## N=1 SUPERSPACE (continue ...)

Covariant derivatives $D_\alpha = \partial_\alpha - i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}$

$$\bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + i\theta^\alpha \partial_{\alpha\dot\alpha}$$

$$\{D_\alpha, \bar D_{\dot\alpha}\} = 2P_{\alpha\dot\alpha} = 2(-i\underline{\underline{\partial_{\alpha\dot\alpha}}})$$

In ordinary curved background $\rightarrow$ covariant derivatives $D_a$

$$D_a = e_a{}^\mu D_\mu$$

$$[D_a, D_b] = \underbrace{T_{ab}{}^c}_{\text{Torsion}} D_c + \underbrace{R_{ab}}_{\text{curvature}}$$

In superspace we start with "curved" indices

$$Z^M = (x^m, \theta^\mu, \bar\theta^{\dot\mu}) \qquad m = \mu\dot\mu \qquad \partial_M = \frac{\partial}{\partial Z^M}$$

$\downarrow$ flat indices

$$D_A = E_A{}^M(\partial_M + \Gamma_M)$$

$(\partial_{\alpha\dot\alpha}, \quad \bar D_{\dot\alpha})$

$$E_A{}^M = \begin{pmatrix} \delta_\alpha{}^\mu & 0 & -i\delta_\alpha{}^\mu\bar\theta^{\dot\mu} \\ 0 & \delta_{\dot\alpha}{}^{\dot\mu} & i\delta_{\dot\alpha}{}^{\dot\mu}\theta^\mu \\ 0 & 0 & \delta_{\alpha\dot\alpha}{}^m \end{pmatrix} \qquad \text{SUPERVIELBEIN}$$

From $\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha} \implies \underbrace{T_{\alpha\dot\alpha}{}^{(\beta\dot\beta)} = -2i\delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}}_{\text{nontrivial Torsion}}$

Covariant derivatives can be used to select superfield components:

$$\phi(x,\theta,\bar\theta) = \varphi(x) + \theta^\alpha \psi_\alpha(x) + \bar\theta_{\dot\alpha} \bar\Psi^{\dot\alpha}(x)$$
$$+ \theta^2 M(x) + \bar\theta^2 \bar M(x) + \theta^\alpha \bar\theta^{\dot\alpha} A_{\alpha\dot\alpha}(x)$$
$$+ \bar\theta^2 \theta^\alpha \lambda_\alpha(x) + \theta^2 \bar\theta_{\dot\alpha} \bar\lambda^{\dot\alpha}(x) + \theta^2 \bar\theta^2 D(x)$$

$$D_\alpha = \partial_\alpha - i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} \qquad \partial_\alpha \theta^\beta = -\delta_\alpha{}^\beta$$
$$\bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + i\theta^\alpha \partial_{\alpha\dot\alpha} \qquad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$$

$$\varphi(x) = \phi(x,\theta,\bar\theta)\Big|_{\theta=\bar\theta=0}$$

$$\psi_\alpha(x) = -D_\alpha\phi\Big|$$

$$\bar\Psi_{\dot\alpha}(x) = -\bar D_{\dot\alpha}\phi\Big|$$

$$M(x) \sim D^2\phi\Big| \qquad \bar M(x) \sim \bar D^2\phi\Big|$$

$$A_{\alpha\dot\alpha}(x) \sim D_\alpha \bar D_{\dot\alpha}\phi\Big|$$

$$\lambda_\alpha(x) \sim \bar D^2 D_\alpha\phi\Big| \qquad \bar\lambda_{\dot\alpha}(x) \sim D^2 \bar D_{\dot\alpha}\phi\Big|$$

$$D(x) \sim D^2\bar D^2\phi\Big|$$

## BEREZIN INTEGRATION

$$\int d\theta^\alpha d\theta_\alpha \sim \int d^2\theta$$

As a toy example, consider 1D superspace $(x,\theta)$

$$f(x,\theta) = a(x) + \theta b(x)$$

We want to define $\int d\theta\, f(x,\theta)$ s.t.

