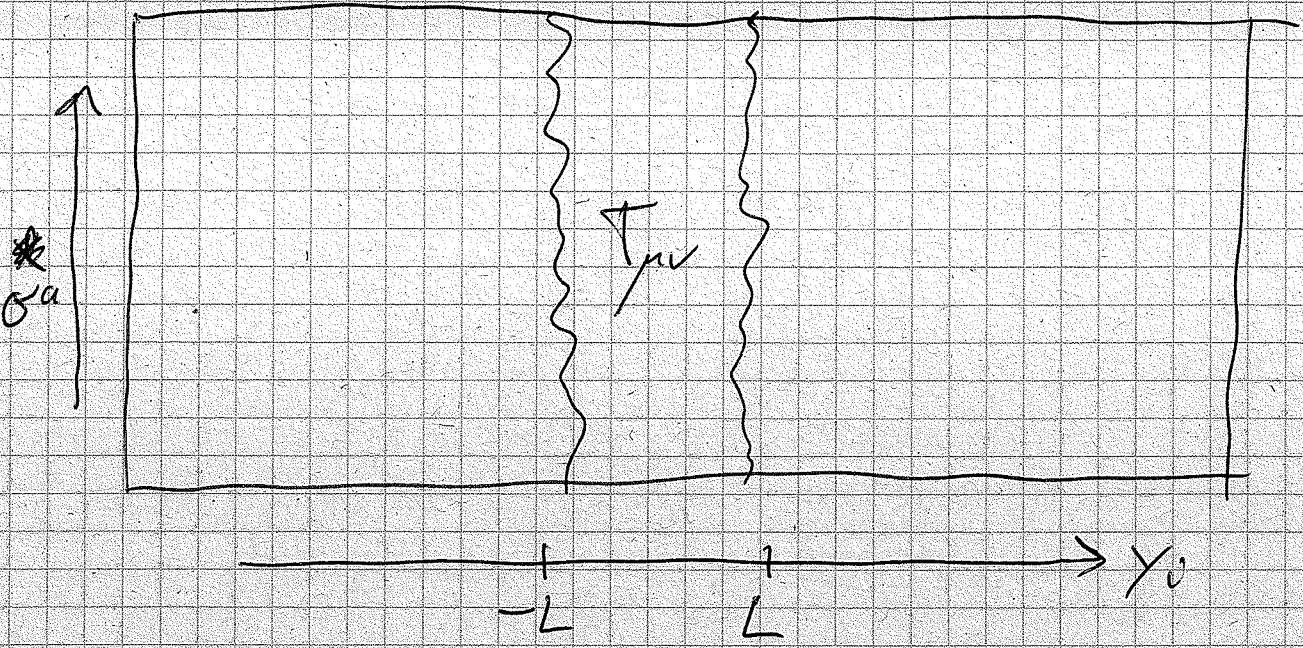
$$ds^2 = \sum_{a,b=0}^{p} \eta_{ab}\, d\sigma^a d\sigma^b + \sum_{i=1}^{n+2} dy_i^2$$

longitudinal transverse

Write $r \equiv \sqrt{\sum_{i=1}^{n+2} y_i^2}$, $n = D-p-3$

- Consider matter distribution localized around origin in the transverse space $y_1 = y_2 = \dots = y_{n+2} = 0$, i.e. $r=0$, in the sense that $T_{\mu\nu}$ only is non-zero for $r < L$

- We assume Newtonian or special relativistic matter (only mild effect on $g_{\mu\nu}$)



For $r \gg L$ the leading contribution to gravitational field comes from monopole moment

→ Now this includes all longitudinal directions σ^a, $a = 0, 1, \ldots, p$

Measured as

$$\mathcal{T}_{ab} = \int d^{n+2}y \, T_{ab} \quad , \; a,b = 0, 1, \ldots, p$$

Called the stress-tensor

"Integrating out" length scale L:

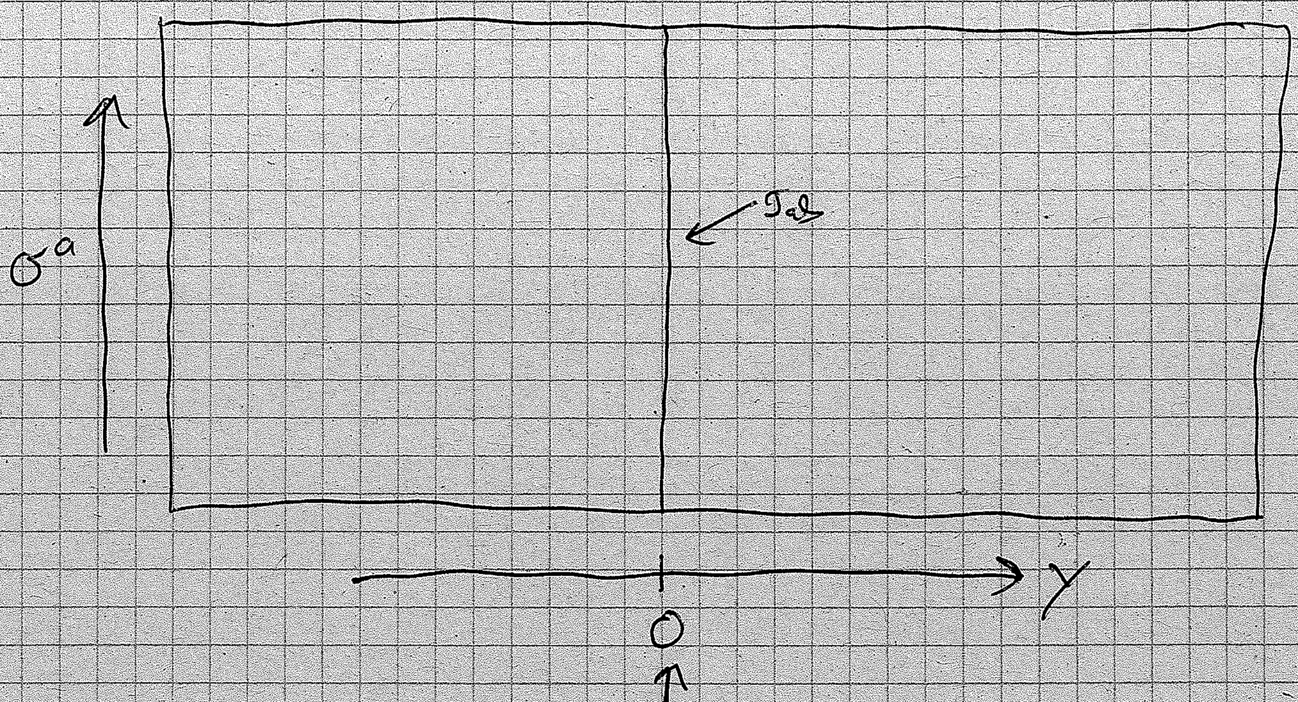
$$T_{ab} = \mathcal{T}_{ab}\, \delta^{D-p-1}(y) \quad , a,b = 0,1,\dots,p$$

$(T_{\mu\nu} = 0$ otherwise$)$

- This captures the monopole moment
- Corresponds to an infinitely thin p-brane with stress tensor $\mathcal{T}_{ab}$

($p=0$: Point particle ~~with~~ mass M)

Generalization of the point particle



Matter sits at $y_1 = y_2 = \dots = y_{n+2} = 0$
(i.e. $r=0$)

Gravitational field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \quad |h_{\mu\nu}| \ll 1$$

Assume $\dfrac{\partial h_{\mu\nu}}{\partial \sigma^a} = 0$

Einstein eqs:

$$\vec{\nabla}^2 h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{D-2} \eta_{\mu\nu} T^\alpha{}_\alpha \right)$$

with $\vec{\nabla}^2 = \dfrac{\partial^2}{\partial y_1^2} + \dfrac{\partial^2}{\partial y_2^2} + \cdots + \dfrac{\partial^2}{\partial y_{n+2}^2}$

$$h_{\mu\nu} = \frac{16\pi G}{n\Omega_{n+1}} \int d^{n+2} y' \, \frac{T_{\mu\nu}(y') - \frac{1}{D-2}\eta_{\mu\nu} T^\alpha{}_\alpha(y')}{\left(\sum_{i=1}^{n+2} (y_i - y_i')^2 \right)^{\frac{n}{2}}}$$

$$\Rightarrow h_{ab} = \frac{16\pi G}{n\Omega_{n+1} r^n} \left(J_{ab} - \frac{1}{D-2} \eta_{ab} J^c{}_c \right)$$

