Supergravity

Marco Zagermann
LACES 2019

Spinor conventions and related background material

Contents

Supersymmetry involves fermions and hence quantities that transform in spinor representations of the Lorentz algebra. In this text, we introduce the spinor conventions used throughout these lectures and provide some additional background material as a reference. Most relevant for the first lectures are sections 1.2 and 2.

Our metric signature in D spacetime dimensions is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1), \tag{1}$$

where the indices $\mu, \nu, \ldots = 0, \ldots, D-1$ are the usual D-dimensional Lorentz indices.

The generators of Lorentz transformations are denoted by $M_{\mu\nu} = -M_{\nu\mu}$ and satisfy the Lorentz algebra $\mathfrak{so}(1, D-1)$,

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}. \tag{2}$$

A spinor representation is a representation of the above Lorentz algebra that does not integrate to an ordinary (i.e. "single-valued") representation of the corresponding Lorentz *group*. Instead, it gives rise only to a "double-valued" representation of the Lorentz group in the sense that spatial rotations by 2π give minus the identity, whereas a rotation by 0 or 4π is represented by the identity operator, even though all these transformations act as the identity on Minkowski spacetime.

Mathematically, this is possible because the Lorentz group [1] is not simply connected, but contains closed loops that cannot be continuously contracted to a point. The universal covering group of the Lorentz group is a group that is locally isomorphic to the Lorentz group but has a different topology so that all closed loops can be continuously contracted to a point. Spinor representations are then single-valued representations of this universal covering group that project to double-valued representations of the Lorentz group itself.

1 Spinors in 4D

In 4D, there are two very common notations for spinor representations: the two-component spinor notation and the four-component spinor notation. In this lecture, we will use the four-component spinor formalism as it is more common in the supergravity literature, generalizes more readily to other spacetime dimensions and is also often used in particle phenomenology. The two-component spinor formalism, on the other hand, is frequently used in texts on global supersymmetry, so we briefly include it here as well to facilitate the translation of one formalism into the other. The following subsection 1.1 is however not necessary for the supergravity course and may be skipped.

1.1 Two-component spinors

The universal covering group of the Lorentz group in 4D is isomorphic to the group $SL(2,\mathbb{C})$, the group of unimodular complex (2×2)-matrices. This isomorphism is easily understood by mapping a real four-vector V^μ to a Hermitian (2×2)-matrix $\hat{V}$ according to

$$\hat{V} := V^\mu \sigma_\mu, \tag{3}$$

where $\sigma_0 = \mathbb{I}_2$ and σ_i $(i=1,2,3)$ are the Pauli matrices. The Minkowski norm can then be expressed as $V^\mu V_\mu = -\det \hat{V}$. Acting with $A \in SL(2,\mathbb{C})$ on $\hat{V}$ according to

$$\hat{V} \rightarrow A\hat{V}A^\dagger, \tag{4}$$

preserves the Hermiticity of $\hat{V}$ as well as its determinant and hence induces a Lorentz transformation on V^μ. As A and $-A$ induce the same Lorentz transformation, $SL(2,\mathbb{C})$ is a double cover of $SO(1,3)_0$.

$SL(2,\mathbb{C})$ has two equivalence classes of irreducible two-dimensional complex representations, corresponding to the defining representation of $SL(2,\mathbb{C})$ and its complex conjugate representation. These two representations are the minimal spinor representations of the 4D Lorentz algebra and are often denoted by $(1/2,0)$ and $(0,1/2)$ or by dotted and undotted two-component spinors, λ_A and $\omega^*_{\dot{A}}$ $(A,\dot{A}=1,2)$.

The Lorentz group generators $M_{\mu\nu}$ act on these spinors via representation matrices $(r(M_{\mu\nu})_A{}^B$ and $(r^*(M_{\mu\nu})_{\dot{A}}{}^{\dot{B}}$ that can be chosen as

$$(1/2,0): \quad r(M_{\mu\nu}) = -\frac{1}{4}(\sigma_\mu\overline{\sigma}_\nu - \sigma_\nu\overline{\sigma}_\mu) \tag{5}$$

$$(0,1/2): \quad r^*(M_{\mu\nu}) = e^{-1}\left[-\frac{1}{4}(\overline{\sigma}_\mu\sigma_\nu - \overline{\sigma}_\nu\sigma_\mu)\right]e, \tag{6}$$

[1] The full isometry group $O(1,D-1)$ of Minkowski spacetime decomposes into four disconnected components. Being a bit sloppy, we mean by Lorentz group here only the component $SO(1,D-1)_0$ of Lorentz transformations that are continuously connected to the identity element. A more accurate term for this subgroup would be the "proper orthochronous Lorentz group".

where $-\overline{\sigma}_i = \sigma_i$ $(i = 1, 2, 3)$ are again the Pauli matrices, $\overline{\sigma}_0 = \sigma_0 = \mathbb{I}_2$, and

$$e \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7}$$

That (6) is indeed the complex conjugate of (5) follows from the identity

$$e^{-1}\sigma^\mu e = \overline{\sigma}^{\mu *} = \overline{\sigma}^{\mu T}. \tag{8}$$

The matrix e is an invariant of $SL(2, \mathbb{C})$ in the sense that

$$A^T e A = e \quad \forall A \in SL(2, \mathbb{C}), \tag{9}$$

which implies that quantities such as $\lambda^T e\omega$ are Lorentz invariant products of two spinors λ_A and ω_A. This is often written as $\lambda_A\omega^A$ or $\lambda^A\omega_A$ using suitable raising and lowering conventions for spinor indices with the matrix e.

All finite-dimensional irreducible representations of $SL(2, \mathbb{C})$ can be obtained from symmetrized tensor products of these elementary building blocks, often denoted by $(n/2, m/2)$, where n and m count the $(1/2, 0)$ and $(0, 1/2)$ factors, respectively. For $(m + n)$ =even/odd, these describe single/double-valued representations of the Lorentz group, corresponding to bosons/fermions.

1.2 Four-component spinors

The four-component spinor formalism in 4D has a direct analogue in any spacetime dimension D, so we will first keep D arbitrary before we specialize to $D = 4$. Starting point for this formalism is a representation of the so-called Clifford-algebra Cliff$(1, D - 1)$ in D spacetime dimensions, i.e. a set of complex matrices γ^μ that satisfy the following anticommutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{I}. \tag{10}$$

Given a set of such γ^μ, the anticommutation relation (10) implies that the matrices

$$\Sigma_{\mu\nu} := \frac{1}{4}[\gamma_\mu, \gamma_\nu] \tag{11}$$

automatically form a representation of the Lorentz algebra (2). This representation is in fact a spinor representation, as one can directly verify by evaluating e.g. the matrix $R(\theta) = e^{\theta\Sigma^{12}}$ for a rotation in the $(1, 2)$-plane by a finite angle θ,

$$R(\theta) = e^{\theta\Sigma^{12}} = \cos(\theta/2)\mathbb{I} + 2\sin(\theta/2)\Sigma^{12}, \tag{12}$$

which shows that indeed $R(2\pi) = -\mathbb{I}$.