- the integral is a linear operator
- SUSY invariant ($\Leftarrow$ the integral to be invariant under supertranslations)

We expect $\int d\theta\,(a+\theta b) = A + \theta B$

Susy invariance $\Rightarrow$ $\underline{\int d\theta\,(a+(\theta+\varepsilon)b)} \equiv \int d\theta\,(a+\theta b)$

$$\int d\theta\,\big((a+\varepsilon b)+\theta b\big)$$

We require $\int d\theta\,(a+\theta b) \sim b$

$\Downarrow$

We define
- $\int d\theta = 0 \quad = \frac{\partial}{\partial\theta}(1)$
- $\int d\theta\,\theta = 1 \quad = \frac{\partial}{\partial\theta}\theta$

$\Rightarrow \int d\theta = \partial_\theta$

Delta-function defined by the usual cond.

$$\int d\theta\,(a+\theta b)\,\delta(\theta-\theta') = a+\theta' b$$

$$\Rightarrow \boxed{\delta(\theta-\theta') = (\theta-\theta')}$$

## Generalising to 4D

$$\int d^2\theta \equiv \frac{1}{2}\int d\theta^\alpha d\theta_\alpha \qquad \int d^2\theta\, 1 = 0$$

$$\int d^2\theta\,\theta_\alpha = 0$$

$$\int d^2\theta\,\underbrace{\theta^2}_{\frac{1}{2}\theta^\beta\theta_\beta} = -1 \equiv D^2\theta^2\Big|_{\theta=\bar\theta=0}$$

$$D^2\theta^2 = \frac{1}{4}D^\alpha D_\alpha(\theta^\beta\theta_\beta) = \frac{1}{4}\cdot 2\,D^\alpha\Big(\underbrace{(D_\alpha\theta^\beta)}_{-\delta_\alpha{}^\beta}\theta_\beta\Big)$$

$$= \frac{1}{2}(-\delta_\alpha{}^\beta)\underbrace{D^\alpha\theta_\beta}_{\delta^\alpha{}_\beta} = -\frac{1}{2}\cdot 2 = -1$$

Same definitions for $\bar\theta$ $\left(\bar\theta^2 = \frac{1}{2}\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\right)$

$$\int d^2\bar\theta = \frac{1}{2}\int d\bar\theta_{\dot\alpha}\, d\bar\theta^{\dot\alpha}$$

$\Rightarrow \int d^2\bar\theta\,\bar\theta^2 = -1 = \bar D^2\bar\theta^2\Big|$

$$\int d^4\theta \equiv \int d^2\theta\, d^2\bar\theta \quad \Rightarrow \quad \int d^4\theta\,\theta^2\bar\theta^2 = 1 \qquad [\theta] = -\tfrac{1}{2} \Rightarrow [d\theta] = \tfrac{1}{2}$$

$$\int d^4\theta\,\phi(x,\theta,\bar\theta) = D(x) = D^2\bar D^2\phi(x,\theta,\bar\theta)\Big|_{\dots}$$

$\theta = \bar\theta = 0$

Delta-function : $\int d^4\theta \; \phi(x,\theta,\bar\theta) \underbrace{\delta^{(4)}(\theta-\theta')}_{\delta^{(2)}(\theta-\theta')\,\delta^{(2)}(\bar\theta-\bar\theta')} = \phi(x,\theta',\bar\theta')$

this is realised by

$$\boxed{\delta^{(4)}(\theta-\theta') = (\theta-\theta')^2 (\bar\theta-\bar\theta')^2}$$