Assume now T_{ab} is of perfect fluid form

$$T_{ab} = (\varepsilon + P) u_a u_b + P \eta_{ab}$$

ε: Energy density of fluid

P: pressure of fluid

u^a: Fluid velocity $\left(u^a u^b \eta_{ab} = -1 \right)$

Then $T_{ab} - \frac{1}{D-2} \eta_{ab} T^c{}_c$

$$= (\varepsilon + P) u_a u_b + \frac{\varepsilon + (n+1)P}{D-2} \eta_{ab}$$

So we have

$$h_{ab} = \frac{16\pi G}{n \Omega_{n+1} r^n} \left((\varepsilon + P) u_a u_b + \frac{\varepsilon + (n+1)P}{D-2} \eta_{ab} \right)$$

Back to black p-brane

- We now measure stress tensor for the black p-brane by considering its grav. field for $r \gg r_0$
- So stress tensor defined in analogy with Newtonian / Special Relativistic matter

Metric: $$ds^2 = -f dt^2 + \sum_{a=1}^{p} dx_a^2 + \frac{dr^2}{f} + r^2 d\Omega_{n+1}^2$$

$$f = 1 - \frac{r_0^n}{r^n}$$

We read off

$$u^a = (1, 0, \ldots, 0)$$

$$\varepsilon + (n+1) P = 0 \Rightarrow \boxed{P = -\frac{1}{n+1} \varepsilon}$$

Equation of state for fluid

$$\frac{16\pi G}{n \Omega_{n+1}} \left(1 - \frac{1}{n+1}\right) \varepsilon = r_0^n$$

$$\Rightarrow \boxed{\varepsilon = \frac{(n+1)\Omega_{n+1} r_0^n}{16\pi G} , \quad P = -\frac{\Omega_{n+1} r_0^n}{16\pi G}}$$

$$\text{Tension} = -P = \frac{\Omega_{n+1} r_0^n}{16\pi G}$$

The black p-brane also has Hawking temp.

$$T = \frac{n}{4\pi r_0}$$

and the entropy density

$$s = \frac{\Omega_{n+1}}{16\pi G} r_0^{n+1}$$

First law of thermodynamics

$$d\mathcal{E} = T\, ds$$

As well as the relation

$$\mathcal{E} + P = T s$$

(known in thermodynamics as the Euler - Gibbs - Duhem equation)

Remark

We can perform a boost along the longitudinal direction

→ Changes u^a but ε, P same

Boosted black p-brane metric

$$ds^2 = \left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2$$

Interlude on black Dp-brane stress tensor:

We can use this method to read off the stress tensor of other ~~...~~ brane geometries as well.

Consider metric for ~~..~~ black Dp-branes in $D=10$ (Einstein frame)

$$ds^2 = H^{-\frac{n}{8}}\left(-f dt^2 + \sum_{i=1}^{p} dx_i^2\right) + H^{\frac{p+1}{8}}\left(\frac{dr^2}{f} + r^2 d\Omega_{n+1}^2\right)$$

with $n = 7-p$,

$$H = 1 + \frac{r_0^n \sinh^2\alpha}{r^n}, \quad f = 1 - \frac{r_0^n}{r^n}$$

This is the metric for N coincident Dp-branes

$$N T_{Dp} = \frac{\Omega_{n+1}}{16\pi G} n r_0^n \cosh\alpha \sinh\alpha$$

and temperature $T = \dfrac{n}{4\pi r_0 \cosh\alpha}$.

Stress tensor is also of perfect fluid form

$$T_{ab} = (\varepsilon + P) u_a u_b + P \eta_{ab}$$

Hence we find

$$h_{00} = \left(+ \frac{n}{8} r_0^n \sinh^2\alpha + r_0^n \right) \frac{1}{r^n}$$

$$h_{ii} = -\frac{n}{8} r_0^n \sinh^2\alpha \, \frac{1}{r^n} \quad , \; i = 1, \ldots, p$$

Hence (since $u^a = (1, 0, \ldots, 0)$)

$$\begin{cases} \dfrac{16\pi G}{n\Omega_{n+1}} \left(\varepsilon + P - \dfrac{\varepsilon + (n+1)P}{8} \right) = \dfrac{n}{8} r_0^n \sinh^2\alpha + r_0^n \\[2ex] \dfrac{16\pi G}{n\Omega_{n+1}} \left(\dfrac{\varepsilon + (n+1)P}{8} \right) = -\dfrac{n}{8} r_0^n \sinh^2\alpha \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \varepsilon + P = \dfrac{\Omega_{n+1}}{16\pi G} n r_0^n \\[2ex] \varepsilon + (n+1) P = -\dfrac{\Omega_{n+1}}{16\pi G} n^2 r_0^n \sinh^2\alpha \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \varepsilon = \dfrac{\Omega_{n+1}}{16\pi G} r_0^n \left(n + 1 + n \sinh^2\alpha \right) \\[2ex] P = -\dfrac{\Omega_{n+1}}{16\pi G} r_0^n \left(1 + n \sinh^2\alpha \right) \end{cases}$$

What is the significance of all this?

Take BH ($p=0$) case

→ Far away from BH we can treat it as if it was a point particle of mass M

How far away? For $r \gg r_0$
$\uparrow$
Schwarzschild rad.

- Suppose now we wish to put the BH in a background geometry with metric $g_{\mu\nu}$ (D-dim. space-time)

- Let R be the "minimum" length scale of this geometry

- Then if $R \gg r_0$ we can to a good approximation treat the BH as a point particle in this geometry

A point particle (of mass M) should follow a geodesic

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

where $x^\mu(\tau)$ is the position of the BH in the backgr. geometry and $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol for $g_{\mu\nu}$

$\Rightarrow$ This is the dynamics of the BH to leading order in an $\frac{r_0}{R}$ expansion!

Higher orders in $\frac{r_0}{R}$?

$\rightarrow$ Take into account effect of BH on the background metric ("backreaction") to first order in M when distance to BH $\gg r_0$

$\rightarrow$ Find corrected geometry near BH

$\rightarrow$ Modified BH geometry $\Rightarrow$ 2nd order modification of geometry far away

$\rightarrow$ 2nd order corrected BH geometry

$\rightarrow$ 3rd order corrected backgr. geometry

⋮

Technique known as Matched asymptotic expansion (MAE)

How does this work for a black p-brane?

$g_{\mu\nu}$: Backgr. geometry with minimum length scale $\gg R$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

For $r_0 \ll R$ (r_0: horizon rad. of black p-brane)

the dynamics of the black p-brane to leading order in $\frac{r_0}{R}$ can be ~~s~~described by the dynamics of an infinitely thin p-brane with the same stress tensor as the black p-brane

Dynamics of infinitely thin p-branes?

- $X^\mu(\sigma)$: Embedding of p-brane in backgr. geom. $g_{\mu\nu}$
  - We ~~assume~~ assume length scale of variation of $X^\mu(\sigma) \gg R$ as well
- $\sigma^a = (\sigma^0, \ldots, \sigma^p)$ worldvolume coord's of p-brane
- $T^{ab}(\sigma)$ stress tensor

Equations for dynamics of infinitely thin p-brane

Intrinsic: $$D_a T^{ab} = 0$$

Extrinsic: $$T^{ab}\left(D_a D_b X^\mu + \Gamma^\mu_{\nu\rho} D_a X^\nu D_b X^\rho\right) = 0$$

D_a: covariant derivative on WV with respect to metric $ds_{WV}^2 = \gamma_{ab}\, d\sigma^a d\sigma^b$

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$

The induced metric of embedding

$\Gamma^\mu_{\nu\rho}$: Christoffel symbol of backgr.

