Susy representations (continue ....)

State	Spin
$\|s\rangle$	s
$\bar{Q}_{\dot{\alpha}}\|s\rangle$	
$\bar{Q}^2\|s\rangle$	

$s=0$	$\|0\rangle$	$s=0$
	$\bar{Q}_{\dot{\alpha}}\|0\rangle$	$s=1/2$
	$\bar{Q}^2\|0\rangle$	$s=0$

Let's consider (φ, ψ, F)

$$\delta\varphi = -\varepsilon^{\alpha}\psi_{\alpha} \qquad \delta\bar{\varphi} = -\bar{\varepsilon}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$$

$$\delta\psi_{\alpha} = -2i\partial_{\alpha\dot{\alpha}}\varphi\,\bar{\varepsilon}^{\dot{\alpha}} - \varepsilon_{\alpha}F \qquad \delta\bar{\psi}_{\dot{\alpha}} = -2i\partial_{\alpha\dot{\alpha}}\bar{\varphi}\,\varepsilon^{\alpha} - \bar{\varepsilon}_{\dot{\alpha}}\bar{F}$$

$$\delta F = 2i\partial_{\alpha\dot{\alpha}}\psi^{\alpha}\bar{\varepsilon}^{\dot{\alpha}} \qquad \delta\bar{F} = 2i\varepsilon^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}$$

$$\delta\varphi = [i(\varepsilon^{\alpha}Q_{\alpha} + \bar{\varepsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}), \varphi]$$

$$\Rightarrow \boxed{[\bar{Q}_{\dot{\alpha}}, \varphi] = 0} \qquad [Q_{\alpha}, \varphi] = i\psi_{\alpha}$$

$$\{Q_{\alpha}, \psi_{\beta}\} = -i\varepsilon_{\alpha\beta}F \qquad \{\bar{Q}_{\dot{\alpha}}, \psi_{\beta}\} = 2\partial_{\beta\dot{\alpha}}\varphi$$

$$[Q_{\alpha}, F] = 0 \qquad [\bar{Q}_{\dot{\alpha}}, F] = -2\partial^{\alpha}{}_{\dot{\alpha}}\psi_{\alpha}$$

Construct a $s=0$ state $\underbrace{\varphi(x)|0\rangle}_{|\tilde{0}\rangle}$ (with $Q_{\alpha}|0\rangle = \bar{Q}_{\dot{\alpha}}|0\rangle = 0$)

$$\bar{Q}_{\dot{\alpha}}(\varphi(x)|0\rangle) = [\bar{Q}_{\dot{\alpha}}, \varphi(x)]|0\rangle = 0$$

<u>State</u> <u>Spin</u>

State	Spin
$|\tilde{0}\rangle = \varphi(x)|0\rangle$	$s=0$

$$Q_\alpha|\tilde{0}\rangle = Q_\alpha\left(\varphi(x)|0\rangle\right)$$
$$= [Q_\alpha, \varphi(x)]|0\rangle$$
$$= i\psi_\alpha(x)|0\rangle \qquad s = 1/2$$

$$Q_\beta Q_\alpha|\tilde{0}\rangle = \{Q_\beta, i\psi_\alpha\}|0\rangle$$
$$= \varepsilon_{\beta\alpha} F(x)|0\rangle \qquad s=0$$

$(\varphi, \psi_\alpha, F)$ supersymmetry multiplet

NB: This is a constrained multiplet, given the condition

$$[\bar{Q}_{\dot\alpha}, \varphi] = 0$$

chiral multiplet

Starting with $(\bar\varphi, \bar\psi_{\dot\alpha}, \bar F)$ we construct a second constrained multiplet $[Q_\alpha, \bar\varphi] = 0$

antichiral multiplet

What happens if we relax any initial constraint?

$\downarrow$

Real scalar multiplet

$$V = (C, \chi, M, N, A_\mu, \lambda, D) \qquad \chi = \begin{pmatrix} \chi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}$$

under the condition $V^\dagger = V$

$$\varepsilon = \begin{pmatrix} \varepsilon_\alpha \\ \bar\varepsilon^{\dot\alpha} \end{pmatrix}$$

$$\begin{cases} \delta C = \bar\varepsilon\gamma^5\chi \\ \delta\chi = (M + \gamma^5 N)\varepsilon - i\gamma^\mu(A_\mu + \gamma^5\partial_\mu C)\varepsilon \\ \delta M = \bar\varepsilon(\lambda - i\gamma^\mu\partial_\mu\chi) \\ \delta N = \bar\varepsilon\gamma^5(\lambda - i\gamma^\mu\partial_\mu\chi) \end{cases} \Big\} ?$$

$$\begin{cases} \delta N = \bar{\varepsilon}\gamma^5(\lambda - i\gamma^\mu\partial_\mu\chi) \\ \delta A_\mu = i\bar{\varepsilon}\gamma^\mu\lambda + \bar{\varepsilon}\partial_\mu\chi \\ \delta\lambda = -i\sigma^{\mu\nu}\varepsilon\partial_\mu A_\nu - \gamma^5\varepsilon D \\ \delta D = -i\bar{\varepsilon}\gamma^\mu\partial_\mu\gamma^5\lambda \end{cases}$$