The complete superspace integral

$$\int \underbrace{d^4x \, d^4\theta}_{-4+2=-2} \; \phi(x,\theta,\bar\theta) \equiv \int d^8z \; \phi(z)$$

$$z = (x,\theta,\bar\theta)$$

## CONSTRAINED SUPERFIELDS

this is done in order to select irr. rep's of SUSY

(1) $\bar D_{\dot\alpha} \phi = 0$ chiral superfield

(2) $D_\alpha \bar\phi = 0$ antichiral superfield

We look for the most general solution to (I)

$$\phi(x_L, \theta, \bar\theta) = \varphi(x_L) + \theta^\alpha \psi_\alpha(x_L) + \theta^2 F(x_L)$$

$$x_L^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - i\theta^\alpha \bar\theta^{\dot\alpha}$$

check: $\bar D_{\dot\alpha} \phi = 0$

$$(\bar\partial_{\dot\alpha} - i\theta^\alpha \partial_{\alpha\dot\alpha})\, \phi(x - i\theta\bar\theta, \theta)$$

$$= \frac{\partial\phi}{\partial x_L^{\alpha\dot\beta}} \frac{\partial x_L^{\alpha\dot\beta}}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha \partial_{\alpha\dot\alpha}\phi$$

$$= \frac{\partial\phi}{\partial x_L^{\alpha\dot\beta}} \cdot (i\theta^\alpha \delta_{\dot\alpha}^{\dot\beta}) - i\theta^\alpha \frac{\partial\phi}{\partial x_L^{\beta\dot\beta}} \frac{\partial x_L^{\beta\dot\beta}}{\partial x^{\alpha\dot\alpha}}$$

$$= i\theta^\alpha \partial^L_{\alpha\dot\alpha}\phi - i\theta^\alpha \partial^L_{\alpha\dot\alpha}\phi = 0$$

It may be convenient to perform change of variables

$$(x, \theta, \bar\theta) \rightarrow (\underline{\underline{x_L}}, \theta, \bar\theta) \qquad \text{(left chiral superspace)}$$

$$\Rightarrow \phi(x,\theta) = \varphi(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x)$$

subject to $\bar\partial_{\dot\alpha} \phi = 0$

Solving constraint (2) we obtain:

$$\bar\Phi(x,\theta,\bar\theta) = \bar\varphi(x_R) + \bar\theta_{\dot\alpha} \bar\Psi^{\dot\alpha}(x_R) + \bar\theta^2 \bar F(x_R)$$

$$x_R^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + i\theta^\alpha \bar\theta^{\dot\alpha}$$

We can go to $(x_R, \theta, \bar\theta)$ variables (right chiral superfield)

$$\bar\Phi(x,\bar\theta) = \bar\varphi(x) + \bar\theta_{\dot\alpha} \bar\Psi^{\dot\alpha}(x) + \bar\theta^2 \bar F(x)$$

$$\partial_\alpha \bar\Phi = 0$$

Exercise: 1) Write down components of (anti) chiral superfields

2) Find susy transfs for components using

$$\delta\phi = \left[ i(\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha} \bar Q^{\dot\alpha}), \phi \right]$$

First step

$$\delta\varphi = \delta\phi\Big| = i(\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha} \bar Q^{\dot\alpha})\phi \Big|$$

$$= (\varepsilon^\alpha D_\alpha + \underbrace{\cancel{\bar\varepsilon_{\dot\alpha} \bar D^{\dot\alpha}}}_{0})\phi \Big| = \varepsilon^\alpha D_\alpha \phi \Big| = \varepsilon^\alpha(-\psi_\alpha)$$

$$\{D_\alpha, \bar D_{\dot\alpha}\} = 2\underbrace{P_{\alpha\dot\alpha}}_{-i\partial_{\alpha\dot\alpha}} \Rightarrow [\bar D_{\dot\alpha}, D^2] \equiv [\bar D_{\dot\alpha}, \tfrac{1}{2} D^\alpha D_\alpha] = 2i D^\alpha \partial_{\alpha\dot\alpha}$$

## ACTION PRINCIPLE IN SUPERSPACE

We can construct functions of superfields

$\Rightarrow$ "super" lagrangian

$$\mathcal{L}\left(\phi, D_\alpha\phi, \bar{D}_{\dot\alpha}\phi, \partial_{\alpha\dot\alpha}\phi, \dots\right)$$

s.t. under susy transfs:

$$\delta\mathcal{L} = \left[i(\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}), \mathcal{L}\right] \qquad \not\Leftarrow$$