Extrinsic eq. (AKA the Carter eq) is the generalization of the geodesic eq for point particles

Intrinsic eq $\quad D^a T_{ab} = 0$

→ Conservation of the stress tensor

(for point particles: M const. in time)

The geometric object

$$K_{ab}{}^{\mu} = D_a D_b X^{\mu} + \Gamma^{\mu}_{\nu\rho} D_a X^{\nu} D_b X^{\rho}$$

- called Extrinsic curvature tensor
- depends only on $g_{\mu\nu}$, $X^{\mu}(\sigma)$ (⇒ geometric)

Extrinsic eq. $\quad T^{ab} K_{ab}{}^{\mu} = 0$

Blackfold:

Consider now a black p-brane with embedding $X^\mu(\sigma)$ in a background space-time with metric $g_{\mu\nu}$.

$g_{\mu\nu}$: length scale R (or larger)

Black p-brane : Thickness scale r_0 (horizon radius)

For $r_0 \ll R$ the leading order dynamics of ~~the~~ black p-brane ~~with embedding~~ is described by the eqs

$$D_a T^{ab} = 0 \ , \qquad T^{ab} K_{ab}{}^{\mu} = 0$$

where

$$T^{ab} = (\varepsilon + P) u^a u^b + P \gamma^{ab}$$

$$\varepsilon(\sigma) = \frac{(n+1)\Omega_{n+1}}{16\pi G} \left(r_0(\sigma)\right)^n \ , \quad P(\sigma) = -\frac{1}{n+1}\varepsilon(\sigma),$$

$u^a(\sigma)$

So the black p-brane is "folded" along the embedding $X^\mu(\sigma)$ of $g_{\mu\nu}$ backgr.

→ We call it a "Blackfold"

$u^a(\sigma)$ ⟸ We allow in general a varying velocity field on the blackfold

This means that ~~we~~ in general that locally on the blackfold we ~~see~~ a boosted black p-brane

More precisely, zooming in to a local neighborhood (size $\ll R$) on the blackfold the ~~~~ space-time metric is

$$ds^2 = \left(\gamma_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 \qquad (*)$$

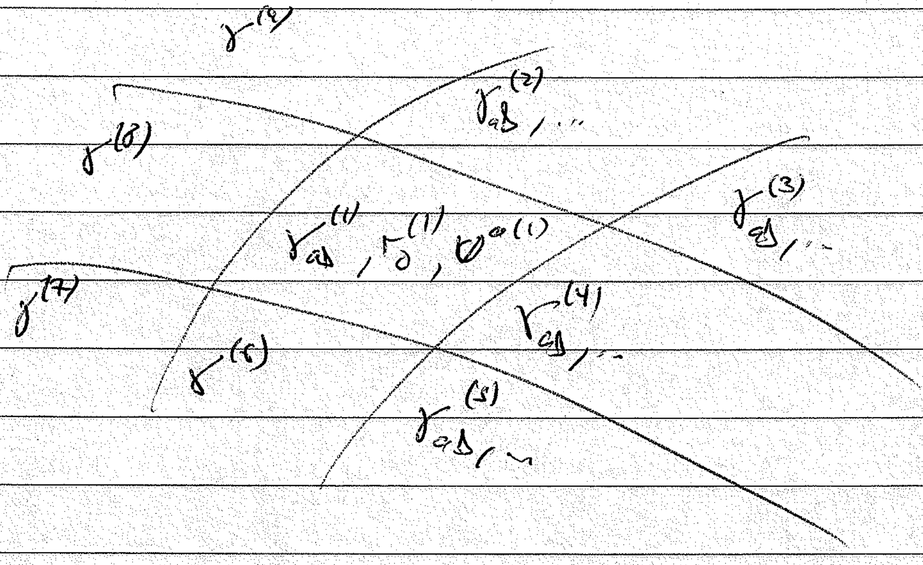
So at large scales $\gtrsim R$ our space-time is described by the metric $g_{\mu\nu}$ with an infinitely thin p-brane with embedding $X^\mu(\sigma)$ and with $r_0(\sigma)$ and $u^a(\sigma)$ such that the

$$D_a T^{ab} = 0 \, , \quad T^{ab} K_{ab}{}^{\rho} = 0$$

Eqs are obeyed.

Instead at small scales $\ll R$ and close to the p-brane we see a constant γ_{ab}, r_0 and u^a with the metric (*)

One can thus think of a blackfold as a $(p+1)$-dimensional submanifold on which we have a patchwork of regions of constant γ_{ab}, r_0, u^a



In each region the metric close to the brane is (*).

Instead the relation ~~between~~ ~~the~~ ~~regions~~ are described by the eqs $D_a J^{ab} = 0$, $J^{ab} K_{ab}{}^{\mu} = 0$.

Similarly to the $p=0$ case one can go to higher orders in $\frac{r_0}{R}$ by order-by-order matching the metric (*) (+ corrections) to the metric $g_{\mu\nu}$ far from the brane (+ corrections)

$\rightarrow$ MAE

Continuing ad infinitum with correcting metric near and metric far from the brane one should end up with an exact black hole geometry

Comments:

- Preceeding BF framework only for neutral black p-brane
  → One can easily generalize this to charged black branes, e.g. D-branes in String Theory. More on this later!

- Even at zeroth order in $\frac{r_0}{R}$ ~~~~
  A very powerful framework with an enormously rich range of applications

To see the power of this framework more clearly we go to special cases
→ Makes it possible to solve the equations!

First special case:

Stationary blackfolds

Stationary Blackfolds:

In general a blackfold does not need to be in equilibrium:

- We can describe blackfolds which are out of thermodynamic equilibrium (over distances $\gtrsim R$)

- We can describe blackfolds with time-dependent embeddings

Now we wish to restrict to blackfolds that are in equilibrium:
→ <u>Stationary blackfolds</u> (incl. static)

How does one characterize a stationary blackfold?

In general all stationary (incl. static) black hole space-times has a vector field k^μ for which

- It is a KVF $(\mathcal{L}_k g)_{\mu\nu} = 0$ i.e.

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$$

- It is null on the EH:

$$g_{\mu\nu} k^\mu k^\nu \big|_{EH} = 0$$

How do we characterize such a vector field for a blackfold?

Remember we have two parts:

- Background metric $g_{\mu\nu}$
- Black p-brane with embedding $X^\mu(\sigma)$ and stress-tensor (perfect fluid)

$$T_{ab} = (\varepsilon + P) u_a u_b + P \gamma_{ab}$$

$$\varepsilon = \frac{(n+1)\Omega_{n+1}}{16\pi G} r_0^n, \quad P = -\frac{1}{n+1}\varepsilon$$

Our definition:

A stationary blackfold (incl. static) is such that

- There exists a vector field k^μ which is a KVF of the backgr. metric, i.e.

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$$

- The pullback of the KVF k^μ to the world-volume

$$k_a = \partial_a X^\mu k_\mu$$

is a KVF on the world-volume, i.e.

$$D_a k_b + D_b k_a = 0$$

- The velocity field u^a is proportional to k^a:

$$u^a = \frac{k^a}{|k|}$$

where $|k| = \sqrt{-\gamma_{ab} k^a k^b}$

Here the normalization follows from requiring $\gamma_{ab} u^a u^b = -1$.

<u>Remark 1:</u>

The last two requirements: $\exists k^a$ on WV:

- $D_a k_b + D_b k_a = 0$
- $u^a = \frac{k^a}{|k|}$

Precisely characterize stationary fluid configurations

→ Necessary for absence of ~~dissipation~~ dissipative effects

Remark 2:

Close to the black ~~brane~~ p-brane we have to leading order the metric

$$ds^2 = \left(\gamma_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1-\frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2$$

The KVF k^μ in these coordinates takes the form

$$k^\mu = \left(\overset{\sigma^a}{k^a}, \overset{r}{0}, \overset{\text{angles}}{\ldots, 0}\right)$$

Thus close to the brane

$$g_{\mu\nu} k^\mu k^\nu = \gamma_{ab} k^a k^b + \frac{r_0^n}{r^n}(u_a k^a)^2$$

For $r = r_0$:

$$g_{\mu\nu} k^\mu k^\nu \Big|_{r=r_0} = -|k|^2 + (u_a k^a)^2$$
$$= -|k|^2 + \left(\frac{1}{|k|} k_a k^a\right)^2 = -|k|^2 + (-|k|)^2$$
$$= 0$$

Hence k^μ is indeed the KVF which is null at the horizon →

→ Existence of such a KVF is a nessecary requirement for a stationary BH space-time.