Thus, a representation of the Clifford algebra Cliff$(1, D-1)$ induces a spinor representation of the Lorentz algebra $\mathfrak{so}(1, D - 1)$.

Specializing now to $D = 4$, the smallest possible complex representation of Cliff$(1, 3)$ is on $\mathbb{C}^4$, i.e., the γ^μ are complex (4×4)-matrices, the well-known Dirac matrices. The elements $\psi \in \mathbb{C}^4$ on which they act are called four-component Dirac spinors. Where needed, we will use indices $\alpha, \beta, \ldots = 1, 2, 3, 4$ to denote the components ψ_α of these spinors or the components $\gamma^\mu_{\alpha\beta}$ of the Dirac matrices.

There are infinitely many choices for the γ^μ, but in 4D they are all equivalent, i.e. any two sets of (4×4)-matrices γ^μ and γ'^μ that satisfy (10) are related by an invertible (4×4)-matrix S

as $\gamma'^{\mu} = S\gamma^{\mu}S^{-1}$. We will rarely make use of particular representations, but for convenience, we will always assume so-called "friendly" representations whose defining properties are

$$\gamma_0^\dagger = -\gamma_0 \tag{13}$$
$$\gamma_i^\dagger = +\gamma_i \qquad (i = 1,2,3) \tag{14}$$
$$\gamma_\mu^T = \pm\gamma_\mu, \tag{15}$$

where the last equation means that each gamma matrix is either symmetric or antisymmetric (the sign need not be the same for all four gamma matrices). A very useful example of a friendly representation is the Weyl representation

$$\gamma_\mu = \begin{pmatrix} 0 & i\overline{\sigma}_\mu \\ i\sigma_\mu & 0 \end{pmatrix}, \tag{16}$$

where $-\overline{\sigma}_i = \sigma_i$ $(i = 1,2,3)$ are the Pauli matrices, and $\overline{\sigma}_0 = \sigma_0 = \mathbb{I}_2$. Other well-known representations such as the Dirac or Majorana representation are also friendly [2].

An important object in the following will be the completely antisymmetrized products of several gamma matrices,

$$\gamma_{\mu_1\ldots\mu_p} := \gamma_{[\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_p]}, \tag{17}$$

where, as usual, the antisymmetrization involves a prefactor $1/p!$, so that, e.g., $\gamma_{\mu\nu} = 1/2[\gamma_\mu, \gamma_\nu]$ etc. Obviously, $\Sigma_{\mu\nu} = 1/2\gamma_{\mu\nu}$.

Note that the Clifford relation (10) implies

$$\gamma_{\mu_1\mu_2\ldots\mu_p} = \begin{cases} \gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_p} & \text{if all } \mu_i \text{ are different} \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

The 16 matrices $\gamma_M = \{\mathbb{I}_4, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\rho}, \gamma_{\mu\nu\rho\sigma}\}$ are linearly independent and form a basis of the complex (4×4) matrices. To understand this, one first notices that the trace of the $\gamma_{\mu_1\ldots\mu_p}$ vanishes for all $p \geq 1$, as one verifies quite easily e.g. in the Weyl representation. As $\gamma_{\mu_1\ldots\mu_p}\gamma_\nu$ is a linear combination of certain matrices $\gamma_{\rho_1\ldots\rho_s}$ with $s \leq p$ unless $p = 1$, one easily verifies that

$$\mathrm{tr}(\gamma_{\mu_1\ldots\mu_p}\gamma_\nu) = \begin{cases} 0 & \text{for } p > 1 \\ \mathrm{tr}(\gamma_{\mu_1}\gamma_\nu) = 4\eta_{\mu_1\nu} & \text{for } p = 1 \end{cases}. \tag{19}$$

Similarly, $\mathrm{tr}(\gamma_{\mu_1\ldots\mu_p}\gamma_{\nu_1\ldots\nu_q})$ vanishes unless $p = q$ and (modulo permutations) $(\mu_1,\ldots,\mu_p) = (\nu_1,\ldots,\nu_q)$. This then implies indeed that the 16 independent matrices $\{\mathbf{1}_4, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\rho}, \gamma_{\mu\nu\rho\sigma}\} =: \{\mathbf{1}_4, \gamma_M\}$ are linearly independent and hence form a basis of the space of complex (4×4) matrices.

As we have seen, an irreducible representation of the Clifford algebra (10) induces a spinor representation (11) of the Lorentz algebra. The $\Sigma_{\mu\nu}$, however, are (4×4)-matrices, so the corresponding spinors $\psi \in \mathbb{C}^4$ on which they act have twice as many components as the two-component spinors discussed in the previous subsection. As the latter provide already irreducible representations of the Lorentz algebra, we expect that the representation given by (11) is in fact *reducible* as a representation of the Lorentz algebra, even though the underlying representation of the Clifford algebra is *irreducible*.

[2] The Dirac representation corresponds to $\gamma_0 = i\sigma_3 \otimes \mathbb{I}_2 \equiv \begin{pmatrix} i\mathbb{I}_2 & 0 \\ 0 & -i\mathbb{I}_2 \end{pmatrix}$, $\gamma_i = \sigma_2 \otimes \sigma_i$, and the Majorana representation is given by $\gamma_0 = i\sigma_2 \otimes \sigma_3$, $\gamma_1 = -\sigma_1 \otimes \mathbb{I}_2$, $\gamma_2 = \sigma_2 \otimes \sigma_2$, $\gamma_3 = \sigma_3 \otimes \mathbb{I}_2$, but they are not really needed for this lecture.