Action : $\underbrace{\int d^4x\, d^4\theta}\ \mathcal{L}(\phi, D\phi, \dots) = S$

under susy transfs the measure is invariant.
Therefore,

$$\delta S = \int d^4x\, d^4\theta\ \delta\mathcal{L} = i\int d^8z\ \left[(\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}), \mathcal{L}\right]$$

$$= i\int d^4x \left\{ \left[D^2\bar{D}^2, \left[\varepsilon^\alpha Q_\alpha, \mathcal{L}\right]\right] + \left[\bar{D}^2 D^2, \left[\bar\varepsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}, \mathcal{L}\right]\right] \right\} \Big|$$

$$= i\int d^4x \left\{ -\left[\mathcal{L}, \cancel{\left[D^2\bar{D}^2, \varepsilon^\alpha Q_\alpha\right]}^{=0}\right] - \left[\varepsilon^\alpha Q_\alpha, \left[\mathcal{L}, D^2\bar{D}^2\right]\right] \right.$$
$$\left. - \left[\mathcal{L}, \cancel{\left[\bar{D}^2 D^2, \bar\varepsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}\right]}^{=0}\right] - \left[\bar\varepsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}, \left[\mathcal{L}, \bar{D}^2 D^2\right]\right] \right\} \Big|$$

$$= i\int d^4x \left\{ \left[\varepsilon^\alpha Q_\alpha, \underbrace{\left[D^2\bar{D}^2, \mathcal{L}\right]}_{(D^2\bar{D}^2\mathcal{L})}\right] + \left[\bar\varepsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}, \underbrace{\left[\bar{D}^2 D^2, \mathcal{L}\right]}_{(\bar{D}^2 D^2\mathcal{L})}\right] \right\} \Big|$$

$$= i\int d^4x \left\{ \varepsilon^\alpha D_\alpha\left(D^2\bar{D}^2\mathcal{L}\right) + \bar\varepsilon_{\dot\alpha}\bar{D}^{\dot\alpha}\left(\bar{D}^2 D^2\mathcal{L}\right) \right\} = 0$$

## FREE ACTION FOR A CHIRAL SUPERFIELD

$$S_0 = \underbrace{\int d^4x\, d^4\theta}_{-2}\ \phi(x_L, \theta)\, \bar\Phi(x_R, \bar\theta) \qquad [\phi] = 1$$

$$= \int d^4x\ D^2\bar{D}^2(\phi\bar\Phi)\ \Big| \qquad \begin{aligned} \bar{D}\phi &= 0 \\ D\bar\Phi &= 0 \end{aligned}$$

$$= \int d^4x \, D^2 \left( \phi \bar{D}^2 \bar{\Phi} \right) \Big| =$$

$$= \int d^4x \left[ D^2\phi \, \bar{D}^2\bar{\Phi} \Big| + D^\alpha\phi \, D_\alpha \bar{D}^2 \bar{\Phi} \Big| + \phi \, D^2 \bar{D}^2 \bar{\Phi} \Big| \right]$$

$$= \int d^4x \left[ - F\bar{F} + D^\alpha\phi \, [D_\alpha, \bar{D}^2] \bar{\Phi} \Big| + \phi \, \{D^2, \bar{D}^2\} \bar{\Phi} \right]$$

We now use $[D_\alpha, \bar{D}^2] = 2i \, \bar{D}^{\dot\alpha} \partial_{\alpha\dot\alpha}$

$$D^2\bar{D}^2 + \bar{D}^2 D^2 = D^\alpha \bar{D}^2 D_\alpha - 4\Box$$

$$= \int d^4x \left[ - F\bar{F} + 2i \, \psi^\alpha \partial_{\alpha\dot\alpha} \bar{\psi}^{\dot\alpha} - 4 \varphi \Box \bar{\varphi} \right]$$

$$= 4 \int d^4x \left[ - \varphi \Box \bar{\varphi} + \frac{i}{2} \psi^\alpha \partial_{\alpha\dot\alpha} \bar{\psi}^{\dot\alpha} - \frac{1}{4} F\bar{F} \right]$$