We now explore further what the conditions for a stationary blackfolds entails

Inserting $\varepsilon = \frac{(n+1)\Omega_{n+1}}{16\pi G} r_0^n$, $P = -\frac{1}{n+1}\varepsilon$ in

$D_a T^{ab} = 0$ one gets

$$(\partial_a \log r_0)\gamma^{ab} = n(\partial_a \log r_0)u^a u^b + (D_a u^a)u^b + u^a D_a u^b$$

Contracting with u_b one gets

$$(n+1)(\partial_a \log r_0)u^a = -D_a u^a$$

Combining the two gives

$$\boxed{\dot{u}_a + \frac{1}{n+1} u_a (D_b u^b) = \partial_a \log r_0}$$

with $\dot{u}^a \equiv u^b D_b u^a$

$\dot{u}^a$: Acceleration of fluid
$D_a u^a$: Expansion of fluid

~~Consider now a stationary fluid~~

$\cancel{u^a = \frac{k^a}{|k|}}$ $\cancel{D_a k_b + D_b k_a = 0}$

Impose now that the fluid is stationary

$$u^a = \frac{k^a}{|k|}, \quad D_a k_b + D_b k_a = 0$$

Contracting the Killing eq with $k^a k^b$ gives $k^a k^b D_a k_b = 0$ which gives

$$k^a \partial_a |k| = 0$$

Using this we find

$$\dot{u}^a = u^b D_b u^a = \frac{k^b}{|k|} D_b \frac{k^a}{|k|} = \frac{k^b}{|k|^2} D_b k^a$$
$$= -\frac{k^b}{|k|^2} D^a k_b = -\frac{1}{2}\frac{1}{|k|^2} D^a k^b k_b$$
$$= \frac{1}{2}\frac{1}{|k|^2}\partial^a |k|^2 = \frac{\partial^a |k|}{|k|} = \partial^a \log |k|$$

Note also $D_a k^a = 0$ so that
$u_a (D_b u^b) = \frac{k_a}{|k|} D_b \frac{k^b}{|k|} = \frac{k^a k^b}{|k|} D_b |k|^{-1} = 0$

Hence we find

$$\partial_a \log |k| = \partial_a \log r_0$$

This means that $\frac{r_0}{|k|}$ is required to be constant on the blackfold world-volume

Interpretation of this?

→ Take another look at $u^a = \frac{k^a}{|k|}$

Consider velocity u^a

→ Means locally a boosted black p-brane

KVF's of black p-brane

(only locally KVF → not at scales $\gtrsim R$)

Time-translation KVF T, $T^2 = -1$

Translations z_i, $z_i^2 = 1$, $i = 1, \dots, p$

(commuting and linear independent)

Velocity $$u^a = \frac{1}{\sqrt{1-V^2}} \left(T^a + \sum_{i=1}^{p} V_i z_i^a \right)$$

with $V^2 = \sum_{i=1}^{p} V_i^2$

Boost velocity

Consider now KVF k^μ.

Assume, we have

$\xi = \frac{\partial}{\partial t}$: time-translation KVF of backgr.

$\chi_i = \frac{\partial}{\partial \phi_i}$, $i=1,\dots,s$: Rotational KVF's of backgr. (commuting)

and assume k^μ is a linear comb. of these:

$$k^\mu = \xi^\mu + \sum_{i=1}^{s} \Omega_i \chi_i^\mu$$

Ω_i ← constants

Take pullback to WV: $\left(V_a = \partial_a X^\mu V_\mu \right)$

$$k^a = \xi^a + \sum_{i=1}^{s} \Omega_i \chi_i^a$$

One has

$$\xi^a = R_0 T^a$$

$$\chi_i^a = R_i Z_i^a \quad , \quad i=1,\dots,s$$

(one can rotate the Z_i's to accomplish this)

Thus

$$k^a = R_0 T^a + \sum_{i=1}^{s} R_i \Omega_i Z_i^a$$

$$= R_0 \left(T^a + \sum_{i=1}^{s} \frac{R_i \Omega_i}{R_0} Z_i^a \right)$$

Norm of k^a:

$$|k|^2 = -k^2 = R_0^2 - \sum_{i=1}^{s} R_i^2 \Omega_i^2$$

From $u^a = \frac{k^a}{|k|}$ we now see

$$V_i = \frac{R_i \Omega_i}{R_0}, \quad i = 1, \dots, s \; ; \quad V_i = 0, \; i = s+1, \dots, p$$

$$V^2 = \sum_{i=1}^{s} V_i^2 = \sum_{i=1}^{s} \frac{R_i^2 \Omega_i^2}{R_0^2}$$

$$|k| = R_0 \sqrt{1 - V^2}$$

- Angular velocities $\leftrightarrow$ Local boosts on brane

Ω_i, $i = 1, \dots, s$, are all constants on the blackfolds WV.

$V_i = \frac{R_i \Omega_i}{R_0}$ signify the corresponding local boost velocity of the black p-brane

Local temperature of black p-brane

$$T_{local} = \frac{n}{4\pi r_0}$$

~~for ds~~

Temperature found for metric

$$ds^2 = -\left(1-\frac{r_0^n}{r^n}\right)dt^2 + \sum_i dx_i^2 + \frac{dr^2}{1-\frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2$$

In coord's

$$ds^2 = \left(\gamma_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1-\frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2$$

temperature is both red-shifted (R_0) and there is a boost factor ($\sqrt{1-V^2}$)

$$T = R_0\sqrt{1-V^2}\, T_{local} = R_0\sqrt{1-V^2}\,\frac{n}{4\pi r_0}$$

We see now that this temp. T is required to be constant over the BF WV in order to have equillibrium (stationarity) since

$$T = \frac{n/k}{4\pi r_0}$$ ← constant from demanding stationarity

Remark 1:

The embedding $X^\mu(\sigma)$ needs to be such that the orbit of k stays within the submanifold spanned by $X^\mu(\sigma)$

In our construction we have furthermore assumed ξ, χ^i, $i=1,\dots,s$, have orbits that stays within the submanifold. This implies

$$R_0 = \text{const.}, \quad R_i = \text{const.}, \quad i=1,\dots,s$$

One could relax this assumption by combining the spatial KVF's into one:

$$\tilde{\Omega} \tilde{\chi}^\mu = \sum_{i=1}^{s} \Omega_i \chi_i^\mu$$

Remark 2:

We saw that for stationary DF's

$D_a J^{ab} = 0 \Rightarrow T = \frac{n|k|}{4\pi r_0}$ const

This also goes the other way T const. $\Rightarrow D_a J^{ab} = 0$

Hence we only need to solve extr. eq.

$$J^{ab} K_{ab}{}^{\mu} = 0$$

Extrinsic eq for stationary blackfolds:

~~Before~~ The extrinsic eq for blackfolds:

$$T^{ab} K_{ab}{}^{\mu} = 0$$

We have $T^{ab} = (\varepsilon + P) u^a u^b + P \gamma^{ab}$, $P = -\frac{1}{n+1}\varepsilon$

Extr. eq.: $(\varepsilon + P) u^a u^b K_{ab}{}^{\mu} + P K^{\mu} = 0$

where we defined the mean curvature vector

$$K^{\mu} = \gamma^{ab} K_{ab}{}^{\mu}$$

Insert eq. of state:

$$\frac{n}{n+1} \varepsilon u^a u^b K_{ab}{}^{\mu} - \frac{1}{n+1} \varepsilon K^{\mu} = 0$$

$$\Leftrightarrow \quad K^{\mu} = n u^a u^b K_{ab}{}^{\mu}$$

For stationary BF's with $u^a = \frac{k^a}{|k|}$ one can show

$$u^a u^b K_{ab}{}^{\mu} = \perp^{\mu}{}_{\nu} \partial^{\nu} \log |k|$$

$\longrightarrow$

where $\perp^\mu{}_\nu$ is the projector orthogonal to the WV:

$$g_{\mu\nu} = h_{\mu\nu} + \perp_{\mu\nu}$$

projects parallel to WV

projects orthogonal to WV

Here $h^{\mu\nu} = \gamma^{ab}\partial_a X^\mu \partial_b X^\nu$

So the extr. eq. is

$$K^\rho = \perp^\rho{}_\nu \partial^\nu \log |k|^n$$

since $\frac{r_0}{|k|}$ is required to be constant we can also write this as

$$K^\rho = \perp^\rho{}_\nu \partial^\nu \log(-P)$$

One can show in general that ~~these~~ are the eqs that follow from varying the action

$$I = \int_{WV} d^{p+1}\sigma \sqrt{-\gamma}\, P$$

Thus we can find stationary blackfolds by extremizing this action!