This is verified most easily in the Weyl representation (16), where $\Sigma_{\mu\nu}$ becomes manifestly reducible,

$$\Sigma_{\mu\nu} = \begin{pmatrix} er^*(M_{\mu\nu})e^{-1} & 0 \\ 0 & r(M_{\mu\nu}) \end{pmatrix}. \tag{20}$$

Here, $r(M_{\mu\nu})$ and $r^*(M_{\mu\nu})$ are the (2×2)-representation matrices (5) and (6), and e is the antisymmetric matrix (7). In the Weyl representation of the gamma matrices, we thus have a natural decomposition of a four-component Dirac spinor ψ_α ($\alpha = 1, \ldots, 4$) into one undotted two-component spinor λ_A in the representation $(1/2, 0)$ and one dotted spinor $\omega^*_{\dot{A}}$ in the representation $(0, 1/2)$ according to

$$\psi = \begin{pmatrix} e \cdot \omega^* \\ \lambda \end{pmatrix}. \tag{21}$$

For supersymmetry, it is convenient to work with minimal spinor representations of the Lorentz algebra, as these typically correspond also to the minimal amount of supersymmetry one can have in the respective spacetime dimension. There are essentially two ways to reduce the number of degrees of freedom of a Clifford algebra spinor so as to obtain irreducible representations (irreps) of the corresponding Lorentz algebra. One possibility is to impose a chirality condition, which leads to Weyl spinors. In the Weyl representation (21), a Weyl spinor ψ would correspond to setting either ω or λ equal to zero. The other possibility is to impose a reality condition, which leads to Majorana spinors. In the Weyl representation (21), this would correspond to setting $\lambda = \omega$.

Note, however, that imposing a Weyl or Majorana condition in 4D does not necessitate the use of the Weyl or the Majorana representation of the gamma matrices. The Weyl condition just takes on a particularly simple form in the Weyl representation, and the Majorana condition leads to a particularly simple result in the Majorana representation (although it is also quite simple in the Weyl representation). We usually do not make use of these simplified forms, however, and instead want to write down the conditions in a covariant way valid for any (friendly) representation so as to be able to make full use of the gamma matrix calculus.

We already mention here that the Weyl and the Majorana condition are not always possible in every spacetime dimension: The Weyl condition can only be imposed in even dimensions, whereas the possibility to impose a Majorana condition shows a somewhat more complicated dependence on the spacetime dimension. Moreover, the Weyl and Majorana condition can often not be imposed simultaneously. In the rest of this section, we resrict ourselves to four spacetime dimensions, where one can impose a Weyl *or* a Majorana condition, but not both of them at the same time.

1.2.1 The Weyl condition

The Weyl condition projects out the part of a four-component spinor that has a particular handedness. In a general representation of the gamma matrices, it is imposed with the γ_5 matrix:

$$\gamma_5 \equiv \gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = +i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{22}$$

The Clifford algebra (10) implies (cf. Ass. # 2, Ex. 2)

$$(\gamma_5)^2 = \mathbb{I}_4 \tag{23}$$

$$\{\gamma_5, \gamma_\mu\} = 0 \Rightarrow [\gamma_5, \Sigma_{\mu\nu}] = 0 \tag{24}$$

so that the chirality projectors

$$P_L \equiv \frac{1}{2}\left(1+\gamma^5\right), \quad P_R \equiv \frac{1}{2}\left(1-\gamma^5\right), \tag{25}$$

can be used to define left and right-handed spinors

$$\psi_L \equiv P_L\psi, \qquad \psi_R \equiv P_R\psi. \tag{26}$$

Because of (24), this projection is consistent with Lorentz covariance (i.e., $P_{R,L}$ and $\Sigma_{\mu\nu}$ commute.), and left and right-handed spinors form separate representations of the Lorentz group. In the Weyl representation, $\gamma_5 = \sigma_3 \otimes \mathbb{I}_2$, i.e., the left and right-handed spinors are nontrivial only for the upper or lower two components of ψ in (21). In other words Weyl spinors are essentially just the two-component spinors introduced earlier with two zero entries attached. Note that $\gamma^5\psi_L = \psi_L$, while $\gamma^5\psi_R = -\psi_R$.

In a friendly representation, $\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu \Rightarrow \Sigma_{\mu\nu}^\dagger\gamma_0 = -\gamma_0\Sigma_{\mu\nu}$, and the *Dirac conjugate* of a general four-component Dirac spinor is defined by

$$\overline{\psi} \equiv i\psi^\dagger\gamma^0 = -i\psi^\dagger\gamma_0, \tag{27}$$

so that bilinears such as $\overline{\psi}\chi$ are Lorentz invariant.

In a friendly representation, $\gamma_5^\dagger = \gamma_5$, and hence $P_{L,R}^\dagger = P_{L,R}$. Moreover $P_L\gamma_0 = \gamma_0 P_R$ and $P_R\gamma_0 = \gamma_0 P_L$, which then implies

$$\overline{\psi_R} = \overline{\psi}P_L, \text{ and } \overline{\psi_L} = \overline{\psi}P_R, \tag{28}$$

where

$$\overline{\psi_R} = \overline{(P_R\psi)} = -i(P_R\psi)^\dagger\gamma_0. \tag{29}$$

Because of this we will often write

$$\overline{\psi}_L := \overline{\psi}P_L = \overline{\psi_R}, \qquad \overline{\psi}_R := \overline{\psi}P_R = \overline{\psi_L}. \tag{30}$$

Apart from defining chiral spinors, the γ_5-matrix also gives rise to some useful gamma matrix identities, in particular,

$$\begin{aligned} \gamma_{\mu\nu\rho\sigma} &= i\hat{\epsilon}_{\mu\nu\rho\sigma}\gamma_5 && (31)\\ \gamma_{\mu\nu\rho} &= i\hat{\epsilon}_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma && (32)\\ \gamma_{\mu\nu} &= -\frac{i}{2}\hat{\epsilon}_{\mu\nu\rho\sigma}\gamma_5\gamma^{\rho\sigma} && (33)\\ \gamma_\mu &= -\frac{i}{6}\hat{\epsilon}_{\mu\nu\rho\sigma}\gamma_5\gamma^{\nu\rho\sigma}, && (34) \end{aligned}$$

where $\hat{\epsilon}_{\mu\nu\rho\sigma}$ denotes the epsilon tensor in 4D Minkowski space with $\hat{\epsilon}_{0123} = -1$. This implies that the 16 matrices $\mathbb{I}_4, \gamma_\mu, \gamma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5$ form a basis of all complex (4×4)-matrices.