This is a reducible representation

Ex: $\quad dV = (\lambda, F_{\mu\nu}, D) \qquad F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$

$$\begin{cases} \delta F_{\mu\nu} = -i\bar{\varepsilon}(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu)\lambda \\ \delta\lambda = -\frac{i}{2}\sigma^{\mu\nu}\varepsilon F_{\mu\nu} - \gamma^5\varepsilon D \\ \delta D = -i\bar{\varepsilon}\gamma^\mu\gamma^5\partial_\mu\lambda \end{cases}$$

Ex2: $\quad \partial V = (M, N, \lambda - i\gamma^\mu\partial_\mu\chi, \partial^\mu A_\mu, D + \Box C)$

$\boxed{N=1 \text{ SUPERSPACE}}$

Let's first review the construction of Minkowski spacetime as the coset

$$\frac{\text{Poincaré}}{\text{Lorentz}} = \{\text{set of equivalence classes of Poincaré elements with equivalence rule defined as}$$

$g, g' \in$ Poincaré $\quad g' \simeq g$ iff $g' = g \circ h$, $\quad h \in$ Lorentz $\}$

Given a particular equivalence class, we choose a representative $L(x)$ s.t.

$$\forall g \in \text{Coset}[x] \; : \; g = L(x) \circ h$$

We choose $\quad L(x) = e^{i x^\mu P_\mu}$

x^μ describes Minkowski spacetime

given $g \in$ Coset$[x] \to g(x)$ we want to move to another element in Coset$[x']$

$$g(x') = L(\xi) \circ g(x) = L(\xi) \circ L(x) \circ h$$
$$= e^{i\xi^\mu P_\mu} \circ e^{i x^\nu P_\nu} \circ h$$
$$\underset{[P_\mu, P_\nu]=0}{=} e^{i(x^\mu + \xi^\mu) P_\mu} \circ h = g(\underbrace{x+\xi})$$

SPACETIME TRANSLATION

What about fields (smooth functions of x^μ : $\phi(x)$)?

We always require ϕ to be a scalar under translations (while it can be anything respect to Lorentz)

$$\phi'(x+\xi) = \phi(x) \iff \phi'(x) = \phi(x-\xi)$$

For ξ infinitesimal parameter

$$\delta_0\phi \equiv \phi'(x) - \phi(x) = \phi(x-\xi) - \phi(x) = \underbrace{-\xi^\mu \partial_\mu \phi} \qquad (1)$$

In general, we define translations on ϕ fields

$$\phi(x+y) = L(y)\,\phi(x)\,L^{-1}(y)$$
$$= e^{iy^\mu P_\mu}\,\phi(x)\,e^{-iy^\mu P_\mu}$$

$$\delta_0\phi = \phi(x-\xi) - \phi(x) = e^{-i\xi^\mu P_\mu}\,\phi(x)\,e^{i\xi^\nu P_\nu} - \phi(x)$$

$$= \underbrace{-i\xi^\mu [P_\mu, \phi]} \qquad (2)$$

Comparing (1) and (2) $\Rightarrow$ $[P_\mu, \phi] = -i\partial_\mu\phi$

Generator P_μ realised on local fields as a differential operator

It immediately follows $\quad [P_\mu, P_\nu] = 0$

Let's redo everything starting with

$$\text{Superpoincaré group} \longrightarrow (P_\mu, M_{\mu\nu}, Q_\alpha, \bar{Q}_{\dot\alpha})$$

We consider a supercoset group

$$\frac{\text{Superpoincaré}}{\text{Lorentz}}$$

We choose a coset representative

$$L(x,\theta,\bar\theta) = e^{i(x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha} \bar{Q}^{\dot\alpha})}$$

$\theta, \bar\theta$ = anticommuting, constant, spinorial parameters

Dimensions (mass dim.):

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\, P_{\alpha\dot\alpha}$$

$$[P_{\alpha\dot\alpha}] = 1 \quad \Longrightarrow \quad [Q] = [\bar{Q}] = \frac{1}{2}$$

$$\Longrightarrow \quad [\theta] = [\bar\theta] = -1/2$$

$$\text{Coset}\,[x,\theta,\bar\theta] = \{\text{set of } g(x,\theta,\bar\theta) = \underset{h \in \text{Lorentz}}{L(x,\theta,\bar\theta)\cdot h}\}$$