~~We return to this action again later when we apply it!~~

Manual for finding stationary BF's

- Start with a KVF of the background metric $g_{\mu\nu}$, e.g.

$$K = \frac{\partial}{\partial t} + \sum_{i=1}^{s} \Omega_i \frac{\partial}{\partial \phi_i} \quad , \Omega_i = \text{const.}$$

- Find a general class of embeddings $X^\mu(\sigma)$ (for p-brane) for which the orbits of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi_i}$ stay within the submanifold spanned by $X^\mu(\sigma)$

~~[illegible]~~ $\Rightarrow$ Means that R_0, R_i are constant

(One can relax the above to a linear combo $\tilde{\Omega}\tilde{\chi} = \sum_{i=1}^{s} \Omega_i \chi_i$)

- Impose $T = \text{const.}$

- Solve extr. eq. $\gamma^{ab} K_{ab}{}^{\mu} = 0$

– Can be done by extremizing the action

$$I = \int_{W_{p+1}} d^{p+1}\sigma \sqrt{-\gamma}\, P$$

Thermodynamics of stationary blackfolds

KVF of backgr. $k = \xi + \sum_{i=1}^{s} \Omega_i \chi_i$

Define $n^a = \frac{1}{R_0} \xi^a$

- The unit normal vector on WV ~~a~~ to B_p: The spatial part of the ~~[illegible]~~ WV

With this we can define in general

$$M = \int_{B_p} dV_{(p)}\, T_{ab} n^a \xi^b$$

$$J_i = -\int_{B_p} dV_{(p)}\, T_{ab} n^a \chi_i^b$$

$$S = -\int_{B_p} dV_{(p)}\, s\, u_a n^a$$

Here $dV_{(p)}$ is defined such that

$$\int_{WV} d^{p+1}\sigma \sqrt{-\gamma}\,(\cdots) = \Delta t \int_{B_p} R_0\, dV_{(p)}\,(\cdots)$$

We have $T^{ab} = (\varepsilon + P) u^a u^b + P \gamma^{ab}$.

Contract with $n_a k_b$:

$$T^{ab} n_a k_b = (\varepsilon + P)(u\cdot n)(u\cdot k) + P\, n\cdot k$$

$$= -T_{local} \cdot s\,(u\cdot n)|k| - R_0 P$$

$$= -T s\,(u\cdot n) - R_0 P$$

$$\Rightarrow T_{ab} n^a \xi^b + \sum_{i=1}^{s} T_{ab} n^a \chi_i^b \Omega_i + T s\,(u\cdot n)$$

$$= -R_0 P$$

So, we get

$$I = \int_{W} d^{p+1}\sigma \sqrt{-\gamma}\, P = \Delta t \int_{B_p} R_0\, dV_{(p)}\, P$$

$$= -\Delta t \left(M - \sum_{i=1}^{s} \Omega_i J_i - TS \right)$$

Euclidean action:

$$I_E = \beta \left(M - \sum_{i=1}^{s} \Omega_i J_i - TS \right)$$

T, Ω_i constants

Extrinsic eq. (from action)

$\Leftrightarrow$ 1st law of thermodynamics

So for stationary blackfolds

$$J^{ab} K_{ab} = 0$$

$$\Leftrightarrow \quad dM = T\,dS + \sum_{i=1}^{s} \Omega_i\, dJ_i$$

Examples of stationary black holes

Example 1: "Rings"

Consider the black string (neutral).

We want to bend that along a closed loop embedded in D-dim. flat space-time. We assume the loop lies within a plane of the space-time.

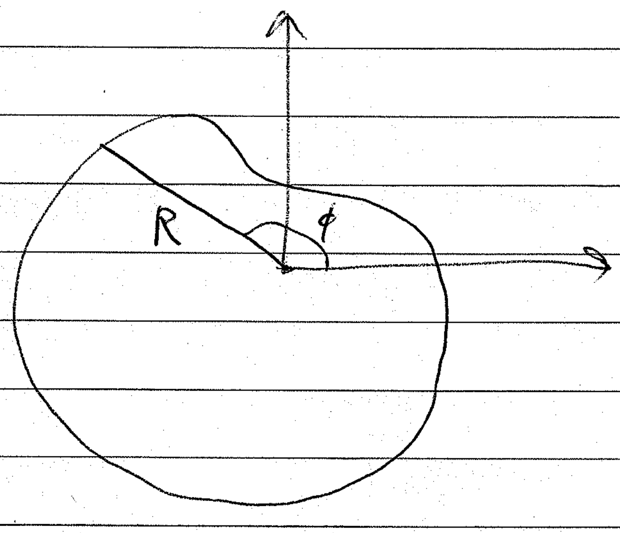
Metric for D-dim. flat space-time:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + \sum_{i=1}^{D-3} dx_i^2$$

Closed loop embedding:

$$t(\tau,\sigma) = \tau \; ; \quad \phi(\tau,\sigma) = \sigma \; ;$$

$$r(\tau,\sigma) = R(\tau,\sigma) \; ; \quad x_i(\tau,\sigma) = 0$$



Are there stationary blackfolds corresponding to a black string bended in a closed loop?

Most general KVF relevant for this:

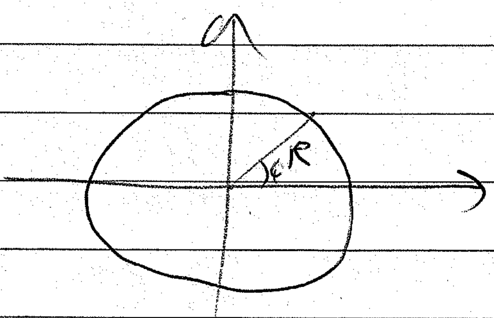
$$k = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$$

Possible with $\Omega = 0$? Means that $R = R(\sigma)$

$\rightarrow$ No, the tension of the string would give a net force, making it time dependent

For $\Omega \neq 0$: $R = \text{const.}$ since we want orbits of both $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ to be along BF WV.

$\Rightarrow$ Only candidates for embeddings are circles



Induced metric: $(\sigma^0 = \tau, \ \sigma^1 = \sigma)$

$$\gamma_{ab} \, d\sigma^a d\sigma^b = -d\tau^2 + R^2 d\sigma^2$$

$$\sqrt{-\gamma} = R$$

WV KVF:

$$k = \frac{\partial}{\partial \tau} + \Omega \frac{\partial}{\partial \sigma} \quad \text{on WV}$$

Normalized spatial KVF: $z = \frac{1}{R} \frac{\partial}{\partial \sigma}$

Hence velocity $V = R\Omega$

Norm of k: $|k| = \sqrt{1 - R^2 \Omega^2}$

(Note: $R_0 = 1 \Leftarrow$ no redshift from backgr.)

Velocity vector: $u = \frac{k}{|k|} = \frac{1}{\sqrt{1-R^2\Omega^2}} \left(\frac{\partial}{\partial \tau} + \Omega \frac{\partial}{\partial \sigma} \right)$

Temperature: $n = D - p - 3 = D - 4$

$$T = \frac{n|k|}{4\pi r_0} = \frac{n\sqrt{1-R^2\Omega^2}}{4\pi r_0} \Rightarrow r_0 = \frac{n\sqrt{1-R^2\Omega^2}}{4\pi T}$$

Pressure:

$$P = -\frac{\Omega_{n+1}}{16\pi G} r_0^n = -\frac{\Omega_{n+1}}{16\pi G} \left(\frac{n}{4\pi T} \right)^n \left(1 - R^2 \Omega^2 \right)^{\frac{n}{2}}$$

Now we should solve the extr. eq.