As an application, we prove a very useful equation that forms the basis of various Fierz identities, namely

$$\epsilon\bar{\eta} = a\mathbf{1}_4 + b_\nu\gamma^\nu + c_{\nu\rho}\gamma^{\nu\rho} + d_\nu\gamma_5\gamma^\nu + f\gamma_5, \tag{35}$$

where

$$\begin{aligned} a &= -\frac{1}{4}\bar{\eta}\epsilon && (36)\\ b_\nu &= -\frac{1}{4}\bar{\eta}\gamma_\nu\epsilon && (37)\\ c_{\nu\rho} &= +\frac{1}{8}\bar{\eta}\gamma_{\nu\rho}\epsilon && (38)\\ d_\nu &= +\frac{1}{4}\bar{\eta}\gamma_5\gamma_\mu\epsilon && (39)\\ f &= -\frac{1}{4}\bar{\eta}\gamma_5\epsilon. && (40)\end{aligned}$$

valid for any anticommuting four-component spinors η and ϵ (not necessarily Majorana). To derive this, one first uses that any complex (4×4)-matrix can be expressed as a linear combination of the 16 independent matrices $\{\mathbf{1}_4, \gamma_\mu, \gamma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5\}$, which then also has to be true for the (4×4)-matrix $\epsilon_\alpha\bar{\eta}_\beta$, i.e., we have

$$\epsilon\bar{\eta} = a\mathbf{1}_4 + b_\nu\gamma^\nu + c_{\nu\rho}\gamma^{\nu\rho} + d_\nu\gamma_5\gamma^\nu + f\gamma_5, \tag{41}$$

with suitable coefficients $a, b_\nu, c_{\nu\rho}, d_\nu, f$. These coefficients can be easily obtained by multiplying (41) with $\mathbf{1}_4, \gamma_\mu, \gamma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5$ and taking the trace each time, using that the only non-vanishing traces are

$$\begin{aligned} \mathrm{tr}(\gamma_\nu\gamma^\rho) &= 4\delta_\nu^\rho && (42)\\ \mathrm{tr}(\gamma^{\mu\nu}\gamma^{\rho\sigma}) &= 4(\eta^{\nu\rho}\eta^{\mu\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) && (43)\\ \mathrm{tr}(\gamma_5\gamma^\mu\gamma_5\gamma^\nu) &= -4\eta^{\mu\nu} && (44)\\ \mathrm{tr}(\gamma_5\gamma_5) &= 4. && (45)\end{aligned}$$

As an example consider

$$\mathrm{tr}(\gamma_\rho\epsilon\bar{\eta}) = (\gamma_\rho)_{\alpha\beta}\epsilon_\beta\bar{\eta}_\alpha = -\bar{\eta}_\alpha(\gamma_\rho)_{\alpha\beta}\epsilon_\beta = -\bar{\eta}\gamma_\rho\epsilon, \tag{46}$$

where the anticommutativity of η and ϵ has been used. From (41) and (42) one then obtains that this must be equal to $b_\nu\mathrm{tr}(\gamma_\rho\gamma^\nu) = 4b_\rho$, which then implies (37). The calculation of the other coefficients in (35) is then completely analogous.

Multiplying (35) with further spinors then gives rise to various Fierz identities that allow one to re-order expressions that are cubic or quartic in fermionic fields.

1.2.2 The Majorana condition

The Majorana condition is a reality condition on a four-component Dirac spinor that can, in a general gamma matrix representation, be written as

$$\psi^* = B\psi, \tag{47}$$

with a suitable matrix B. This condition is self-consistent (i.e., $\psi^{**} = \psi$) and Lorentz covariant if B satisfies

$$B^*B = \mathbb{I}_4, \qquad \gamma_\mu^* = B\gamma_\mu B^{-1} \Rightarrow \Sigma_{\mu\nu}^* B = B\Sigma_{\mu\nu}. \tag{48}$$

If one defines the Majorana conjugate as $\psi^c := B^{-1}\psi^*$, the Majorana condition reads $\psi^c = \psi$. In the Weyl basis (16) one can choose

$$B = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}, \tag{49}$$

so that a Majorana spinor would be of the form (21), but with $\lambda = \chi$. From this we see that a Majorana spinor describes the same number of independent degrees of freedom as a Weyl spinor.

Another equivalent, but for many purposes more convenient, way to write the Majorana condition is via the so-called charge conjugation matrix, C, which satisfies

$$C^T = -C, \qquad \gamma_\mu^T = -C\gamma_\mu C^{-1}. \tag{50}$$

In a friendly representation, one can moreover choose C such that it also satisfies

$$C^{-1} = -C = C^\dagger, \tag{51}$$

as we will always assume.

In terms of C, the charge conjugate spinor is defined as

$$\psi^c := C\overline{\psi}^T = iC\gamma^{0T}\psi^*, \tag{52}$$

and a *Majorana spinor* is defined as

$$\psi^c = \psi. \tag{53}$$

This is equivalent to (47) if we identify

$$B = (iC\gamma_0^T)^{-1} \tag{54}$$

so that in terms of B charge conjugation reads

$$\psi^c = B^{-1}\psi^*. \tag{55}$$

The advantage of C is that for a Majorana spinor the Dirac conjugate can be written as

$$\overline{\psi} = \psi^T C. \tag{56}$$

Notice that using (50) one finds the symmetry properties

$$\begin{aligned} &C^T = -C, && (C\gamma^{\mu\nu\rho})^T = -(C\gamma^{\mu\nu\rho}), && (C\gamma^{\mu\nu\rho\sigma})^T = -(C\gamma^{\mu\nu\rho\sigma}), \\ &(C\gamma^\mu)^T = (C\gamma^\mu), && (C\gamma^{\mu\nu})^T = (C\gamma^{\mu\nu}). \end{aligned} \tag{57}$$

For *anti*-commuting Majorana spinors, this then implies

$$\overline{\psi}_1 M\psi_2 = \begin{cases} +\overline{\psi}_2 M\psi_1 & \text{for } M = \mathbb{I}_4, \gamma_{\mu\nu\rho}, \gamma_{\mu\nu\rho\sigma} \\ -\overline{\psi}_2 M\psi_1 & \text{for } M = \gamma_\mu, \gamma_{\mu\nu} \end{cases} \tag{58}$$

*Unless stated otherwise, we will, in the following, always use (anticommuting) Majorana spinors, but often also take in addition the chiral projections ψ_L and ψ_R of these Majorana spinors, which therefore are **not** independent.* More specifically, we have, in our conventions,

$$(\psi_L)^c = \psi_R, \qquad (\psi_R)^c = \psi_L. \tag{59}$$

To show this, we use

$$B^{-1}\gamma_5^* = -\gamma_5 B^{-1} \Leftrightarrow B^{-1}P_L^* = P_R B^{-1} \tag{60}$$

so that

$$(\psi_L)^c = (P_L\psi)^c = B^{-1}P_L^*\psi^* = P_R B^{-1}\psi^* = P_R\psi^c = P_R\psi = \psi_R. \tag{61}$$

Note that (59) implies that ψ_R is no longer a Majorana spinor, because that would require $(\psi_R)^c$ being equal to ψ_R. Thus, in 4D, a four-component spinor cannot be simultaneously chiral and Majorana. Nevertheless, it makes sense to talk about the projection ψ_R or ψ_L of a given Majorana spinor ψ.