$(x^{\alpha\dot\alpha}, \theta^\alpha, \bar\theta^{\dot\alpha})$ = coordinates of <u>N=1</u> <u>SUPERSPACE</u>

left - multiplication

$$g(x',\theta',\bar\theta') = L(\xi,\varepsilon,\bar\varepsilon)\circ g(x,\theta,\bar\theta)$$

$$= \underbrace{L(\xi,\varepsilon,\bar\varepsilon)\circ L(x,\theta,\bar\theta)}\cdot h$$

$$i(\xi^\mu P_\mu + \varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar{Q}^{\dot\alpha})$$

$$L(\xi,\varepsilon,\bar\varepsilon)\circ L(x,\theta,\bar\theta) = e^{i(\xi^\mu P_\mu + \varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha})} \times e^{i(x^\nu P_\nu + \theta^\beta Q_\beta + \bar\theta_{\dot\beta}\bar Q^{\dot\beta})}$$

$e^A \cdot e^B$ with $[A,[A,B]]=0$

$[B,[A,B]]=0$

$$e^A\cdot e^B = e^{A+B+\frac{1}{2}[A,B]}$$

$$L(\xi,\varepsilon,\bar\varepsilon)\circ L(x,\theta,\bar\theta) =$$

$$\exp\Big\{i\Big[(x+\xi)^{\alpha\dot\alpha}\underline{P_{\alpha\dot\alpha}} + (\theta+\varepsilon)^\alpha Q_\alpha + (\bar\theta+\bar\varepsilon)_{\dot\alpha}\bar Q^{\dot\alpha}$$

$$+\frac{1}{2}\underbrace{[i\varepsilon^\alpha Q_\alpha, i\bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}]}_{\substack{\varepsilon^\alpha\bar\theta_{\dot\alpha}\{Q_\alpha,\bar Q^{\dot\alpha}\} \\ -2\varepsilon^\alpha\bar\theta^{\dot\alpha}P_{\alpha\dot\alpha}}} + \frac{1}{2}[i\bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha}, i\theta^\alpha Q_\alpha]\Big]\Big\}$$

$$= \exp\Big\{i\Big[(x^{\alpha\dot\alpha}+\xi^{\alpha\dot\alpha} + i\varepsilon^\alpha\bar\theta^{\dot\alpha} + i\bar\varepsilon^{\dot\alpha}\theta^\alpha)P_{\alpha\dot\alpha}$$

$$+ (\theta+\varepsilon)^\alpha Q_\alpha + (\bar\theta+\bar\varepsilon)_{\dot\alpha}\bar Q^{\dot\alpha}\Big]\Big\}$$

$$= L\underbrace{(x+\xi+i\varepsilon\bar\theta+i\bar\varepsilon\theta,\ \theta+\varepsilon,\ \bar\theta+\bar\varepsilon)}$$

$\downarrow$

Supertranslation

$$\begin{cases} x' = x+\xi+i\varepsilon\bar\theta+i\bar\varepsilon\theta \\ \theta' = \theta+\varepsilon \\ \bar\theta' = \bar\theta+\bar\varepsilon \end{cases}$$

We can realize transformations of superPoincaré group on new "objects" $\phi(x,\theta,\bar\theta)$ that are smooth functions of superspace coordinates

$$\phi(x,\theta,\bar\theta) \longrightarrow \underline{\text{SUPERFIELDS}}$$

Under supertranslations ϕ transforms as

$$\phi(x + \underbrace{\xi + i\varepsilon\bar\theta + i\bar\varepsilon\theta}_{\delta x},\ \theta+\varepsilon,\ \bar\theta+\bar\varepsilon) \equiv$$

$$L(\xi,\varepsilon,\bar\varepsilon)\,\phi(x,\theta,\bar\theta)\,L^{-1}(\xi,\varepsilon,\bar\varepsilon) \qquad (3)$$

Superlocality cond: ϕ has to behave as a scalar under supertranslations

$$\phi'(x',\theta',\bar\theta') = \phi(x,\theta,\bar\theta)$$

$$\Leftrightarrow \quad \phi'(x,\theta,\bar\theta) = \phi(x-\delta x,\ \theta-\varepsilon,\ \bar\theta-\bar\varepsilon)$$

$$\delta_0\phi \equiv \phi'(x,\theta,\bar\theta) - \phi(x,\theta,\bar\theta)$$

$$= \phi(x-\delta x,\ \theta-\varepsilon,\ \bar\theta-\bar\varepsilon) - \phi(x,\theta,\bar\theta)$$