## Superfield EOM

In the ordinary case we define the functional derivative

$$\frac{\delta\varphi(x)}{\delta\varphi(x')} = \delta^{(4)}(x - x')$$

$$S = \frac{1}{2} \int d^4x \, \partial_\mu\varphi(x) \, \partial^\mu\varphi(x) = -\frac{1}{2} \int d^4x \, \varphi(x) \Box \varphi(x)$$

$$\delta S = \int d^4x' \, \frac{\delta S}{\delta\varphi(x')} \delta\varphi(x') =$$

$$= -\frac{1}{2} \int d^4x' \, d^4x \, \frac{\delta \left( \varphi(x) \Box \varphi(x) \right)}{\delta\varphi(x')} \delta\varphi(x')$$

$$= -\frac{1}{2} \cdot 2 \int d^4x' \, d^4x \, \underbrace{\frac{\delta\varphi(x)}{\delta\varphi(x')}}_{\delta^{(4)}(x - x')} \left( \Box \varphi(x) \right) \delta\varphi(x')$$

$$= - \int d^4x \left( \Box\varphi(x) \right) \delta\varphi(x) = 0 \quad \forall \, \delta\varphi$$

$$\Downarrow$$

$$\Box\varphi = 0$$

Superspace functional derivative

$$\frac{\delta\phi(x,\theta,\bar\theta)}{\delta\phi(x',\theta',\bar\theta')} = \delta^{(4)}(x-x')\,\delta^{(4)}(\theta-\theta')$$
$$= \delta^{(4)}(x-x')\,(\theta-\theta')^2\,(\bar\theta-\bar\theta')^2$$

when $\phi$ is an unconstrained superfield

$\phi \to \phi + \delta\phi$

$$\delta S = \int d^8z' \,\frac{\delta S}{\delta\phi(z')}\,\delta\phi(z') = \int d^8z' \int d^8z \,\frac{\delta\mathcal{L}(\phi(z))}{\delta\phi(z')}\,\delta\phi(z')$$

$$= \int d^8z'\, d^8z \,\frac{\partial\mathcal{L}}{\partial\phi}\,\underbrace{\frac{\delta\phi(z)}{\delta\phi(z')}}_{\delta^{(4)}(x-x')\,\delta^{(4)}(\theta-\theta')}\,\delta\phi(z')$$

$$= \int d^8z \,\frac{\partial\mathcal{L}}{\partial\phi(z)}\,\delta\phi(z) = 0 \qquad \forall\, \delta\phi$$

$$\Rightarrow \boxed{\frac{\delta\mathcal{L}}{\delta\phi} = 0} \qquad \Leftarrow \text{EoM for superfields}$$

Important: for chiral superfield $\bar D_{\dot\alpha}\phi = 0$

the functional variation is defined as

$$\frac{\delta\phi(z)}{\delta\phi(z')} = \underset{\Uparrow}{\bar D^2}\,\delta^{(8)}(z-z') = \delta^{(4)}(x-x')\,(\theta-\theta')^2$$

Why?

Chirality constraint $\quad \bar D_{\dot\alpha}\phi = 0 \;\rightsquigarrow\; \phi = \bar D^2 \chi$

$\delta\phi \rightsquigarrow \delta\chi$

$$\frac{\delta\phi(z)}{\delta\chi(z')} = \frac{\delta(\bar D^2\chi(z))}{\delta\chi(z')} = \bar D^2\,\delta^{(8)}(z-z')$$

Free action for chirals $\quad S_0 = \int d^8z \, \phi \bar{\phi}$

$$\left.\begin{aligned} \frac{\delta}{\delta \phi(z)} &: \quad \bar{D}^2 \bar{\phi} = 0 \\ \frac{\delta}{\delta \bar{\phi}(z)} &: \quad D^2 \phi = 0 \end{aligned}\right\} \text{ free EOM for superfield}$$