From the definition of γ_5 and (50), one also gets

$$C\gamma_5 = \gamma_5^T C \Leftrightarrow CP_L = P_L^T C \tag{62}$$

so that for a Majorana spinor ψ,

$$\overline{\psi}_L = \overline{\psi}P_L = \psi^T CP_L = \psi^T P_L^T C = (\psi_L)^T C \tag{63}$$

even though ψ_L is not Majorana. From this we can obtain more symmetry properties for the chiral projections that are very similar to those for the Majorana spinors themselves,

$$\begin{aligned}
&\overline{\chi}_L\psi_L = \chi_L^T C\psi_L = -\psi_L^T C^T\chi_L = \overline{\psi}_L\chi_L,\\
&\overline{\chi}_L\gamma^\mu\psi_R = -\overline{\psi}_R\gamma^\mu\chi_L \quad \overline{\chi}_L\gamma^{\mu\nu}\psi_L = -\overline{\psi}_L\gamma^{\mu\nu}\chi_L\\
&\overline{\chi}_L\gamma^{\mu\nu\rho}\psi_R = \overline{\psi}_R\gamma^{\mu\nu\rho}\chi_L.
\end{aligned} \tag{64}$$

Finally, under charge conjugation

$$(\gamma_\mu)^c = \gamma_\mu, \quad (\gamma_5)^c = -\gamma_5 \tag{65}$$

in the sense that $(\gamma^\mu\psi)^c = \gamma^\mu\psi^c$ etc.

In all the subsequent formulae, the hermitian conjugate $+h.c.$ of a field operator is denoted with a superscript *, whereas the superscript † is reserved for matrix expressions when also a transposition is involved. On ordinary complex numbers and classical fields, the hermitian conjugation acts as complex conjugation, where, however, the order of anticommuting spinor fields is exchanged to mimic the effect of hermitian conjugation of operators. This results in a minus sign when the original spinor order is restored. Fortunately, the effect of this hermitian conjugation can simply be obtained by writing down the charge conjugate expression with all the rules obtained so far, including (65), but without exchanging the order of the spinors. As an example, we show $(\overline{\psi}_L\gamma^\mu\chi_R)^* = (\overline{\psi}_L\gamma^\mu\chi_R)^c = \overline{\psi}_R\gamma^\mu\chi_L$:

$$(\overline{\psi}_L\gamma^\mu\chi_R)^* = (\psi_L^T C\gamma^\mu\chi_R)^* = -\psi_L^\dagger C^*\gamma^{\mu*}\chi_R^* = -\psi_L^\dagger C^*\gamma^{\mu*}B(\chi_R)^c = -\psi_L^\dagger C^* B\gamma^\mu\chi_L. \tag{66}$$

Inserting (54), $C^* = C$ and $(\gamma^{0T})^{-1} = -C(\gamma^0)^{-1}C^{-1} = -C^{-1}(\gamma^0)^{-1}C$, this becomes

$$(\overline{\psi}_L\gamma^\mu\chi_R)^* = -i\psi_L^\dagger(\gamma^0)^{-1}\gamma^\mu\chi_L = \overline{(\psi_L)}\gamma^\mu\chi_L = \overline{\psi}_R\gamma^\mu\chi_L = (\overline{\psi}_L\gamma^\mu\chi_R)^c, \tag{67}$$

where in the second equation we used $(\gamma^0)^{-1} = -\gamma^0$, which follows from $(\gamma^0)^2 = \eta^{00}\mathbb{I}_4 = -\mathbb{I}_4$.

2 The simplest globally supersymmetric field theory in 4D

Both as an illustration of our spinor conventions and because it will serve as a starting point of the supergravity lecture, we give here a short account of the simplest globally supersymmetric field theory in 4D, namely the free and massless Wess-Zumino model of one chiral multiplet.

In superspace notation this model would correspond to

$$\mathcal{L}_{WZ} = \int d^4\theta \; \Phi^*\Phi, \tag{68}$$

where Φ is a chiral superfield and Φ^* its complex conjugate. In other words, we assume a canonical Kähler potential $K(\Phi, \Phi^*) = \Phi^*\Phi$ and a vanishing superpotential $W(\Phi) = 0$.

Switching to our four-component notation, the component fields of Φ are $(\phi, \chi, \mathcal{F})$, where $\phi(x)$ is a complex scalar field, $\chi(x)$ is an anticommuting four-component Majorana spinor (which has the same number of degrees of freedom as a complex two-component spinor and its complex conjugate) and the complex auxiliary scalar field $\mathcal{F}(x)$. Choosing suitable field normalizations, the Lagrangian for these component fields is

$$\mathcal{L}_{WZ} = -(\partial_\mu\phi)(\partial^\mu\phi^*) - \overline{\chi}\partial\!\!\!/\chi + |\mathcal{F}|^2, \tag{69}$$

where, as usual, $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. Modulo total derivative terms, this Lagrangian is invariant under the supersymmetry transformations

$$\delta\phi = \overline{\chi}_L\epsilon_L, \qquad \delta\phi^* = \overline{\chi}_R\epsilon_R \tag{70}$$

$$\delta\chi_L = \frac{1}{2}(\partial\!\!\!/\phi)\epsilon_R + \frac{1}{2}\mathcal{F}\epsilon_L, \qquad \delta\chi_R = \frac{1}{2}(\partial\!\!\!/\phi^*)\epsilon_L + \frac{1}{2}\mathcal{F}^*\epsilon_R \tag{71}$$

$$\delta\mathcal{F} = \overline{\epsilon}_R\partial\!\!\!/\chi_L, \qquad \delta\mathcal{F}^* = \overline{\epsilon}_L\partial\!\!\!/\chi_R, \tag{72}$$

where the supersymmetry parameter ϵ is an anticommuting four-component Majorana spinor[3]. These supersymmetry transformations satisfy the supersymmetry algebra

$$[\delta_\eta, \delta_\epsilon]\phi(x) = \frac{1}{2}(\overline{\epsilon}\gamma^\mu\eta)\partial_\mu\phi(x), \tag{73}$$

and similar for the other fields, where η denotes another supersymmetry parameter.