We introduce $\underline{\text{spinorial}}$ $\underline{\text{derivatives}}$: $\partial_\alpha \equiv \dfrac{\partial}{\partial\theta^\alpha}$

$$\bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$$

s.t. $\begin{cases} \partial_\alpha\theta^\beta = -\delta_\alpha{}^\beta \\ \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta} \end{cases}$

$$\Rightarrow \delta_0\phi = -\delta x^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\phi - \varepsilon^\alpha\partial_\alpha\phi - \bar\varepsilon_{\dot\alpha}\bar\partial^{\dot\alpha}\phi$$

$$= -(\xi^{\alpha\dot\alpha} + i\varepsilon^\alpha\bar\theta^{\dot\alpha} + i\bar\varepsilon^{\dot\alpha}\theta^\alpha)\partial_{\alpha\dot\alpha}\phi - \varepsilon^\alpha\partial_\alpha\phi - \bar\varepsilon_{\dot\alpha}\bar\partial^{\dot\alpha}\phi$$

$$= \underline{-\xi^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\phi - \varepsilon^\alpha(\partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha})\phi - \bar\varepsilon_{\dot\alpha}(\bar\partial^{\dot\alpha} + i\theta^\alpha\partial_\alpha{}^{\dot\alpha})\phi} \qquad (4)$$

(4)

From definition (3) we also have

$$\delta_\alpha \phi = \phi(x-\xi, \theta-\varepsilon, \bar\theta-\bar\varepsilon) - \phi(x,\theta,\bar\theta)$$

$$= L(-\xi, -\varepsilon, -\bar\varepsilon)\, \phi(x,\theta,\bar\theta)\, L(\xi, \varepsilon, \bar\varepsilon)$$

$$= \underbrace{-i\xi^{\alpha\dot\alpha}[P_{\alpha\dot\alpha}, \phi] - i\varepsilon^\alpha [Q_\alpha, \phi] - i\bar\varepsilon_{\dot\alpha}[\bar Q^{\dot\alpha}, \phi]} \qquad (5)$$

Comparison between (4) and (5) leads to

$$\begin{cases} [P_{\alpha\dot\alpha}, \phi] = -i\partial_{\alpha\dot\alpha}\phi \\ [iQ_\alpha, \phi] = (\partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha})\phi \\ [i\bar Q^{\dot\alpha}, \phi] = (\bar\partial^{\dot\alpha} - i\theta^\alpha \partial_\alpha{}^{\dot\alpha})\phi \end{cases}$$

$$\Downarrow$$

$$P_{\alpha\dot\alpha} = -i\partial_{\alpha\dot\alpha} \qquad Q_\alpha = -i(\partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha})$$

$$\bar Q_{\dot\alpha} = -i(\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha})$$

You can check that $\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$

SUPERFIELDS

$$\begin{aligned} \phi(x,\theta,\bar\theta) = {} & \varphi(x) + \theta^\alpha \psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x) \\ & + \theta^2 M(x) + \bar\theta^2 N(x) + \theta^\alpha\bar\theta^{\dot\alpha} A_{\alpha\dot\alpha}(x) \\ & + \bar\theta^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) \\ & + \theta^2\bar\theta^2 D(x) \end{aligned}$$

this is nothing but the field content of the real scalar multiplet.

$$\delta\phi = [i(\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha}), \phi]$$

$$= \text{-----}$$ (using the explicit realization of $Q, \bar Q$ as differential operators in superspace)

Exercise: find susy transfs for components

SUSY COVARIANT DERIVATIVES

As ∂_μ is a covariant derivative for ordinary translations ($[\partial_\mu, P_\nu]=0$)

we would like to check whether $\partial_\alpha, \bar\partial_{\dot\alpha}$ are covariant derivatives respect to supertranslations

$$\{\partial_\alpha, \bar Q_{\dot\alpha}\} = \{\partial_\alpha, -i\bar\partial_{\dot\alpha} - \theta^\beta\partial_{\beta\dot\alpha}\} = -\underbrace{(\partial_\alpha\theta^\beta)}_{-\delta_\alpha{}^\beta}\partial_{\beta\dot\alpha}$$

$$= \partial_{\alpha\dot\alpha} \neq 0 \;!$$

no good!

We need to construct new derivatives $D_\alpha, \bar D_{\dot\alpha}$ s.t.

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar Q_{\dot\beta}\} = 0$$

$$\{\bar D_{\dot\alpha}, Q_\beta\} = \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0$$

$$\begin{cases} D_\alpha = \partial_\alpha - i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} \\ \bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + i\theta^\alpha\partial_{\alpha\dot\alpha} \end{cases}$$

$$\{D_\alpha, \bar D_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$$

$$\{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0$$

these are good covariant derivatives (check it!)

$$D_\alpha = iQ_\alpha - \underbrace{2i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}}$$

$$\bar D_{\dot\alpha} = i\bar Q_{\dot\alpha} + \underbrace{2i\theta^\alpha\partial_{\alpha\dot\alpha}}$$

they differ by a total spacetime derivative