Action method:

$$I = \int d\sigma d\sigma \sqrt{-\gamma}\, P = -\frac{\Omega_{n+1}}{16\pi G}\left(\frac{n}{4\pi T}\right)^n \int d\sigma d\sigma\, R\left(1-R^2\Omega^2\right)^{\frac{n}{2}}$$

Extremize:

$$0 = \delta\left(R\left(1-R^2\Omega^2\right)^{n/2}\right) = \delta R\left(1-R^2\Omega^2\right)^{\frac{n}{2}}$$
$$+ R\frac{n}{2}\left(1-R^2\Omega^2\right)^{\frac{n}{2}-1}\left(-\Omega^2 2R\right)\delta R$$

$$\Leftrightarrow \quad 0 = 1 - R^2\Omega^2 - nR^2\Omega^2 = 1-(n+1)R^2\Omega^2$$

$$\Leftrightarrow \quad R = \frac{1}{\sqrt{n+1}\,\Omega}$$

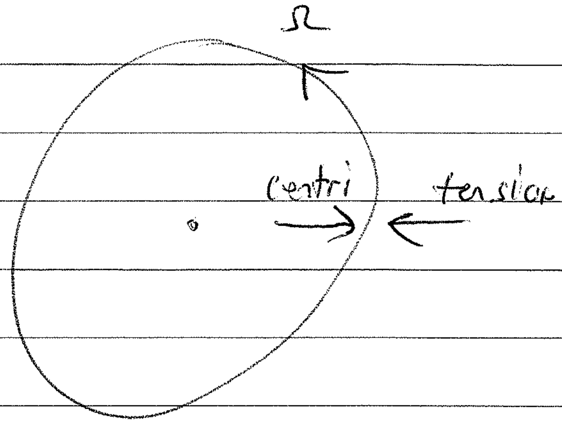
We found a stationary blackfold!

$\boxed{\text{A rotating black ring}}$

→ ~~Note~~ A solution for any dimension $D \geq 5$ (since we need at least codim. 3 corresponding to $n \geq 1$)

Rotation ensures balance of forces:

Force of black string tension = Centrifugal force



This exemplifies why $\exists$ extended black objects for $D \geq 5$ leads to non-uniqueness (given asymptotic charges) even for asymptotically flat space-times since the above means we find asymptotically flat stationary BH's with event horizon topology $S^1 \times S^{D-3}$ in addition to the Myers-Perry BH's with EH topology S^{D-2}.

The above valid in limit $r_0 \ll R$
$\rightarrow$ separation of scales on EH

$$\text{radius}(S^1) \gg \text{radius}(S^{D-3})$$

Solution to all orders?
Found in $D=5$ by Emparan and Reall
(years before BF method was invented)

Note:

One can also solve directly $J^{ab} K_{ab}{}^{\mu} = 0$ by noticing that the embedding immediately gives

$$K_{11}{}^{r} = \frac{1}{R}$$

as the only non-zero component, so eltr. eq. reduces to

$J^{11} = 0$ ← Zero tension condition ⇔ Balance of forces

$$J^{11} = 0 \Leftrightarrow \frac{n}{n+1} \varepsilon (u^1)^2 - \frac{1}{n+1} \varepsilon \gamma^{11} = 0$$

$$\Leftrightarrow n (u^1)^2 = \gamma^{11} \Leftrightarrow n \frac{\Omega^2}{1-R^2\Omega^2} = \frac{1}{R^2}$$

$$\Leftrightarrow n R^2 \Omega^2 = 1 - R^2 \Omega^2 \Leftrightarrow R = \frac{1}{\sqrt{n+1}\,\Omega}$$

Example 2 "Odd-spheres"

The rotating black ring is part of a much larger family of solutions that we now find

We want to bend the neutral black $(2k+1)$-brane (so $p = 2k+1$) on a S^{2k+1} sphere (geometrically round) in a D-dim. flat space-time

Metric for D-dim. flat space-time:

$$ds^2 = -dt^2 + dr^2 + r^2 \sum_{i=1}^{k+1} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right) + \sum_{i=1}^{D-2k-3} dx_i^2$$

where the director cosines μ_i are constrained by

$$\sum_{i=1}^{k+1} \mu_i^2 = 1$$

We are looking for a stationary blackfold, Relevant KVF of backgr. is

$$k = \frac{\partial}{\partial t} + \sum_{i=1}^{k+1} \Omega_i \frac{\partial}{\partial \phi_i}$$

Hence ~~the~~ ~~most~~ we can write a general embedding ~~as~~ as

$$t(\sigma) = \sigma^0 \; ; \quad \phi_i(\sigma) = \sigma^i \; , \; i = 1, \dots, k+1 \; ;$$

$$\mu_i(\sigma) = \sigma^{i+k+1} \; , \; i = 1, \dots, k \; ;$$

$$r(\sigma) = R(\sigma^{k+2}, \dots, \sigma^{2k+1}) = R(\mu_1, \dots, \mu_k)$$

as candidate for an embedding ~~as~~ of a stationary blackfold.

However, we now restrict ourselves for the case of a geometrically round S^{2k+1} sphere

$$R(\mu_1, \dots, \mu_k) = \text{const.}$$

i.e. the embedding $\quad r(\sigma) = R \quad (= \text{const.})$

Induced metric:

$$\gamma_{ab} d\sigma^a d\sigma^b = -dt^2 + R^2 \sum_{i=1}^{k+1} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right)$$

$$\sqrt{-\gamma}\, d^{p+1}\sigma = R^{2k+1} d\sigma^0 \, d\Omega_{2k+1}$$

$$= R^p d\sigma^0 d\Omega_p \; , \quad p = 2k+1$$

We furthermore set all angular velocities to be equal:

$$\Omega_i = \Omega , \quad i=1,\dots,k+1$$

So $k = \frac{\partial}{\partial t} + \Omega \sum_{i=1}^{k+1} \frac{\partial}{\partial \phi_i}$

$$|k| = \sqrt{-\gamma_{ab} k^a k^b} = \sqrt{1 - \Omega^2 \sum_{i=1}^{k+1} \mu_i^2 R^2}$$

$$= \sqrt{1 - R^2 \Omega^2}$$

$$u = \frac{k}{|k|} = \frac{1}{\sqrt{1-R^2\Omega^2}} \left(\frac{\partial}{\partial t} + \Omega \sum_{i=1}^{k+1} \frac{\partial}{\partial \phi_i} \right)$$

Temp:

$$T = \frac{n|k|}{4\pi r_0} = \frac{n\sqrt{1-R^2\Omega^2}}{4\pi r_0} \Rightarrow r_0 = \frac{n\sqrt{1-R^2\Omega^2}}{4\pi T}$$

Pressure: $P = -\frac{\Omega_{n+1}}{16\pi G} r_0^n = -\frac{\Omega_{n+1}}{16\pi G}\left(\frac{n}{4\pi T}\right)^n \left(1-R^2\Omega^2\right)^{\frac{n}{2}}$

Action:

$$I = \int d^{p+1}\sigma \sqrt{-\gamma}\, P = -\frac{\Omega_{n+1}}{16\pi G}\left(\frac{n}{4\pi T}\right)^n \Omega_p \int d\sigma^0 R^p \left(1-R^2\Omega^2\right)^{\frac{n}{2}}$$

Extremize:

$$0 = \delta\left(R^p \left(1-R^2\Omega^2\right)^{\frac{n}{2}}\right)$$

$$\Leftrightarrow \quad R = \sqrt{\frac{p}{n+p}} \cdot \frac{1}{\Omega}$$

New stationary BF's:

Rotating ~~B~~ geometrically round odd-spheres S^{2k+1}

Solutions when $n = D-2k-4 \geq 1$

Rotation balances force of tension of the black $(2k+1)$-brane

Event horizon topology:

$$S^{2k+1} \times S^{D-2k-3}$$

Example 3: "Products of odd-spheres"

Further generalization:

One can make products of odd-spheres

D-dim. space-time metric:

$$ds^2 = -dt^2 + \sum_{a=1}^{l} \left(dr_{(a)}^2 + r_{(a)}^2 \left(d\Omega_{p(a)}^{(a)}\right)^2 \right) + \sum_{i=1}^{D-p-l-1} dx_i^2$$

KVF: $k = \frac{\partial}{\partial t} + \sum_{a=1}^{l} \Omega_{(a)} \sum_i \frac{\partial}{\partial \phi_i^{(a)}}$

$p_{(a)}$ odd, $p = \sum_{a=1}^{l} p_{(a)}$

Embedding with $r_{(a)} = R_{(a)}$ (= const.)