In this course, we will always work with the physical on-shell fields only, i.e. we will always work with theories after all auxiliary fields have been integrated out. In the case at hand, the auxiliary field $\mathcal{F}$ has the field equation $\mathcal{F} = 0$, so we can here simly drop $\mathcal{F}$ from the theory and work with the on-shell multiplet (ϕ, χ) with the on-shell Lagrangian

$$\mathcal{L}_{WZ} = -(\partial_\mu\phi)(\partial^\mu\phi^*) - \overline{\chi}\partial\!\!\!/\chi \tag{74}$$

and the on-shell transformation laws

$$\delta\phi = \overline{\chi}_L\epsilon_L, \qquad \delta\phi^* = \overline{\chi}_R\epsilon_R \tag{75}$$

$$\delta\chi_L = \frac{1}{2}(\partial\!\!\!/\phi)\epsilon_R, \qquad \delta\chi_R = \frac{1}{2}(\partial\!\!\!/\phi^*)\epsilon_L. \tag{76}$$

Note that the on-shell Lagrangian (74) is still invariant (modulo total derivatives) under these on-shell transformation laws, as we will explicitly verify below. The only disadvantage of using

[3] In terms of the supersymmetry operator, Q_α, which is an operator valued four-component Majorana spinor that satisfies $\{Q_\alpha, \overline{Q}_\beta\} = -2iP_\mu(\gamma^\mu)_{\alpha\beta}$, the supersymmetry variations $\delta\psi(x)$ of a general field operator $\psi(x)$ are defined by $i\delta\psi(x) := \frac{1}{2}[\overline{\epsilon}Q, \psi(x)]$.

only on-shell fields is that the supersymmetry algebra (73) is only satisfied on-shell, i.e. there will be extra terms in (73) that only vanish upon using the equations of motion. This is no problem for us as we are anyway only interested in the Lagrangians for the physical fields. Moreover, for increasing amounts of supersymmetry, a full off-shell formulation of the theory can become quite complicated or may even not be known.

We now close this section by explicitly verifying the invariance of the on-shell action (74) under the on-shell supersymmetry transformations (75), (76). In order to prepare our discussion of local supersymmetry and supergravity, however, we will a priori allow for a spacetime-dependent supersymmetry parameter $\epsilon = \epsilon(x)$.

To begin with, we write

$$\begin{aligned} \mathcal{L}_{WZ} &= -(\partial_\mu \phi)(\partial^\mu \phi^*) - \overline{\chi} \partial\!\!\!/ \chi && (77) \\ &= -(\partial_\mu \phi)(\partial^\mu \phi^*) - \overline{\chi}_L \partial\!\!\!/ \chi_R - \overline{\chi}_R \partial\!\!\!/ \chi_L. && (78) \end{aligned}$$

Using (64), the order of the spinors in the last term in (78) can be exchanged, and a subsequent partial integration brings this term to the form of the second term in (78):

$$-\overline{\chi}_R \partial\!\!\!/ \chi_L \equiv -\overline{\chi}_R \gamma^\mu (\partial_\mu \chi_L) = +\overline{\partial_\mu \chi_L} \gamma^\mu \chi_R = -\overline{\chi}_L \partial\!\!\!/ \chi_R + \text{tot. div.}, \quad (79)$$

so that, up to a total derivative,

$$\mathcal{L}_{WZ} = \underbrace{-(\partial_\mu \phi)(\partial^\mu \phi^*)}_{\mathcal{L}_{bos}} \underbrace{-2\overline{\chi}_L \partial\!\!\!/ \chi_R}_{\mathcal{L}_{fer}}. \quad (80)$$

To verify the invariance under the on-shell transformations (75), (76), we find for the bosonic part

$$\begin{aligned} \delta \mathcal{L}_{bos} &= -(\partial_\mu \delta\phi)(\partial^\mu \phi^*) - (\partial_\mu \phi)(\partial^\mu \delta\phi^*) && (81) \\ &= +\delta\phi \, \Box \phi^* + c.c. + \text{tot. div.} && (82) \\ &= +\overline{\chi}_L \epsilon_L \, \Box \phi^* + c.c. + \text{tot. div.} && (83) \\ & && (84) \end{aligned}$$

On the other hand, the fermionic part is

$$\begin{aligned} \delta \mathcal{L}_{fer} &= -2 \left[\overline{\chi}_L \partial\!\!\!/ \delta\chi_R + \overline{\delta\chi}_L \partial\!\!\!/ \chi_R \right] && (85) \\ &= -2 \left[\overline{\chi}_L \partial\!\!\!/ \delta\chi_R + \overline{\chi}_R \partial\!\!\!/ \delta\chi_L \right] && (86) \\ &= -2 \left[\overline{\chi}_L \partial\!\!\!/ \delta\chi_R + c.c. \right] && (87) \\ &= -\left[\overline{\chi}_L \gamma^\mu \partial_\mu (\partial\!\!\!/ \phi^* \epsilon_L) + c.c. \right] && (88) \\ &= -\left[\overline{\chi}_L (\partial\!\!\!/ \partial\!\!\!/ \phi^*) \epsilon_L + \overline{\chi}_L \gamma^\mu (\partial\!\!\!/ \phi^*)(\partial_\mu \epsilon_L) + c.c. \right], && (89) \end{aligned}$$

where, in the second equality, we have again used (64) to change the order of the two spinors, followed by a partial integration (cf. eq. (79)).

Using now $\partial\!\!\!/ \partial\!\!\!/ = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2} \gamma^\mu \gamma^\nu (\partial_\mu \partial_\nu + \partial_\nu \partial_\mu) = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu = \Box \, \mathbb{I}_4$, we see that the first term in (89) together with its conjugate precisely cancels $\delta \mathcal{L}_{bos}$, and we are left with

$$\delta \mathcal{L}_{WZ} = -(J^\mu_L \partial_\mu \epsilon_L + J^\mu_R \partial_\mu \epsilon_R) \quad (90)$$

with the "supercurrents"

$$J^\mu_L := \overline{\chi}_L \gamma^\mu (\partial\!\!\!/ \phi^*), \qquad J^\mu_R := \overline{\chi}_R \gamma^\mu (\partial\!\!\!/ \phi). \quad (91)$$

Thus, in *global* supersymmetry, where $\epsilon = $ const., the action $S_{WZ} = \int d^4x \, \mathcal{L}_{WZ}$ is indeed invariant under supersymmetry (as usual we neglect any boundary terms that might arise from the varous partial integrations we did along the way).

3 Clifford algebras and spinors in arbitrary D

As discussed at the beginning of section 1.2, Clifford algebra representations yield spinor representations of the corresponding Lorentz algebra. This works the same in any spacetime dimension D, but the dimension of the spinor representation and the possibility to reduce it by imposing a chirality or/and a Majorana condition are strongly D-dependent. The D-dependence of the existence of chirality and Majorana conditions also leads to D-dependent R-symmetry groups, which in turn contribute to a rich variety of possible scalar manifold geometries in the respective spacetime dimensions. It is the purpose of this section to classify the representations of Clifford algebras, the minimal spinor representations of the corresponding Lorentz groups as well as the resulting R-symmetry groups. This generalizes the discussion of spinors in four dimensions given in section 1.2.

Further information on this topic and many explicit proofs can be found, e.g., in

- P.C. West, "Supergravity, brane dynamics and string duality", hep-th/9811101.
- A. Van Proeyen, "Tools for supersymmetry," Ann. U. Craiova Phys. **9** (1999) no.I, 1 [hep-th/9910030].

Starting point is the Clifford algebra in D Lorentzian dimensions, where we will now choose capital letters for the gamma matrices,

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \qquad (\mu, \nu, \ldots = 0, 1, \ldots, D-1) \tag{92}$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1) \tag{93}$$

Just as in 4D, the relation (92) implies that

$$\Sigma_{\mu\nu} \equiv \frac{1}{4}[\Gamma_\mu, \Gamma_\nu] = \frac{1}{2}\Gamma_{\mu\nu} \tag{94}$$

form a representation of the Lorentz algebra. The exponentials $\exp\left[\frac{\omega^{\mu\nu}\Sigma_{\mu\nu}}{2}\right]$ with $\omega_{\mu\nu}$ finite rotation angles or boost paramters then form a double-valued representation of the Lorentz group $SO(1, D-1)_0$.

3.1 Irreducible representations of Cliff$(1, D-1)$

The structure of the irreducible representations of the Clifford algebras is slightly different for even and odd dimensions:

3.1.1 Even dimensions

Up to equivalence there is, just as in 4D, exactly one nontrivial irreducible representation of Cliff$(1, D-1)$ (see e.g. hep-th/9811101). It has complex dimension $2^{D/2}$, i.e., the Γ_μ are complex $(2^{D/2} \times 2^{D/2})$-matrices, generalizing the (4×4)-matrices in 4D. Explicit forms of these representations can be built up by successive tensor products of the irreps of lower-dimensional Clifford algebras, starting with the case $D = 2$ (see e.g., hep-th/9910030), but we do not need them in these lectures.

3.1.2 Odd dimensions

If D is odd, irreps of Cliff$(1, D-1)$ can be obtained from an irrep of Cliff$(1, D-2)$ by defining the analogue of the γ_5 matrix in 4D:

$$\Gamma_* \equiv (-i)^{\frac{D+1}{2}} \Gamma_0 \Gamma_1 \dots \Gamma_{D-2} \tag{95}$$

This matrix satisfies

$$\begin{aligned} (\Gamma_*)^2 &= \mathbb{1} && (96)\\ \{\Gamma_*, \Gamma_\mu\} &= 0 \qquad \forall \mu = 0, \dots, D-2 && (97) \end{aligned}$$

so that either of

$$\Gamma_{D-1}^{(\pm)} \equiv \pm \Gamma_* \tag{98}$$

can be used as the remaining gamma matrix to complement $\{\Gamma_0, \dots, \Gamma_{D-2}\}$ to a representation of Cliff$(1, D-1)$. One thus obtains *two* inequivalent representations of Cliff$(1, D-1)$ for odd D, one for each sign in (98).

3.2 Irreducible spinor representations of $SO(1, D-1)_0$

Thus far, we have discussed the irreps of Cliff$(1, D-1)$ and described how these induce double-valued spinor representations of the corresponding Lorentz groups $SO(1, D-1)_0$. Just as in four dimensions, however, the spinor representations of the Lorentz group so-obtained are in general *not* irreducible, even though they descend from irreducible representations of Cliff$(1, D-1)$. In order to obtain an irreducible spinor representation of $SO(1, D-1)_0$, one in general has to impose additional constraints, which may be of the following type:

1. Chirality condition

2. Reality condition

3. Chirality and a reality condition

The possibilities to impose one of the above is strongly dimension dependent, as we will now describe.

3.2.1 Chirality conditions

For even D, we can always impose the following chirality condition to define a left or right handed *Weyl spinor*:

$$\Gamma_* \psi_{\substack{L\\R}} = \pm \psi_{\substack{L\\R}}. \tag{99}$$

Note that that this condition is Lorentz covariant because of $[\Sigma_{\mu\nu}, \Gamma_*] = 0$, just as in 4D.

For odd D, on the other hand, there is no non-trivial analogue of Γ_*, because

$$\underbrace{\Gamma_0 \Gamma_1 \dots \Gamma_{D-2}}_{\sim \Gamma_{D-1}^{(\pm)}} \Gamma_{D-1}^{(\pm)} \sim (\Gamma_{D-1}^{(\pm)})^2 \sim \mathbb{1}. \tag{100}$$

Thus a nontrivial chirality condition can only be imposed in *even* D.

3.2.2 Reality conditions

It is again useful to distinguish between even and odd dimensions:

Even dimensions As discussed above, for even D there is only one equivalence class of irreps of Cliff$(1, D-1)$ generated by matrices Γ_μ. Hence, the complex conjugate matrices $\pm\Gamma_\mu^*$, which also satisfy the Clifford algebra, must be equivalent to the matrices Γ_a, i.e., there has to be a matrix B such that

$$\Gamma_\mu^* = \eta B \Gamma_\mu B^{-1} \tag{101}$$

for both signs $\eta = \pm 1$.

Odd dimensions If D is odd, we can obviously find a matrix B that also satisfies (101) for the first $(\mathrm{D}-1)$ gamma matrices with $\eta = \pm 1$. What is non-trivial, however, is to extend (101) also to the remaining gamma matrix $\Gamma_{D-1}^{(\pm)} = \pm\Gamma_*$ (cf. eq. (95)), i.e., to have

$$(\Gamma_*)^* = \eta B \Gamma_* B^{-1}. \tag{102}$$

Indeed, using the definition (95) and (101) for $\Gamma_0, \ldots, \Gamma_{D-2}$, one easily shows

$$(\Gamma_*)^* = (-1)^{\frac{D+1}{2}} B \Gamma_* B^{-1}, \tag{103}$$

which is consistent with (102) only for one sign:

$$\eta = (-1)^{\frac{D+1}{2}} = \begin{cases} -1 \text{ for D=5 mod 4} \\ +1 \text{ for D=3 mod 4} \end{cases} \tag{104}$$

Obviously, the defining equations (101) and (102) define B only up to an arbitrary rescaling. We may thus choose the overall scaling such that

$$|\det B| = 1 \qquad \text{(choice)} \tag{105}$$

With this normalization, one has [4]

$$\begin{aligned} B^* B &= \epsilon \mathbb{1} && (106) \\ \epsilon &= \pm 1. && (107) \end{aligned}$$

The important point now is that this parameter ϵ is not arbitrary, but is instead fixed by the values of η and D. Concretely, for $D = 2n$ or $D = 2n+1$, one finds [5]

$$\epsilon = -\eta\sqrt{2}\cos\left[\frac{\pi}{4}(1 + \eta 2n)\right], \tag{110}$$

[4] This can be proven by inserting (101) and (102) into their own complex conjugate to derive $[B^*B, \Gamma_\mu] = 0$ for all μ and similarly for all symmetrized products of gamma matrices. As these together with the unit matrix span all complex matrices, B^*B commutes with all matrices and thus, according to Schur's lemma must be proportional to the unit matrix. Using the complex conjugate of (106) and the choice (105), one then derives $\epsilon = \pm 1$.