Action: $I \propto \int d\sigma \prod_{b=1}^{l} R_{(b)}^{p_b} \left(1 - \sum_{a=1}^{l} R_{(a)}^2 \Omega_{(a)}^2\right)^{\frac{n}{2}}$

l eqs (one for each $R_{(a)}$):

$$R_{(a)} = \sqrt{\frac{p_{(a)}}{n+p}} \frac{1}{\Omega_{(a)}}, \quad a = 1, \ldots, l$$

EH topology

$$\left(\prod_{a=1}^{l} S^{p(a)}\right) \times S^{n+1}$$

$n = D - p - 3$

One can now make a table of the asymptotically flat stationary BH's

$\underline{D=4}$: S^2

$\underline{D=5}$: S^3, $S^1 \times S^2$

$\underline{D=6}$: S^4, $S^1 \times S^3$, $T^2 \times S^2$

$\underline{D=7}$: S^5, $S^1 \times S^4$, $T^2 \times S^3$,
$S^3 \times S^2$, $T^3 \times S^2$

$\underline{D=8}$: S^6, $S^1 \times S^5$, $T^2 \times S^4$,
$S^3 \times S^3$, $T^3 \times S^3$, $S^1 \times S^3 \times S^2$

$\underline{D=9}$: S^7, $S^1 \times S^6$, $T^2 \times S^5$, $S^3 \times S^4$, $T^3 \times S^4$,
$T^4 \times S^3$, $S^1 \times S^3 \times S^3$,
$S^5 \times S^2$, $T^2 \times S^3 \times S^2$

$\underline{D=10}$: S^8, $S^1 \times S^7$, $T^2 \times S^6$, $S^3 \times S^5$, $T^3 \times S^5$,
$T^4 \times S^4$, $S^1 \times S^3 \times S^4$, $S^5 \times S^3$, $T^2 \times S^3 \times S^3$,
$S^1 \times S^5 \times S^2$, $S^3 \times S^3 \times S^2$

$\underline{D=11}$: S^9, $S^1 \times S^8$, $T^2 \times S^7$, $S^3 \times S^6$, $T^3 \times S^6$,
$T^4 \times S^5$, $S^1 \times S^3 \times S^5$, $S^5 \times S^4$, $T^2 \times S^3 \times S^4$,
$T^5 \times S^4$, $S^1 \times S^5 \times S^3$, $S^3 \times S^3 \times S^3$, $T^3 \times S^3 \times S^3$,
$S^7 \times S^2$, $T^2 \times S^5 \times S^2$, $S^1 \times S^3 \times S^3 \times S^2$

etc.

(Black) Dp-branes on (products of) odd-spheres:

If we consider the backgr. to be $D=10$ flat space-time (Minkowski space) with all other SUGRA fields turned off (dilaton constant, no Ramond-Ramond fields, etc.) then the effective eqs for a black Dp-brane folded on an embedding $X^\mu(\sigma)$ of a $(p+1)$-dim. submanifold are

$$D_a T^{ab} = 0 , \quad T^{ab} K_{ab}{}^{\mu} = 0$$

with T^{ab} of perfect fluid form and with

$$\varepsilon = \frac{\Omega_{n+1}}{16\pi G} r_0^n \left(n+1+n\sinh^2\alpha \right)$$

$$P = -\frac{\Omega_{n+1}}{16\pi G} r_0^n \left(1+n\sinh^2\alpha \right)$$

These are the leading order eqs for a Dp-brane blackfold in the limit where

$$\left(r_0^n \sinh^2\alpha \right)^{\frac{1}{n}} , r_0 \ll \underset{\substack{\uparrow \\ \text{scale of} \\ \text{embedding}}}{R}$$

In the stationary case all that changes is that $\gamma^{ab} K_{ab}{}^{\mu} = 0$ has a modified γ^{ab}. Hence action is still

$$I = \int d^{p+1}\sigma \sqrt{-\gamma}\, P$$

but we should vary $X^{\mu}(\sigma)$ for fixed charge

$$N\, T_{Dp} = \frac{\Omega_{n+1}}{16\pi G} n r_0^n \sinh\alpha \cosh\alpha$$

Solving the general case of a product of odd-spheres

$$\prod_{a=1}^{l} S^{p(a)} \quad , \; p(a) \text{ odd}$$

we find

$$R_{(a)} = \frac{1}{\Omega_{(a)}} \sqrt{\frac{p_{(a)}(1+n\sinh^2\alpha)}{n+p(1+n\sinh^2\alpha)}}$$

$(n = 7-p)$

So we can for instance wrap a D3-brane on an S^3. This would (for non-zero temp.) give an event horizon with topology $S^3 \times S^5$

Perturbations of a blackfold:

We now go beyond the case of stationary blackfold and look at linear perturbations

$X^\mu(\sigma)$; $g_{\mu\nu}$ both vary at scales $\gtrsim R$

BF approx : $R \gg r_{\text{thickness}}$ (r_0 for black p-brane)

But consider now perturbations with wave-length λ such that

$$R \gg \lambda \gg r_{\text{thickness}}$$

Since $\lambda \ll R$: The p-brane looks flat at these length scales

$\Rightarrow$ We can set $g_{\mu\nu} = \eta_{\mu\nu}$ and

$$X^\mu(\sigma) = \begin{cases} \sigma^\mu, & \mu = 0, 1, \dots, p \\ 0, & \mu = p+1, \dots, D-1 \end{cases}$$

Since $\lambda \gg r_{\text{thickness}}$: The perturbations are described by the BF eqs

$$D_a T^{ab} = 0, \quad T^{ab} K_{ab}{}^\mu = 0$$

We consider first a general perfect fluid ~~[illegible]~~ stress tensor and then specialize to black p-branes and black Dp-branes, respectively.

~~[illegible]~~

Unperturbed: ε, P constant, $u^a = (1, 0, \ldots, 0)$

Perturbations:

$$\delta\varepsilon \, , \quad \frac{dP}{d\varepsilon}\delta\varepsilon \, , \quad \delta u^a = (0, v^i) \, ,$$

$$\delta X^m = \xi^m \, , \quad m = p+1, \ldots, D-1 \quad \text{(transverse directions)}$$

Perturbed stress tensor

$$T^{ab} = \left(\varepsilon + P + \left(1 + \frac{dP}{d\varepsilon}\right)\delta\varepsilon\right)(u^a + \delta u^a)(u^b + \delta u^b)$$
$$+ \left(P + \frac{dP}{d\varepsilon}\delta\varepsilon\right)\eta^{ab}$$

~~[illegible]~~

$$T^{00} = \varepsilon + \delta\varepsilon \, , \quad T^{ii} = P + \frac{dP}{d\varepsilon}\delta\varepsilon \, ,$$
$$T^{0i} = (\varepsilon + P) v^i$$

Perturbed extr. curv. tensor

$$K_{ab}{}^m = \partial_a \partial_b \xi^m$$

Notice decoupling of intrinsic/extrinsic perturbations!

Extr. eq.: $g^{ab} K_{ab}{}^{m} = 0$

$$\Rightarrow \left(\mathcal{E} \partial_0^2 + P \partial_i^2 \right) \varepsilon^m = 0$$

Solution:

$$\xi^m = A^m e^{-i\omega\sigma^0} e^{ik_i\sigma^i}$$

ω: frequency
k: wave number
$\frac{\omega}{k}$: velocity of propagation

provided: $\mathcal{E}\omega^2 + P k^2 = 0$

(with $k \equiv \sqrt{k_i k_i}$)

So we find $\frac{\omega^2}{k^2} = - \frac{P}{\mathcal{E}}$

$\Rightarrow$ Real number for $P < 0$, $\mathcal{E} > 0$
$\Rightarrow$ Means that the BF is stable under such perturbations

I.e.
Neutral black p-brane

$$\mathcal{E} = \frac{\Omega_{n+1}}{16\pi G}(n+1) r_0^n \,, \quad P = - \frac{1}{n+1}\mathcal{E}$$

Black Dp-brane

$$\mathcal{E} = \frac{\Omega_{n+1}}{16\pi G} r_0^n \left(n + 1 + n \sinh^2 \alpha \right)$$

$$P = - \frac{\Omega_{n+1}}{16\pi G} r_0^n \left(1 + n \sinh^2 \alpha \right)$$

Intrinsic eq: $D_a J^{ab} = 0$

$$\begin{cases} 0 = \partial_a J^{a0} = \partial_0 J^{00} + \partial_i J^{i0} \\ 0 = \partial_0 J^{0i} + \partial_j J^{ji} \end{cases}$$

$$\Rightarrow \quad \partial_0^2 J^{00} = -\partial_0 \partial_i J^{i0} = \partial_i \partial_j J^{ji}$$

We get: $\partial_0^2 \delta\varepsilon = \frac{dP}{d\varepsilon} \partial_i^2 \delta\varepsilon$

Solution:

$$\delta\varepsilon = A\, e^{-i\omega s} e^{i k_i \sigma^i}$$

provided: $\omega^2 - \frac{dP}{d\varepsilon} k^2 = 0$

So: $\frac{\omega^2}{k^2} = \frac{dP}{d\varepsilon}$

For neutral black p-branes: $\frac{dP}{d\varepsilon} = -\frac{1}{n+1}$

$\Rightarrow$ Means the BF is unstable to intrinsic perturbations!

What is this instability?

ω is imaginary, so write $\omega = i\Omega$

$$\delta\varepsilon = A\, e^{\Omega\sigma} e^{ik_i \sigma^i}$$

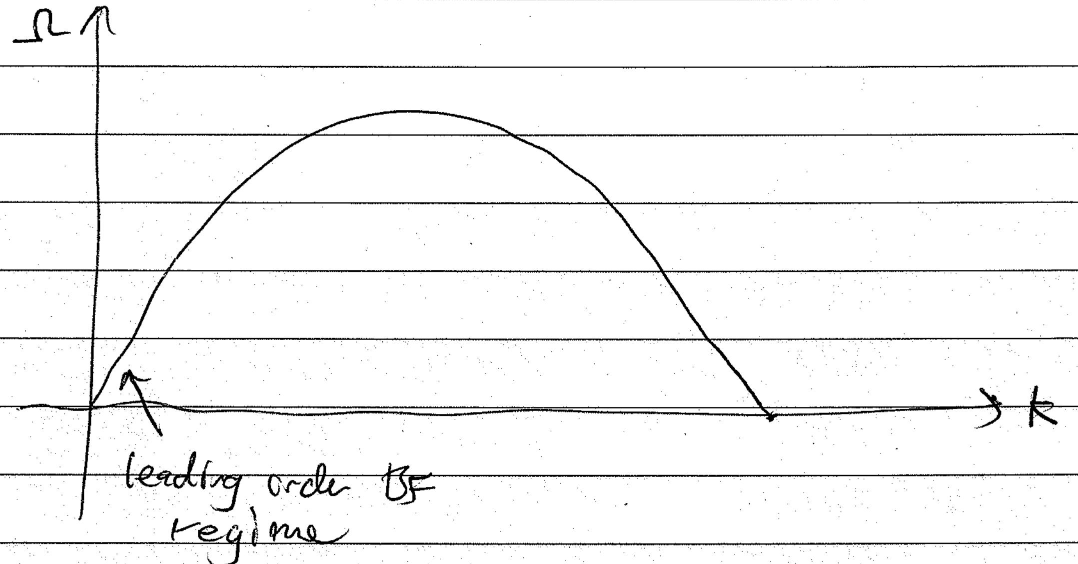
Mode grows if $\Omega > 0$

We see that $\Omega = \frac{1}{\sqrt{n+1}} k$

This is precisely the dispersion relation of the Gregory-Laflamme instability in the long-wave length limit, i.e. $k \to 0$ limit!

GL found by very complicated analysis this instability
(differential eq ~ 1 page)

Full dispersion relation:



~~Emparan~~, Camps, Emparan & Haddad found k^2 correction by including viscosity correction to stress tensor.

~~Black p-Branes~~

General result:

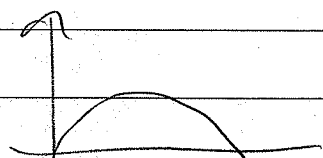
$$\Omega = \sqrt{-\frac{dP}{d\varepsilon}}\, k - \left(\left(1 - \frac{1}{p}\right)\frac{\eta}{s} + \frac{\zeta}{2s} \right)\frac{k^2}{T} + O(k^3)$$

p-brane
not pressure

$$\frac{\eta}{s} = \frac{1}{4\pi} \quad , \quad \frac{\zeta}{s} = \frac{1}{2\pi}\left(\frac{1}{p} - \frac{dP}{d\varepsilon}\right)$$

universal relation for shear viscosity η	conjecture (checked for black p-branes) for bulk viscosity ζ

So for $\frac{dP}{d\varepsilon} < 0$ we get a curve of shape



already by including k^2 term

What about black Dp-branes?

$$\mathcal{E} = \frac{\Omega_{n+1}}{16\pi G} r_0^n (n+1+n\sinh^2\alpha) , \quad P = -\frac{\Omega_{n+1}}{16\pi G} r_0^n (1+n\sinh^2\alpha)$$

We should compute $\frac{dP}{d\mathcal{E}}$ for fixed charge:

$$N T_{Dp} = \frac{\Omega_{n+1}}{16\pi G} n r_0^n \cosh\alpha \sinh\alpha$$

Write $q \equiv r_0^n \cosh\alpha \sinh\alpha$

Then $r_0^n \sinh^2\alpha = \sqrt{q^2 + \frac{1}{4}(r_0^n)^2} - \frac{1}{2} r_0^n$

$$\Rightarrow \frac{d(r_0^n \sinh^2\alpha)}{dr_0^n} = \frac{1}{2\sqrt{}} \cdot \frac{1}{2} r_0^n - \frac{1}{2}$$

$$= \frac{r_0^n}{4\sqrt{}} - \frac{1}{2}$$

$$\frac{d\mathcal{E}}{dr_0^n} = \frac{\Omega_{n+1}}{16\pi G}\left[n+1 + n\left(\frac{r_0^n}{4\sqrt{}} - \frac{1}{2}\right)\right]$$

$$= \frac{\Omega_{n+1}}{16\pi G}\frac{1}{4\sqrt{}}\left[(2n+4)\sqrt{} + n r_0^n \right]$$

$$= \left[(2n+4) r_0^n \sinh^2\alpha + (n+2) r_0^n + n r_0^n \right]$$

$$= \frac{1}{2\sqrt{}}\left[(n+2) r_0^n \sinh^2\alpha + (n+1) r_0^n \right] > 0$$

$$\frac{dP}{d\mathcal{E}} > 0 \iff \frac{dP}{d(r_0^n)} \cdot \left(\frac{d\mathcal{E}}{dr_0^n}\right)^{-1} > 0$$

$$\iff \frac{dP}{dr_0^n} > 0 \iff -1 - n\left(\frac{r_0^n}{4\sqrt{\quad}} - \frac{1}{2}\right) > 0$$

$$\iff \frac{n}{2} - 1 - n\frac{r_0^n}{4\sqrt{\quad}} > 0$$

$$\iff (n-2)\sqrt{\quad} - \frac{n}{2} r_0^n > 0$$

$$\iff (n-2)\left(\sqrt{\quad} - \frac{1}{2} r_0^n\right) - r_0^n > 0$$

$$\iff (n-2)\sinh^2\alpha > 1 \iff \sinh^2\alpha > \frac{1}{n-2}$$

So for sufficiently large charge the GL instability dissappears!

Old result → but here derived without complicated diff. eqs. for perturbations of the geometry of the Dp-brane