[5] This can be proven, e.g., with the help of the charge conjugation matrix C. In a friendly representation (so that for $\Gamma_\mu \Gamma_\mu^\dagger = \mathbb{1}$ (no sum)), $C \equiv B^T \Gamma_0$ satisfies, because of (101) and (106),

$$\begin{aligned} \Gamma_\mu^T &= -\eta C \Gamma_\mu C^{-1} && (108) \\ C^T &= -\eta \epsilon C. && (109) \end{aligned}$$

The matrices $(C\Gamma_{\mu_1 \ldots \mu_p})$ then have a definite symmetry under transposition. This symmetry depends on p, ϵ and η. On the other hand, the set of all matrices $\Gamma_{\mu_1 \ldots \mu_p}$ plus the unit matrix form a complete basis of all complex $(2^{[D/2]} \times 2^{[D/2]})$-matrices. As the number of linearly independent antisymmetric and symmetric of such matrices is fixed to be $2^{[D/2]}(2^{[D/2]}-1)/2$ and $2^{[D/2]}(2^{[D/2]}+1)/2$, respectively, one can determine the possible values of ϵ as a function of D and η (which, for odd dimensions, is itself fixed by D). (cf. e.g. hep-th/9811101).

and one arrives at the possible values for ϵ and η shown in table 1.

The Majorana condition What makes the possible values of ϵ so important, is that it determines whether one can impose a Majorana condition on a spinor, which, in terms of B reads

$$\psi^* = \alpha B\psi \qquad \text{(Majorana condition)}, \tag{111}$$

where α is an arbitrary phase. This condition is consistent with Lorentz invariance, because $\Gamma^*_{\mu\nu} = B\Gamma_{\mu\nu}B^{-1}$. A Majorana spinor thus furnishes a complete representation of the Lorentz algebra and has only half as many degrees of freedom as an unconstrained complex Dirac spinor. The consistency of (111) with $\psi^{**} = \psi$, however, imposes the consistency condition

$$\epsilon = +1 \qquad \text{(for Majorana condition)}, \tag{112}$$

limiting the possibility of Majorana spinors to certain dimensions, as indicated in table 1.

Symplectic Majorana spinors If $\epsilon = -1$, one can impose a so-called symplectic Majorana condition. To this end, one needs an even number of Dirac spinors ψ_i, $(i, j, \ldots = 1, \ldots, 2N)$ and an antisymmetric real matrix Ω_{ij} with $\Omega^2 = -\mathbb{1}_{2N}$ and imposes

$$(\psi_i)^* = \Omega_{ij} B\psi_j. \tag{113}$$

As one needs at least two Dirac spinors to impose the symplectic Majorana condition, it does not lead to a reduction of the minimal number of degrees of freedom relative to a single Dirac spinor. The symplectic Majorana condition is however convenient, because it makes the action of the R-symmetry group (which in these dimensions involve symplectic groups, see table 1) manifest.

3.3 Majorana and Weyl condition

In some dimensions, the Majorana and the Weyl condition can be imposed simultaneously. This reduces the number of independent degrees of freedom to one quarter relative to an unconstrained Dirac spinor. Imposing (we set the phase $\alpha = 1$ for simplicity)

$$\psi^* = B\psi \tag{114}$$
$$\Gamma_*\psi = \pm\psi, \tag{115}$$

at the same time, obviously requires the consistency condition

$$(\Gamma_*)^* = B\Gamma_* B^{-1} \tag{116}$$

which is possible only if $D = 4n - 2$. But as there are no Majorana spinors in $D = 6, 14, \ldots$, Majorana-Weyl spinors can only exist for

$$D = 2 \bmod 8 \qquad \text{(Condition for Majorana-Weyl spinors)}. \tag{117}$$

Note, in particular, that in 4D, one can have Majorana spinors *or* Weyl spinors, but not Majorana-Weyl spinors, as mentioned earlier.

Analogously, in dimensions in which $\epsilon = -1$ allows a symplectic Majorana condition, one can sometimes also simultaneously impose a Weyl condition, and the corresponding spinors are then called symplectic Majorana-Weyl spinors. These are the dimensions $D = 6 \bmod 8$

D	η	ϵ	min. spinor type	min. # of real supercharges	R-symmetry group
2	$+1$ -1	$+1$ $+1$	MW	1	$SO(N_L) \times SO(N_R)$
3	$+1$	$+1$	M	2	$SO(N)$
4	$+1$ -1	$+1$ -1	M or W	4	$U(N)$
5	-1	-1	SM	8	$Usp(2N)$
6	$+1$ -1	-1 -1	SMW	8	$Usp(2N_L) \times Usp(2N_R)$
7	$+1$	-1	SM	16	$Usp(2N)$
8	$+1$ -1	-1 $+1$	M or W	16	$U(N)$
9	-1	$+1$	M	16	$SO(N)$
10	$+1$ -1	$+1$ $+1$	MW	16	$SO(N_L) \times SO(N_R)$
11	$+1$	$+1$	M	32	$SO(N)$
12	$+1$ -1	$+1$ -1	M or W	64	$U(N)$
...	...	...	...	...	...

Table 1: The possible values for η and ϵ together with the resulting minimal spinor types, the minimal number of real supercharges and the general form of the R-symmetry groups. (M = Majorana, SM = Symplectic Majorana, W= Weyl, MW= Majorana-Weyl, SMW = Symplectic Majorana-Weyl).

The minimal amount of supersymmery in each spacetime dimension is generated by a spinor operator that corresponds to the minimal spinor representation of the Lorentz group in the respective spacetime dimension. Extended supersymmetries then correspond to multiples of such minimal spinors. The R-symmetry group of the corresponding supersymmetry algebra has to respect these reality and chirality conditions and thus depends on the minimal spinor type as shown in table 1. If the scalar fields of a given type of multiplet transform nontrivially under the R-symmetry group (or a factor thereof), the holonomy group of the scalar manifold typically contains this group (factor) as a factor. Especially for large amounts of supersymmetry, this already strongly constrains the possible scalar manifolds.

For more than 32 real supercharges, one always has states with helicity $h > 2$ in the supermultiplets, which, for Lorentzian signature, limits supersymmetric field theories to $D \leq 11$.